Invariant Geometric Flows

in Affine-related Geometries

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Outline

◇ Introduction

◇ Invariant geometric flows

- Centro-equiaffine geometry
- Affine geometry
- Centro-affine geometry
- Centro-equiaffine symplectic geometry
- Affine symplectic geometry

1. Introductions

◇ Questions

- Invariant differential equations under certain Lie groups?
- Geometric realizations of integrable systems?
- Geometric formulation of various properties to integrable systems?
- Heat flows in various geometries?
- Symmetries and solutions of the geometric flows?

◇ Integrable curve flows

Di Rios (1906), Hasimoto (1972), Lamb (1977), Lakshmanan (1979), Langer and Perline (1991), Doliwa and Santini (1994), Goldstein and Petrich (1991), Nakayama, Segur and Wadati (1992), Pinkall (1995), Terng and Uhlenbeck (1998-2008), Chou and Qu (2002, 2003), Marí Beffa, Sanders and Jingping (2000), Marí Beffa (2002-, Anco et al (2006-), Calini, Ivey, Beffa (2013), Olver (2008), Valiquette (2012), Song, Qu (2012), Musso, (2012), Musso (2009), Asadi and Sanders (2009), Terng, Wu (2015-2021)

◇ Euclidean curve shortening flow (CSF)

Mullins, Gage,Hamilton (1986), Grayson (1987-1989), Angenent (1991), Chou, X.P. Zhu (1997-2005), Altschuler (1991), Hamilton (1995), Oaks, Abresch, Langer (1986) .

◇ Affine CSF

Sapiro, Tannenbaum (1994), Angenent, Sapiro, Tannenbaum (1998), Calabi, Olver, Tannenbaum (1996), Andrews (1999), Chou, Li (2002).

◇ Centro-equiaffine CSF

Wo, Wang, Qu, (2018),

◇ Centro-affine CSF

Olver, Qu, Yang, (2020), Yang, Qu (2021)

2. Works by Olver et al

◇ Symmetries, Invariant theory, Hamiltonian systems

- P. Olver, Applications of Lie Groups to Differential Equations, 2nd edn, Springer-Verlag, New-York, 1993;
- P. Olver, Equivalence, Invariants, and Symmetry, Cambridge Univ. Press, Cambridge, UK, 1995;

◇ Invariant differential equations

Consider the evolution equation

$$u_t = K(x, u, u_x, \cdots, u^{(n)}). \tag{1}$$

Theorem 2.1. (Olver, 1993) Let G be a transformation group acting on $E \simeq X \times U$. Suppose $L(x, u^{(n)})$ is a G-Lagrangian such that $E(L) \neq 0$. Then every G-invariant evolution equation has the form

$$u_t = \frac{L}{E(L)} I,$$

where I is an arbitrary differential invariant of G.

Theorem 2.2. (Olver, Sapiro and Tannenbaum, 1994) Let G be a subgroup of projective group $SL(3)$. Let $ds = Ldx$ denote the G-invariant one-form of lowest-order and I its fundamental differential invariant. Then every G-invariant evolution equation has the form

$$u_t = \frac{u_{xx}}{L^2} J, \tag{2}$$

where J is an arbitrary differential invariant of G.

Ex. 2.1. The WKI (Wadati-Konno-Ichikawa) equation

$$u_t = \left(\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right)_x$$

is an special example of (2) by choosing

$g = \sqrt{1+u_x^2}$, $J = \left(\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right)_x \frac{1+u_x^2}{u_{xx}}$.

Ex. 2.2. The curve shortening equation in Euclidean geometry

$$u_t = \frac{u_{xx}}{1+u_x^2}$$

and the affine curve shortening equation

$$u_t = (u_{xx})^{\frac{1}{3}}$$

are the special examples of (2).

