ODEs with non fiber-preserving Symmetry

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Joint work with
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Symmetry Invariants Applications
Peter Olver - 70
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Introduction: contact symmetry of scalar ODEs

Among the maximally symmetric scalar ODEs (over $\mathbb{C}$) of ord > 2 only the equation $y''' = 0$ has a contact symmetry algebra, namely $\mathfrak{sp}(4, \mathbb{C})$. For every $n > 3$ the ordinary differential equation $y^{(n)} = 0$ has the symmetry algebra $\mathfrak{gl}(2, \mathbb{C}) \ltimes S^{n-1}\mathbb{C}^2$ consisting of point transformations preserving fibers of $J^0 = \mathbb{C}^2(x, y) \to \mathbb{C}^1(x)$.

There are indeed, ODEs with non-point symmetries. Consider, for instance, the following two equations (in jet-notation $y^{(n)} = y_n$):

$$(i) \quad y_4 = 3y_3^2/y_2, \qquad (ii) \quad y_4 = 3y_3^2/y_2 + y_3^2/y_2^2.$$

Then (i) has (three) contact non-point symmetries but is trivializable, while (ii) has (only one) contact non-point symmetry and is not even linearizable. However the Legendre transformation $(x, y, y_1) \mapsto (-y_1, y - xy_1, x)$ maps these into ODEs with fiber-preserving symmetry:

$$(i) \quad y_4 = 0, \qquad (ii) \quad y_4 = y_3^2.$$

Essentially point and fiber-preserving symmetry

A subalgebra of the algebra of point vector fields in $J^0 = \mathbb{C}^2(x, y)$ is essentially fiber-preserving if it is conjugate to a subalgebra preserving the foliation $\{x = \mathrm{const}\}$. A subalgebra of the algebra of contact vector fields in $J^1 = \mathbb{C}^3(x, y, y_1)$ is essentially point if it is conjugate to a (prolonged) subalgebra of vector fields in J^0. The rest is called essentially contact algebras.

Question: which ODE have essentially contact symmetry algebras?

Among ODEs of order $n > 3$ with submaximal symmetry dimension (equal to $n + 2$ for $n \neq 5, 7$ and $n + 3$ for $n = 5, 7$; max symmetry dim $= n + 4$) all have essentially fiber-preserving point symmetries except for three :

$$L_4[y] = 3y_2y_4 - 5y_3^2 = 0, \quad L_5[y] = 9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0,$$
$$L_7[y] = 10y_3^3y_7 - 70y_3^2y_4y_6 - 49y_3^2y_5^2 + 280y_3y_4^2y_5 - 175y_4^4 = 0,$$

where the symmetry algebra $\mathfrak{g}$ is $\mathfrak{aff}(2, \mathbb{C})$, $\mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{sp}(4, \mathbb{C})$ (resp: point, point, contact).

Irreducible algebras of vector fields

It turns out that there are many more essentially contact and point symmetry algebras, yet they are in a certain sense rare.

A Lie algebra of contact vector fields on $J^1(\mathbb{C},\mathbb{C})$ is irreducible if there exists no invariant foliation by Legendrian curves. Reducible Lie algebras are prolongations of essentially point transformations.

Theorem (Sophus Lie)

Assume the symmetry algebra of a scalar ODE of order $n > 2$

$$y_n = f(x, y, ..., y_{n-1})$$

is irreducible. Then it contains Lie subalgebra $\mathfrak{g}$ which is spanned by

$$\partial_x,\ \partial_y,\ x\partial_y + \partial_{y_1},\ x^2\partial_y + 2x\partial_{y_1},\ -x\partial_x + y_1\partial_{y_1},\ 2y_1\partial_x + y_1^2\partial_y.$$

Can this be used to classify all ODEs with essentially contact symmetry? Yes, the latter are $\mathfrak{g}$-invariant differential equations.

Differential Invariants and Invariant Differential Equations

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Abstract. This paper surveys recent results on the classification of differential invariants of transformation groups, and their applications to invariant differential equations and variational problems.

