Associative submanifolds in some nearly parallel G_2-manifolds

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Nearly parallel G_2-structures

A G_2-structure on a 7-manifold M is a 3-form φ that is *non-degenerate* in a specific algebraic sense. A G_2-structure φ on M induces a Riemannian metric g_φ and a volume form vol_φ on M.

A G_2-structure φ is called *nearly-parallel* if it satisfies the first-order equation:

$$d\varphi = 4 *_\varphi \varphi.$$

Motivation for the study of nearly parallel G_2-structures comes from holonomy theory: **nearly parallel G_2-manifolds are models for conically singular manifolds with holonomy** Spin(7). More precisely, an 8-dimensional Riemannian cone $\left(M \times \mathbb{R}_+, g_c = dt^2 + t^2 g_M\right)$ has $\mathrm{Hol}(g_c) \subset \mathrm{Spin}(7)$ if and only if M admits a nearly parallel G_2-structure with $g_M = g_\varphi$.

Manifolds with holonomy contained in Spin(7) are Ricci flat, and it follows that if φ is nearly parallel then g_φ is Einstein with positive scalar curvature.

The prototypical nearly parallel G_2-manifold is the 7-sphere S^7 endowed with its Spin(7)-invariant G_2-structure, which induces the round metric. Other examples include *Sasaki-Einstein* 7-manifolds, as well as *3-Sasakian* 7-manifolds and their squashings, described later.

There is a classification of homogeneous nearly parallel G_2-structures. All examples are either 3-Sasakian, squashed 3-Sasakian, or Sasaki-Einstein, except for one outlier: the *Berger space* $\mathrm{SO}(5)/\mathrm{SO}(3)$.

Associative submanifolds

An oriented 3-dimensional submanifold N of a manifold M with G_2-structure φ is called an *associative submanifold* if $\varphi|_N = \mathrm{vol}|_N$.

If φ is nearly parallel, then the cone on N is a calibrated submanifold of the cone on M for the Cayley calibration $\Phi = r^3\, dr \wedge \varphi + r^4 *_\varphi \varphi$, so N is a minimal submanifold of M, although in general it will not be volume minimizing. Therefore, **associative submanifolds in nearly parallel G_2-manifolds are models for conically singular Cayley submanifolds.**

Further motivation for the study of associative submanifolds comes from the program of Donaldson and Segal to construct invariants of G_2-structures by combining counts of associative submanifolds and G_2-instantons.

The associative condition is a non-linear first-order PDE system. Local existence of solutions follows from an application of the Cartan-Kähler theorem, but in general it is hard to produce global solutions.

Our approach: ruled submanifolds

A natural condition on a submanifold is that it be ruled (foliated) by some special class of curves in the ambient space. The study of surfaces in $\mathbb{R}^3$ ruled by lines is a classical topic in differential geometry. For instance, Catalan proved in 1842 that the helicoid is the only non-planar ruled minimal surface in $\mathbb{R}^3$.

In the context of calibrated geometry and special holonomy, Bryant, Fox, Lotay, and Joyce have studied ruled calibrated submanifolds of Euclidean spaces and spheres. In these cases the choice of ruling curve is straightforward: lines or circles.

We apply similar techniques to study associative submanifolds in a nearly parallel G_2-manifold M. The first difficulty to overcome is to determine an appropriate class of ruling curves. Motivated by the general theory, we choose ruling curves that are also geodesics, but the exact choice depends on the specific ambient manifold M we are working with.

With a specific choice of ruling curves in hand, Let N denote the space of these curves and let Γ denote the map that sends a point $p \in N$ to the curve $C \subset M$. A surface $S \subset N$ then gives rise to a 3-fold $\Gamma(S) \subset M$.

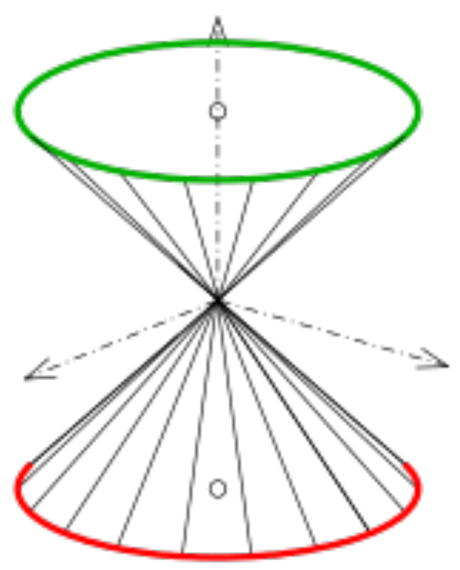
Key question: What conditions on $S \subset N$ ensure that the 3-fold $\Gamma(S) \subset M$ is associative?