Ex. 2.3. The curve shortening equation in centro-equiaffine geometry

$$u_t = \frac{u_{xx}}{(xu_x - u)^2}$$

and the equation (equivalent to KdV equation)

$$u_t = \left(\frac{u_{xx}}{(xu_x-u)^3}\right)_x, \quad g = xu_x - u, \quad J = \left(\frac{u_{xx}}{(xu_x-u)^3}\right)_x \frac{(xu_x-u)^2}{u_{xx}}.$$

are the special examples of (2).

◇ Moving coframe method

Definition 2.1. (Fels and Olver, 98,99) Given a transformation group G acting on a manifold M, a moving frame is a smooth G-equivariant map

$$\rho : M \to G.$$

Theorem 2.3. (Fels and Olver, 98,99) If G acts on M, then a moving frame exists in a neighborhood of a point $x \in M$ if and only if G acts freely and regularly near x.

Theorem 2.4. (Fels and Olver, 98,99) If $\rho(x)$ is a moving frame, then the components of the map $I\colon M \to M$ defined by $I(x) = \rho(x) \cdot x$ provide a complete set of invariants for the group.

◇ Klein geometries on the plane

According to the Erlangen program, a Klein geometry is the theory of geometric invariants of a transformation group. Lie did the classification for complex vector fields, and the classification for real vector fields was carried out by Gonzalez-Lopez, Kamran, Olver (1992) and Olver (Equivalence, Invariants, and Symmetry, 1995).

$E_\alpha(2)$	$\partial_x, \partial_u, u\partial_x - x\partial_u + \alpha(x\partial_x + u\partial_u)$	3	$R \ltimes R^2$
$SL(2)$	$x\partial_u, u\partial_x, x\partial_x - u\partial_u$	3	$\mathfrak{sl}(2)$
$SO(3)$	$u\partial_x - x\partial_u, (1+\tilde{x})\partial_x + 2xu\partial_u,$ $2xu\partial_x + (1-\tilde{x})\partial_u$	3	$\mathfrak{so}(3)$
$Sim(2)$	$\partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u$	4	$R^2 \ltimes R^2$
$SA(2)$	$\partial_x, \partial_u, x\partial_x - u\partial_u, x\partial_u, u\partial_x$	5	$\mathfrak{sa}(2)$
$A(2)$	$\partial_x, \partial_u, x\partial_x, u\partial_u, x\partial_u, u\partial_x$	6	$\mathfrak{a}(2)$
$SO(3,1)$	$\partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u,$ $\tilde{x}\partial_x + 2xu\partial_u, 2xu\partial_u - \tilde{x}\partial_u$	6	$\mathfrak{so}(3,1)$
$SL(3)$	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u,$ $x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u$	8	$\mathfrak{sl}(3)$

Primitive Lie algebras of vector fields in R^2

where $\tilde{x} = x^2 - u^2$.

$S(1,k)$	$\partial_x, x\partial_x + \alpha u\partial_u, \partial_u, x\partial_u, \cdots, x^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1) \times R^k$
$S'(1,k)$	$\partial_x, x\partial_x + (ku + x^k)\partial_u, \partial_u, x\partial_u, \cdots, x^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1) \times R^k$
$S^2(1,k)$	$\partial_x, x\partial_x, u\partial_u, \partial_u, x\partial_u, x^2\partial_u, \cdots, x^{k-1}\partial_u$	$k+3$	$\mathfrak{c}(1) \times R^k$

3a: Imprimitive Lie algebras of vector fields on R^2 reducible to $S(1)$

$S(1,k)$	$\partial_x, x\partial_x + \alpha u\partial_u, \partial_u, x\partial_u, \cdots, x^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1) \times R^k$
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3a: Imprimitive Lie algebras of vector fields on R^2 reducible to $S(1)$

◇ Invariant variational bicomplex and geometric flows

Olver (2008) developed the invariant variational bicomplex approach to study the invariant geometric flows. Consider the flow determined by

$$\frac{\partial\gamma}{\partial t} = \sum_{j=1}^{p} I^j t_j + \sum_{\alpha=1}^{q} J^\alpha n_\alpha. \qquad (3)$$

Olver proved the following result.