Consider a group of transformations acting on a jet space coordinatized by the independent variables, the dependent variables, and their derivatives. Scalar functions which are not affected by the group transformations are known as differential invariants. Their importance was emphasized by Sophus Lie, [**13**], who showed that every invariant system of differential equations, [**14**], and every invariant variational problem, [**17**], could be directly expressed in terms of the differential invariants. As such they form the basic building blocks of many physical theories, where one begins by postulating the invariance of the equations, or the variational principle, under a prescribed symmetry group. Lie

The proper formulation of the relations between differential invariants and invariant differential equations, in the case of scalar ODEs is the following statement (from the same paper of Peter Olver, p7):

to order $r-1$, and computing the determinant of a suitable maximal $r \times r$ minor. By a slight abuse of terminology, we shall call this maximal minor the Lie determinant in this case.

Theorem 11. *Suppose G is an r-dimensional transformation group acting on $M \subset X \times U \simeq \mathbb{C} \times \mathbb{C}$. Then every invariant differential equation can either be written in terms of the fundamental differential invariants or by the vanishing of the associated Lie determinant.*

Example 12. Consider the four parameter group generated by $\partial_x, x\partial_x, \partial_u, u\partial_u$, which is Case 2.9 for $k = 1$. The second prolongations of these vector fields are ∂_x,

This statement, in various versions, can be found in Sophus Lie mehr erkennen wir, dass das Gleichungensystem, welches durch Nullsetzen jener $(r-m+1)$-reihigen Determinanten erhalten wird, alle Transformationen der Gruppe $X_1f \cdots X_rf$ gestattet. Wir haben demnach das

Theorem 39. *Erzeugen die r unabhängigen infinitesimalen Transformationen*

$$X_kf = \sum_1^n {}_i\, \xi_{ki}(x_1 \cdots x_n) \frac{\partial f}{\partial x_i} \qquad (k=1\cdots r)$$

eine r-gliedrige Gruppe, so erhält man durch Nullsetzen aller $(r-m+1)$*-reihigen Determinanten der Matrix*

$$(1) \qquad \begin{vmatrix} \xi_{11}(x) & \cdot & \cdot & \xi_{1n}(x) \\ \cdot & & \cdot\ \cdot & \cdot \\ \xi_{r1}(x) & \cdot & \cdot & \xi_{rn}(x) \end{vmatrix}$$

stets ein Gleichungensystem, welches alle Transformationen der Gruppe $X_1f \cdots X_rf$ *gestattet; das gilt für jede Zahl* $m \leq r$, *vorausgesetzt nur, dass es überhaupt Werthsysteme* $x_1 \cdots x_n$ *giebt, welche alle die bewussten* $(r-m+1)$*-reihigen Determinanten zum Verschwinden bringen.*

Relative invariants

Let $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$ be the Lie algebra sheaf of a connected Lie (pseudo)group. A function R is a relative differential invariant if

$$g^* R = \lambda_g\, R\; \forall g \in \mathcal{G} \quad \Leftrightarrow \quad \mathscr{L}_\xi R = w(\xi) R\; \forall \xi \in \mathfrak{g}.$$

(we tacitly omited the prolongation sign), where $w \in \mathfrak{g}^* \otimes \mathcal{P}(\mathcal{E})$. Absolute differential invariants are relative with weight $w = 0$.

Multiplication/division of relative invariants adds/subtracts their weights, so the set of weights $\mathcal{W} = \{w\}$ is a linear space. Because absolute invariants are rational in higher jets, we assume that relative invariants are polynomials, taking values in $\mathfrak{g}$-module $\mathcal{P}(\mathcal{E})$.

For any relative differential invariant R the equation given by $R = 0$ is $\mathcal{G}$-invariant. It contains singular orbits, if R is genuinely relative. (An orbit is regular if its neighborhood is fibred by orbits.)

Thus any invariant subset of codimension 1 is given by a relative differential invariant. (More generally, orbits of codimension k are described by relative invariant tensors; perhaps reducible.)

