From Geometric Invariants to Integrable Systems via Lie Groups

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The tangential flow is a centro-affine realization of the Volterra model.

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Moving frames and invariants for curves and polygons

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Definition

A (left or right invariant) moving frame of order k is an equivariant map

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A differential invariant for parametrized curves is a map

$$I : J^{(k)}(\mathbb{R}, M) \to \mathbb{R}$$

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Note: if ρ is a left (res. right) moving frame, so is ρg (resp. $g\rho$), g invariant .

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and its entries generate all differential invariants under minimal assumptions (Hubert 07).

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where $C^\infty(S^1, H)$ acts on $C^\infty(S^1, \mathfrak{g})$ via the left (resp. right) gauge action.

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Theorem

(Mansfield, MB, Wang, 13) Assume $M = G/H$, H subgroup of G and isotropic group of $p \in M$. The set of left (resp. right) N-periodic Maurer-Cartan matrices can be identified with an open subset (often dense) of

$$G^N/H^N$$

H^N acts on G^N via the left (resp. right) gauge action $(h_n, g_n) \to h_n^{-1} g_n h_{n+1}$.

Hamiltonian structures

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The most common Hamiltonian system, with Hamiltonian function $h(\mathbf{q}, \mathbf{p})$, $\mathbf{q}, \mathbf{p} \in \mathbb{R}^m$, is

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with $z(t,x) \in \mathbb{R}^m$ smooth and P a linear differential operator, $\mathcal{F}(z) = \int_{S^1} f(z)dx$ the Hamiltonian, and $\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{F}(z+\epsilon y) = \int_{S^1}\langle \delta F, y\rangle dx$ the variational derivative

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It also applies to differential-difference Hamiltonian systems of the form

$$(v_n)_t = P_n\delta_n h$$

with v_n a periodic bi-infinite sequence, $h : M^N \to \mathbb{R}$ the Hamiltonian, and $\delta_n h$ its variational derivative in the n-direction

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It also applies to differential-difference Hamiltonian systems of the form

$$(v_n)_t = P_n\delta_n h$$

with v_n a periodic bi-infinite sequence, $h : M^N \to \mathbb{R}$ the Hamiltonian, and $\delta_n h$ its variational derivative in the n-direction. The Poisson bracket (satisfying #1 and Jacobi) would be defined as

$$\{f,h\}(v) = \sum_n \langle \delta_n f, P_n \delta_n h\rangle$$

with $\langle,\rangle$ an inner product and P_n a linear *difference* operator.

Examples

1. Korteweg de Vries equation

$$v_t = v_{xxx} + 3vv_x = \left(\frac{d^3}{dx^3} + v\frac{d}{dx} + \frac{d}{dx}v\right)\delta\frac{1}{2}v^2, \quad h = \frac{1}{2}\int_{S^1} v^2$$

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2. Volterra equation

$$\begin{aligned} (u_n)_t &= u_n(u_{n+1} - u_{n-1}) \quad \text{or} \quad \text{if} \quad u_n = p_n p_{n-1} \\ (p_n)_t &= p_n^2(p_{n+1} - p_{n-1}) = p_n^2(\mathcal{T} - \mathcal{T}^{-1})p_n^2\, \delta \ln p_n, \quad h = \sum_n \ln p_n \end{aligned}$$

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3. Toda Lattice

$$(u_n)_{tt} = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1})$$

$$\text{or} \qquad \text{if} \quad q_n = (u_n)_t, \quad p_n = \exp(u_n - u_{n+1})$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix}_t = \begin{pmatrix} p_n(q_n - q_{n+1}) \\ p_{n-1} - p_n \end{pmatrix} = \begin{pmatrix} p_n(\mathcal{T}^{-1} - \mathcal{T})p_n & p_n(1 - \mathcal{T})q_n \\ -q_n(1 - \mathcal{T}^{-1})p_n & \mathcal{T}^{-1}p_n - p_n\mathcal{T} \end{pmatrix}\delta q_n, \; h = \sum_n q_n.$$

BiHamiltonian systems

Definition

Two Hamiltonian structures defined by Poisson brackets $\{,\}_1$ and $\{,\}_2$ are compatible, whenever $\{f,h\} = \{f,h\}_1 + \{f,h\}_2$ is also Poisson.