Theorem 2.5. (Olver, 2008) The flow (3) is intrinsic if and only if

$$\mathcal{D}_j I^i + \sum_{k=1}^{p} Y^i_{jk} I^k + \sum_{\alpha=1}^{q} \mathcal{B}^i_{j\alpha}(J^\alpha) = 0.$$

For $p = 1$-dimensional submanifolds, given a differential invariant k, there is the following result.

Theorem 2.6. (Olver, 2008) Under the arc-length preserving flow,

$$k_t = \mathcal{R}_k(J), \quad \text{where } \mathcal{R}_k = \mathcal{A}_k - k_s \mathcal{D}^{-1}\mathcal{B}$$

is the characteristic operator associated with k. More generally, the time evolution of $k_n = \mathcal{D}^n k$ is given by the arc-length differentiation:

$$\frac{\partial k_n}{\partial t} = \mathcal{R}_{k_n}(J) = \mathcal{D}^n \mathcal{R}_k(J).$$

◇ Local symplectic invariants for curves

According to Darboux Theorem, symplectic submanifolds have no local differential invariants. It is of interest to study local symplectic invariants in affine geometry. In (Kamran, Olver, Tenenblat), they construct the local affine symplectic invariants by the standard Schmidt orthogonal scheme or the moving frame method. They proved the fundamental theorem for curve in affine symplectic geometry.

Theorem 2.7. Let $H_2, \cdots, H_n$ and $K_1, \cdots, K_n$ be $2n-1$ smooth real valued functions defined on an interval I with $H_j \neq 0$ for all $2 \leq j \leq n-1$. There exists, up to a rigid symplectic motion of (R^{2n}, Ω), a unique symplectic regular curve $z : I \to R^{2n}$, parametrized by symplectic arc-length, whose local symplectic invariants are the given functions $H_2, \cdots, H_n$ and $K_1, \cdots, K_n$.

Definition 2.2. (Kamran, Olver, Teneblat, 2009) The affine symplectic arc-length parameter s is defined by

$$s = \int_{x_0}^{x} < u_x; u_{xx} >^{\frac{1}{3}} dx.$$

For any such curve, we can associate an adapted symplectic frame $(a_1, \cdots, a_n, a_{n+1}, \cdots, a_{2n})$. They satisfy the Frenet formulae

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix}_s = B \begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix}, \tag{4}$$

with

$$B = \begin{pmatrix} & \begin{matrix} 1 & & & & \\ & H_2 & & & \\ & & H_3 & & \\ & & & \ddots & \\ & & & & H_n \end{matrix} \\ \begin{matrix} K_1 & 1 & & & \\ 1 & K_2 & 1 & & \\ & 1 & K_3 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & K_n \end{matrix} & \end{pmatrix}.$$

Theorem 2.8. (Valiquette, 2012) The invariant curve flow in $AS(4, R)$

$$\gamma_t = a_1$$

gives the following bi-Hamiltonian system

$$K_t = \mathcal{P}_1 \delta \mathcal{H}_1 = \mathcal{P}_2 \delta \mathcal{H}_2, \tag{5}$$

with

$$\mathcal{H}_1 = \int (\frac{1}{4} I^2 - H) dx,$$
$$\mathcal{H}_2 = \int (\frac{1}{8} H^3 + \frac{1}{16} I'^2 + J H^2 - \frac{3}{2} H I) dx.$$

Indeed, (5) can be written explicitly

$$
\begin{aligned}
H_t &= -H''' + 3HI' + \frac{3}{2}IH' - 3(JH^2)', \\
I_t &= -\frac{1}{4}I''' + \frac{3}{2}II' - 3J', \\
J_t &= -J''' + 3J^2H' + \frac{3}{2}IJ' + \frac{3}{4}(\frac{I''}{J})' - 3(\frac{(H'J)'}{H})'.
\end{aligned}
$$

◇ Applications of affine geometric flows

- Discrete affine curve flows in Computer vision (Olver, Sapiro, Tenenbaum, 1998);
- Affine invariant and geometirc flows in image denoising, edge detection,(Olver, Sapiro, Tenenbaum, 1999). ;

◇ Classification of integrable equations

- Integrable equations on associated algebras (Olver, Sokolov, 1998)
- Non-abelian integrable systems of the DNLS-types (Olver, Sokolov, 1998)

◇ Questions

Geometric formulations to the resulting equations?