Technical detail: singularities

Our main goal is to construct *smooth* associatives submanifolds n a nearly parallel G_2-manifold M using the process described above. However, the 3-fold $\Gamma(S)$ may be singular even when the surface $S \subset N$ is smooth.

This issue already appears in the classical case of ruled submanifolds in $\mathbb{R}^3$. For instance, a cone in $\mathbb{R}^3$ is the image of a smooth curve in the space of lines in $\mathbb{R}^3$, but it has a singularity at its apex.

We deal with this issue by identifying the geometric properties of S that cause singularities to occur in $\Gamma(S)$ and constructing surfaces $S \subset N$ without these properties. On the other hand, there is interest in examples of singular associative submanifolds and our approach also allows us to construct ruled associatives with certain kinds of singularities.



Associatives in squashed 3-Sasakian manifolds

A Riemannian 7-manifold is said to be *3-Sasakian* if its metric cone has holonomy contained in Sp(2). Every smooth 3-Sasakian manifold M can be realized as a the total space of an S^3 or SO(3)-bundle over a positive self-dual Einstein orbifold X and M is also a circle bundle over the *twistor space* Z of X.

We have $\mathrm{Sp}(2) \subset \mathrm{Spin}(7)$, so any 3-Sasakian 7-manifold carries a nearly-parallel G_2-structure. Varying the 3-form φ by scaling the fibers of $M \to X$, one obtains a 1-parameter family of G_2-structures, exactly one other of which is nearly-parallel. This alternative nearly-parallel G_2-structure is called the *squashed* structure on M.

The ruling curves we consider are the geodesics in M tangent to the fibers of $M \to X$. We call these *Hopf circles.* The space of Hopf circles in M is diffeomorphic to $Z \times S^2$.

Theorem: There is an almost complex structure J on $Z \times S^2$ such that any Hopf-ruled associative in M is locally the Γ-image of some J-holomorphic curve S. Conversely, for each J-holomorphic curve S in $Z \times S^2$ that does not lie in an $S^2 \times S^2$-fiber over X, there is a discrete subset $D \subset S \subset S$ such that $\Gamma(S \setminus D)$ is a Hopf-ruled associative submanifold of M.

The prime examples of 3-Sasakian 7-manifolds are the 7-sphere S^7 and the Aloff-Wallach space $N_{1,1}$. In both cases, we are able to construct many J-holomorphic curves with $D = \emptyset$ and produce smooth associative submanifolds.

Associatives in the Berger space

The group SO(3) acts irreducibly on $\mathbb{R}^5 \cong \mathrm{Sym}^2_0(\mathbb{R}^3)$, giving rise to a non-standard inclusion $\mathrm{SO}(3) \subset \mathrm{SO}(5)$. The *Berger space* B is the resulting homogeneous space $\mathrm{SO}(5)/\mathrm{SO}(3)$. The Berger space has a number of remarkable properties: it is an isotropy irreducible space, the homogeneous metric on B has positive curvature, and topologically B is an S^3-bundle over S^4 and a rational homology sphere.

B carries a unique SO(5)-invariant nearly-parallel G_2-structure. In fact, the cone on B was the first known example of an explicit metric with holonomy group equal to Spin(7).

B can be realized explicitly as the space of *Veronese surfaces* in the 4-sphere S^4. The set of Veronese surfaces in S^4 tangent to a given 2-plane $P \in T_pS^4$ is a geodesic $C \subset B$. We call such curves $\mathcal{C}$-curves. The space of $\mathcal{C}$-curves is diffeomorphic to the Grassmannian $\mathrm{Gr}_2(TS^4) \cong \mathrm{SO}(5)/\mathrm{T}^2$.

Theorem: There is an SO(5)-invariant almost complex structure J on $\mathrm{Gr}_2(TS^4)$ such that any $\mathcal{C}$-ruled associative in B is locally the Γ-image of some J-holomorphic curve S. Conversely, for each J-holomorphic curve S in $\mathrm{Gr}_2(TS^4)$ there is a dense subset $S^\circ \subset S$ such that $\Gamma(S^\circ)$ is a $\mathcal{C}$-ruled associative submanifold of B.

As a consequence of this result, we are able to construct infinitely many diffeomorphism types of immersed associative submanifolds in B.

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References: *Associative Submanifolds of the Berger Space*, with Jesse Madnick (to appear in Communications in Analysis and Geometry)
Associative Submanifolds of Squashed 3-Sasakian Manifolds, with Jesse Madnick (in preperation)