Flags of subalgebras under contractions of Lie algebras

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Definitions

Lie algebras can be defined in terms of their structure constants. If a basis $(e_1,\dots,e_n)$ of V is fixed, the algebra is given by Lie brackets of basis elements:

$$[e_i,e_j]=c_{ij}^k e_k,$$

c_{ij}^k are called *structure constants* of the algebra $\mathfrak g$.
Given a matrix-valued function $U\colon (0,1]\to \mathrm{GL}(n)$, $\varepsilon\mapsto U_\varepsilon$, we define the transformed Lie bracket $\mu_\varepsilon=\mu\circ U_\varepsilon=[\cdot,\cdot]_\varepsilon\colon V\times V\to V$ according to $\mu_\varepsilon(x,y):={U_\varepsilon}^{-1}\mu(U_\varepsilon x,U_\varepsilon y)$, $x,y\in V$. If for any $x,y\in V$ there exists the limit

$$\lim_{\varepsilon\to+0}\mu(x,y)_\varepsilon=:\mu_0(x,y)$$

then $\mu_0=:[\cdot,\cdot]_0$ is a well-defined Lie bracket. The Lie algebra $\mathfrak g_0=(V,[\cdot,\cdot]_0)$ is called a *contraction* of the Lie algebra $\mathfrak g$.

Contraction Criteria

Different criteria exist to demonstrate impossibility of contraction.

Many of criteria make no distinction between the pairs of algebras $(\mathfrak g,\mathfrak g_0)$ and $(\mathfrak g\oplus m\mathfrak g_1,\mathfrak g_0\oplus m\mathfrak g_1)$. But sometimes $\mathfrak g\oplus m\mathfrak g_1\to\mathfrak g_0\oplus m\mathfrak g_1$ even when $\mathfrak g\not\to\mathfrak g_0$.

In particular, there is no contraction between complex 3D Lie algebras $\mathfrak g_{3.3}$: $[e_1,e_3]=e_1$, $[e_2,e_3]=e_2$ and $\mathfrak g_{3.1}=\mathfrak h_1$: $[e_2,e_3]=e_1$, but $\mathfrak g_{3.3}\oplus\mathfrak g_1\to\mathfrak g_{3.1}\oplus\mathfrak g_1$.

The situation for real Lie algebras that have the same complexifications is similar:

$$\mathfrak{sl}(2,\mathbb C)\to\mathfrak p^{\mathbb C}(1,1),\quad \mathfrak{sl}(2,\mathbb R)\to\mathfrak p(1,1)\quad\text{but}\quad \mathfrak{so}(3)\not\to\mathfrak p(1,1),$$

where $\mathfrak p(1,1)=A_{3.4}^{-1}$: $[e_1,e_3]=e_1$, $[e_2,e_3]=-e_2$.

Triangular Matrices Suffice

A Lie algebra $\mathfrak g$ is sequentially contracted to a Lie algebra $\mathfrak g_0$ if and only if in the fixed basis $(e_1,\dots,e_n)$ of the underlying space V there exists the sequence $\{L_p,\,p\in\mathbb N\}$ of nondegenerate lower triangular $n\times n$ matrices and an orthogonal (resp. unitary) $n\times n$ matrix Q in the real (resp. complex) case such that

$$C\circ L_p\to C_0\circ Q\quad\text{as}\quad p\to\infty.$$

References

[1] Popovych D.R., Flags of subalgebras in contracted Lie algebras, *Reports of the National Academy of Sciences of Ukraine* (2021), no. 4, 9–17. (in Ukrainian)

Flags of Subalgebras

Theorem. Suppose that a Lie algebra $\mathfrak g_0$ is a contraction of the Lie algebra $\mathfrak g$, $\mathfrak g\to\mathfrak g_0$, and the algebra $\mathfrak g$ contains a flag of subalgebras

$$\{0\}=\mathfrak s^0\subset\mathfrak s^1\subset\mathfrak s^2\subset\dots\subset\mathfrak s^m\subset\mathfrak s^{m+1}=\mathfrak g.$$