◇ **Invariant geometric flows** For a submanifold M, Let α_i, $i = 1, 2, \cdots, n$ be its frame. A submanifold flow is governed by

$$\frac{\partial \gamma}{\partial t} = \sum_{i=1}^{n} f_i \alpha_i,$$

where f_i are the velocities of the flow depending on the curvatures of the curve and its derivatives with respect to the arclength. For the plane curve, the flow is

$$\frac{\partial \gamma}{\partial t} = fN + gT, \qquad (6)$$

where f and g are normal and tangent velocities, respectively.

3. Integrable flows in $CEA(n, R)$

The geometry is invariant under the centro-equiaffine transformations

$$x' = Ax,$$

where $A \in SL(n, R)$.

- n-dimensional centro-equiaffine geometry $CEA(n, R)$

The centro-equiaffine arclength $d\tilde{s}$ is

$$d\tilde{s} = [\gamma, \gamma_p, \cdots, \gamma_p^{(n-1)}]^{\frac{2}{n(n-1)}} dp.$$

The centro-equiaffine curvatures ϕ_i, $i = 1, 2, \cdots, n-1$ are defined by

$$\phi_i = [\gamma, \gamma_{\tilde{s}}, \cdots, \gamma_{\tilde{s}}^{(i-2)}, \gamma_{\tilde{s}}^{(i)}, \cdots, \gamma_{\tilde{s}}^{(n-1)}, \gamma_{\tilde{s}}^{(n)}].$$

The frame $\vec{f} = (\gamma, \gamma_{\tilde{s}}, \cdots, \gamma_{\tilde{s}}^{(n-1)})$ is the centro-equiaffine frame. (Calini, Ivey, Beffa, 2013; Terng, Wu, 2019)

On the planar case, the centro-equiaffine tangent and normal vectors: $T = \gamma_{\tilde{s}}$ and $N = -\gamma$. The centro-equiaffine Serret-Frenet formulas

$$T_{\tilde{s}} = \phi N, \quad N_{\tilde{s}} = -T.$$

- The KdV flow

Theorem 3.1. (Pinkall, 1995) The Pinkall's flow

$$\gamma_t = \phi_{\tilde{s}} N + \phi T.$$

is equivalent to KdV equation

$$\phi_t + \phi_{\tilde{s}\tilde{s}\tilde{s}} + 6\phi\phi_{\tilde{s}} = 0$$

- Geometrical Hamiltonian structure

In the following, $\mathcal{G}$ and $\mathcal{H}$ are functionals on $\mathcal{L}g^*$, and $\delta\mathcal{G}/\delta L$, $\delta\mathcal{H}/\delta L$ are the gradients at the point $L \in \mathcal{L}g^*$.

Theorem 3.2. (Calini, Ivey, Beffa, 2009) The following compatible Poisson structures on $\mathcal{L}g^*$, defined by

$$\{\mathcal{H},\mathcal{G}\}(L) = \int_{S^1} \mathrm{trace}\left(\left((\frac{\delta H}{\delta L})_x + \left[L, \frac{\delta H}{\delta L}\right]\right)\frac{\delta \mathcal{G}}{\delta L}\right) dx,$$

$$\{\mathcal{H},\mathcal{G}_0\}(L) = \int_{S^1} \mathrm{trace}\left([A, \frac{\delta H}{\delta L}]\frac{\delta \mathcal{G}}{\delta L}\right) dx, \qquad (7)$$

where $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in the projective case, can be reduced to the quotient M/LN to produce the second and first Hamiltonian structures, respectively, for KdV equation.