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Semisimple Lie groups have natural Poisson structures both on $\mathfrak{g}^*$ (continuous) and on the group (discrete)

Poisson brackets on $C^\infty(\mathbb{R}, \mathfrak{g})$ and Poisson Lie Groups

Semisimple Lie groups have natural Poisson structures both on $\mathfrak{g}^*$ (continuous) and on the group (discrete). If $\mathcal{F} : C^\infty(\mathbb{R}, \mathfrak{g}) \to \mathbb{R}$, then $\delta\mathcal{F} \in C^\infty(\mathbb{R}, \mathfrak{g}^*)$, $\mathfrak{g}^* \simeq \mathfrak{g}$

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Theorem

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$$C^\infty(S^1, \mathfrak{g})/C^\infty(S^1, H).$$

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The bracket is Poisson, and its symplectic leaves are (discrete) gauge orbits (Semenov-Tian-Shansky, 85).

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Theorem

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$$G^N / H^N.$$

(MB and Wang, 13) The equations

$$\begin{cases} (a_n^i)_t = \dfrac{a_{n+1}^{i+1}}{a_{n+1}^1} - \dfrac{a_n^{i+1}}{a_{n-i}^1}, & i = 1, 2, \cdots, m-1 \\ (a_n^m)_t = \dfrac{1}{a_{n+1}^1} - \dfrac{1}{a_{n-m}^1} \end{cases}$$

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are a discretization of so called Generalized KdV systems . They have a projective/ $\mathbb{R}^m$ centro-affine geometric realization of the form

$$(\gamma_n)_t = -\frac{m}{m+1}\gamma_n - \frac{a_{n-1}^2}{a_{n-1}^1}\gamma_{n+1} - \frac{a_{n-1}^3}{a_{n-1}^1}\gamma_{n+2} - \cdots - \frac{a_{n-1}^m}{a_{n-1}^1}\gamma_{n+m-1} + \frac{1}{a_{n-1}^1}\gamma_{n+m}.$$

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Case $m = 2$: if $\mathcal{T} a_n^k = a_{n+1}^k$ and $\mathcal{R} = \mathcal{T}^{-1} + 1 + \mathcal{T}$ (invertible if $N \neq 3s$ for any s)

$$\{f, h\}_2(\mathbf{a}) = \sum_n \begin{pmatrix} \frac{\partial f}{\partial a_n^1} & \frac{\partial f}{\partial a_n^2} \end{pmatrix} \begin{pmatrix} \mathcal{T}^{-1} a_n^2 - a_n^2 \mathcal{T} & \mathcal{T} - \mathcal{T}^{-2} \\ \mathcal{T}^2 - \mathcal{T}^{-1} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial a_n^1} \\ \frac{\partial h}{\partial a_n^2} \end{pmatrix}$$

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Lifting the biHamiltonian picture to projective polygons

Consider invariant vector fields in the space of centro-affine twisted polygons $(\mathbb{R}^m)^N$, invariant under the centro-affine group preserving constant arc-length $\ell = \det(\gamma_n, \gamma_{n+1}, \dots, \gamma_{n+m-1}) = 1$

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$$L_n = a_n^0 + a_n^1 \mathcal{T} + \cdots + a_n^m \mathcal{T}^m - \mathcal{T}^{m+1}$$

with $L_n(\gamma_n) = 0$

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Finally, define

$$\omega_1 = \sum_n d_L\theta_n, \qquad \omega_2 = \sum_n d\theta_n.$$

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(Calini, MB, 22) Both ω_1 and ω_2 are closed forms on the space of projective vector fields preserving ℓ, (i.e., they are pre-symplectic).

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Corollary

(Calini, MB, 22) The Hamiltonian structures $\{,\}_1$ and $\{,\}_2$ are compatible if, and only if $d\omega_2 = \sum_n d^2\theta_n = 0$. The discretizations of generalized KdV are biHamiltonian and Liouville-integrable.

GRACIAS! THANKS! MERCI!