Then the algebra $\mathfrak g_0$ contains a flag of subalgebras

$$\{0\}=\mathfrak s_0^0\subset\mathfrak s_0^1\subset\mathfrak s_0^2\subset\dots\subset\mathfrak s_0^m\subset\mathfrak s_0^{m+1}=\mathfrak g_0$$

such that $\dim\mathfrak s_0^a=\dim\mathfrak s^a$ and $\mathfrak s^a\to\mathfrak s_0^a$, $a=1,\dots,m$. If $\mathfrak s^a$ is an ideal in $\mathfrak s^b$, $1\leqslant a<b\leqslant m+1$, then $\mathfrak s_0^a$ can be chosen to be an ideal in $\mathfrak s_0^b$, and $\mathfrak s^b/\mathfrak s^a\to\mathfrak s_0^b/\mathfrak s_0^a$. More generally, provided $[\mathfrak s^a,\mathfrak s^b]\subseteq\mathfrak s^c$ for some $a,b,c\in\{1,\dots,m\}$, we have $[\mathfrak s_0^a,\mathfrak s_0^b]_0\subseteq\mathfrak s_0^c$, where also $\dim[\mathfrak s_0^a,\mathfrak s_0^b]_0\leqslant\dim[\mathfrak s^a,\mathfrak s^b]$. Similar assertions hold for arbitrary composition of commutators on any tuple of subalgebras $\mathfrak s^a$.

Simple consequences

The properties of commutativity, nilpotency, solvability and unimodularity are stable under contractions.

Hence $\mathfrak s_0^a$ inherits the respective properties of $\mathfrak s^a$.

Corollary. The dimensions of the following objects do not decrease under contractions of Lie algebras:

- maximal Abelian subalgebras,
- maximal nilpotent subalgebras,
- maximal solvable subalgebras,
- maximal Abelian ideals,
- maximal nilpotent ideals (nilradicals),
- maximal solvable ideals (radicals).

Corollary. The dimensions of the elements of the derived series and of the descending central series do not increase under contractions of Lie algebras.

Denote by $\mathrm d_{\mathfrak g}(h)$ the highest dimension for a subalgebra $\mathfrak s$ of $\mathfrak g$ such that $\dim[\mathfrak s,\mathfrak g]\leqslant h$.

Corollary. Suppose that a Lie algebra $\mathfrak g_0$ is a contraction of a Lie algebra $\mathfrak g$, $\mathfrak g\to\mathfrak g_0$, and $\dim\mathfrak g=\dim\mathfrak g_0=n<\infty$. Then for any $0\leqslant h\leqslant n$, one has

$$\mathrm d_{\mathfrak g}(h)\leqslant\mathrm d_{\mathfrak g_0}(h).$$

Future Research

$d_{\mathfrak g}(0)$ is dimension of center and is used as a criterion.

Can at least $d_{\mathfrak g}(1)$ serve as a meaningful criterion?

Nilpotent algebras

We use the classification of contractions of 5D nilpotent Lie algebras by Grunewald&O'Halloran (1988) and the classifications of 6D nilpotent Lie algebras by Morozov (1958) and Magnin (1986).

Lemma. *(i) There is are no contraction from any of the algebras*

$$\mathfrak g_{6.2},\ \mathfrak g_{6.4},\ \mathfrak g_{6.5},\ \mathfrak g_{6.9},\ \mathfrak g_{6.9}^{\mathbb R},\ \mathfrak g_{6.10},\ \mathfrak g_{6.11},\ \mathfrak g_{6.12},\ \mathfrak g_{6.14},\ \mathfrak g_{6.17},\ \mathfrak g_{6.18}$$

to the algebra $(2\mathfrak g_3)^{\mathbb R}$.

(ii) There is are no contraction from any of the algebras

$$\mathfrak g_{6.12},\quad \mathfrak g_{6.14},\quad \mathfrak g_{6.17},\quad \mathfrak g_{6.18}$$

to the algebra $\mathfrak g_{6.5}^{\mathbb R}$.

Consider 5D nilpotent Lie algebras

$$\mathfrak g_{5.2}\colon\quad [e_1,e_2]=e_4,\ [e_1,e_3]=e_5$$
$$\mathfrak g_{5.4}\colon\quad [e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_2,e_3]=e_5$$

We prove that all 5D subalgebras of $(2\mathfrak g_3)^{\mathbb R}$ are isomorphic to the algebra $\mathfrak g_{5.2}$.

Relying on the full list of contractions between 5D nilpotent real Lie algebras by Grunewald and O'Halloran, we find 5D subalgebras of the listed 6D algebras that have no contraction to $\mathfrak g_{5.2}$.

Applying our theorem gives the part (i) of the lemma.

Similarly, all 5D subalgebras of $\mathfrak g_{6.5}^{\mathbb R}$ are isomorphic to either of the algebras $\mathfrak g_{5.2}$ and $\mathfrak g_{5.4}$, leading to the part (ii) of the lemma.

Conclusion

We prove a theorem that describes the behavior of subalgebra flags of Lie algebras under contractions and can be applied as a new criterion for non-existence of contraction.

A weaker version of the theorem is obtained for flags of subspaces.

Using the theorem, we prove the non-existence of contractions for a number of pairs of six-dimensional nilpotent real Lie algebras, for which the earlier known criteria do not work.

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