They are equivalent respectively to

$$\{h, f\}(\phi) = \int_{S^1} \frac{\delta f}{\delta \phi}(D^3 + 2\phi D + 2D\phi)\frac{\delta h}{\delta \phi} dx,$$

and

$$\{h, f\}_0(\phi) = 2\int_{S^1} \frac{\delta f}{\delta \phi} D \frac{\delta h}{\delta \phi} dx.$$

- The CH flow

Theorem 3.3. (Chou, Qu, 2002) The Camassa-Holm equation

$$m_t + 4u_{\tilde{s}}m + 2um_{\tilde{s}} + 2u_{\tilde{s}} = 0, \quad \phi = m = u - u_{\tilde{s}\tilde{s}}.$$

is equivalent to the flow

$$\gamma_t = -u_{\tilde{s}}N - 2uT.$$

- The Boussinesq flow

Theorem 3.4. (Chou, Qu, 2003) The Boussinesq equation

$$\beta_{tt} + \beta_{\tilde{s}\tilde{s}\tilde{s}\tilde{s}} - 4(\beta\beta_{\tilde{s}})_{\tilde{s}} = 0.$$

is equivalent to the flow in 3-dimensional geometry $CEA(3, R)$:

$$\gamma_t = -2\beta\gamma + 3\gamma_{\tilde{s}\tilde{s}}.$$

- $A^{(1)}_{n-1}$-KdV flow

Theorem 3.5. (Calini, Ivey, Marï Beffa, 2013; Terng, Wu, 2019) $A^{(1)}_{n-1}$-KdV hierarchy

$$L_t = [(L^{\frac{2}{n}})_+, L]$$

is equivalent to the flow in n-dimensional centro-equiaffine geometry:

$$\gamma_t = gZ_{j,0}(u)e_1,$$

where g and u are the centro-equiaffine moving frame and curvature, $e_1 = (1, 0, \cdots, 0)^t \in R^n$.

4. CSF in centro-equiaffine geometry

Consider the heat flow in $CEA(R,2)$

$$\gamma_t = \gamma_{\tilde{s}\tilde{s}}. \tag{8}$$

In terms of the curvature ϕ and the support function $h = -(\gamma, n)$, the heat flow can be written as

$$\phi_t = \phi_{\tilde{s}\tilde{s}} + 4\phi^2, \tag{9}$$

or

$$h_t = -\frac{1}{h^2(h_{\theta\theta} + h)}, \tag{10}$$

where $\tilde{s}$ is the arc-length parameter, θ is the angle between the tangent vector and a fixed direction.

Theorem 4.1. (Wo, Wang, Qu, 2018) Any convex curves governed by the centro-equiaffine heat flow (8) with the initial value γ_0 converges smoothly to an ellipse centered at the origin.

Theorem 4.2. (Wo, Wang, Qu, 2018) (The centro-equiaffine isoperimetric inequality) For any closed convex curve γ, there exists a point (x_0, y_0), such that the centro-equiaffine isoperimetric inequality

$$A \int_\gamma \phi d\sigma \leq 2\pi^2, \tag{11}$$

holds, where A is area enclosed by the curve γ, ϕ is the centro-equiaffine curvature of γ with respect to (x_0, y_0) and equality holds if and only if γ is an ellipse. Furthermore, if the curve γ is origin symmetric, then inequality (11) also holds for $(x_0, y_0) = (0, 0)$.

5. Geometric flows in Affine geometry $AF(n, R)$

The geometry is invariant under the unimodular affine transformations

$$x' = Ax + B,$$

where $A \in SL(n, R)$, $B \in R^n$.

• The Sawada-Kotera flow in $AF(2, R)$

Theorem 5.1. (Chou,Qu, 2002; Olver, 2008) The curve flow in $AF(2, R)$

$$\gamma_t = -3\mu_\rho N + (\mu_{\rho\rho} - \mu_\rho^2)T$$

is equivalent to the S-K equation

$$\mu_t + \mu_5 + 5(\mu\mu_{\rho\rho})_\rho + 5\mu^2\mu_\rho = 0.$$

This is based on the operator identity:

$$\begin{aligned}\Omega &= (\partial_y^3 + 2Q\partial_y + 2\partial_y Q) \cdot (2\partial_y^3 + 2\partial_y^2 Q \partial_y^{-1} \\ &\quad + 2\partial_y^{-1} Q\partial_y^2 + Q^2 \partial_y^{-1} + \partial_y^{-1} Q^2) \\ &= 2(\partial_y^4 + 5Q\partial_y^2 + 4Q_y\partial_y + Q_{yy} + 4Q^2 \\ &\quad + 2Q_y\partial_y^{-1}Q)\left(\partial_y^2 + Q + Q_y\partial_y^{-1}\right).\end{aligned}$$

(Chou, Qu, 2002)

6. CSF in affine geometry

Consider the heat flow in $Af(R,2)$

$$\gamma_t = \gamma_{\rho\rho}.$$

In terms of the affine curvature μ and local graph $(x, u(x,t))$, the heat flow are equivalent to the equations

$$\mu_t = \mu_{\rho\rho} + \mu^2, \tag{12}$$

$$u_t = u_{xx}^{\frac{1}{3}}. \tag{13}$$

Theorem 6.1 (Angenent, Sapiro, Tanenbaum, 1997) The curve governed by the affine heat flow (or affine shortening flow) (12) preserves convexity and shrinks any closed convex curve to an elliptic point. Furthermore, the family of dilated normalized curves converges in the Hausdorff metric to an ellipse.

7. Flows in centro-affine geometry $CA(n, R)$

The geometry is invariant under the centro-affine transformation

$$x' = Ax$$

where $A \in GL(n, R)$. For the planar case, we have the following result.

- The mKdV flow

Theorem 7.1. (Chou, Qu, 2003; Qu, Yang, 2021) The mKdV flow in the centro-affine geometry

$$\gamma_t = \frac{1}{2}(\kappa_s - \frac{1}{4}\kappa^2)\gamma_s + \kappa\gamma.$$

gives the mKdV equation

$$\kappa_t = \kappa_{sss} - \frac{3}{8}\kappa^2\kappa_s + 4\kappa_s.$$

- A geometric formulation of Miura transformation

Theorem 7.2. (Chou, Qu, 2003) The curvature ϕ in RP^1 is related to the curvature k in $CA(2,R)$ by the Miura transformation

$$\phi = -\frac{1}{2}\kappa_s - \frac{1}{8}\kappa^2 - 2.$$

8. CSF in centro-affine geometry

Consider the heat flow in $CA(2, R)$ (Olver, Qu, Yang, 2020)

$$\gamma_t = \gamma_{ss}. \tag{14}$$

In terms of the curvature κ, the heat flow can be written as the equations

$$\kappa_t = \kappa\kappa_s, \tag{15}$$

where s is the centro-affine arc-length parameter.

Theorem 8.1. (Olver, Qu, Yang, 2020) The centro-affine curve evolution process

$$\begin{aligned} \gamma_t &= \gamma_{ss}, \\ \gamma(s,0) &= \gamma_0, \end{aligned}$$

is equivalent to the initial problem of the inviscid Burgers' equation:

$$\begin{aligned} \frac{\partial \kappa}{\partial t} - \kappa \frac{\partial \kappa}{\partial s} &= 0, \\ \kappa(s,0) &= \kappa_0(s), \end{aligned}$$

where $\varphi(s)$ is the signed centro-affine curvature κ of the initial curve γ.

Theorem 8.2. (Olver, Qu, Yang, 2020) The curve family $\gamma(\bar{p}, t)$ converges smoothly to the origins as $t \to \infty$.

Thank you!

Happy birthday to Peter!