ODEs with essentially contact symmetry

Simplest relative differential invariants of the 6D Lie algebra $\mathfrak{g}$ are

$$R_3 = y_3,\ R_5 = 3y_3y_5 - 5y_4^2,\ R_6 = 9y_3^2y_6 - 45y_3y_4y_5 + 40y_4^3,\ \dots$$

Rel inv of weight k, $\mathcal{R}^k = \{P : \mathscr{L}_X P = -k\mathscr{L}_X(x)P\ \forall X \in \mathfrak{g}\}$.

Proposition (Generation of relative differential invariants wrt $\mathfrak{g}$)

The algebra $\mathcal{R} = \oplus_{k>0}\mathcal{R}^k$ is generated by R_3, R_5 and the rel inv derivation $\nabla_k = y_3 D_x - \frac{k}{3}y_4 : \mathcal{R}^k \to \mathcal{R}^{k+4}$, w/localization by y_3.

Theorem (Generation of absolute differential invariants wrt $\mathfrak{g}$)

The field of rational abs differential invariants is generated by the differential invariant $I_5 = \frac{R_5^3}{R_3^8}$ and invariant derivation $\nabla = \frac{R_5}{R_3^3}D_x$.

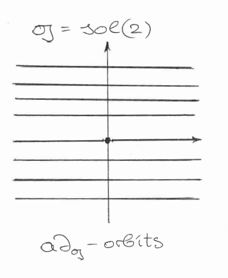
Singular orbits are contained in $\mathcal{S}^k = \pi_{k,3}^{-1}(\mathcal{S}^3)$, $\mathcal{S}^3 = \{R_3 = 0\}$.

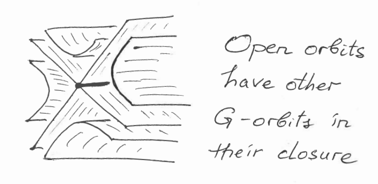
Corollary (ODE with essentially contact symmetry)

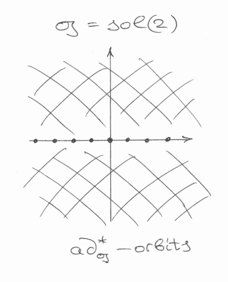
Such ODEs are $\sim R_3 = 0$ or $f(I_5, \nabla(I_5), \dots, \nabla^{n-5}(I_5)) = 0$.

ODEs with essentially point symmetry

Scalar ODEs with essentially point symmetry have quite similar description, however for ODE systems (no contact symmetries by Lie-Bäcklund) the situation is more complicated due to possibility of singular orbits of higher codimension.







These are some effects of orbit partitions...

Example: Poincaré transformations on 3D Minkowski space

There are several minimal Lie algebras of vector fields in $\mathbb{C}^3$, which are either primitive or preserve 1- but not 2-dim foliation, e.g.

$$\mathfrak{g} = \langle \partial_x, \partial_y, \partial_z, x\partial_y + y\partial_x, x\partial_z + z\partial_x, y\partial_z - z\partial_y \rangle.$$

Simplest relative differential invariants are:

$$R_1 = y_1^2 + z_1^2 - 1, \quad R_2 = (y_1^2 - 1)z_2^2 - 2y_1 z_1 y_2 z_2 + (z_1^2 - 1)y_2^2,$$
$$R_{3a} = z_2 y_3 - y_2 z_3, \; R_{3b} = R_1 D_x(R_2) - 3D_x(R_1)R_2.$$

Theorem

The algebra of absolute differential invariants is generated by the differential invariants $I_2 = \frac{R_2}{R_1^3}, I_{3a} = \frac{R_{3a}}{R_1^3}$ *and the invariant derivation* $\nabla = \frac{R_{3b}}{R_1^5} D_x$.

Singular orbits: $\{R_1 = 0\}$ and $\mathcal{S}^k = \pi_{k,2}^{-1}(\mathcal{S}^2)$, $\mathcal{S}^2 = \{R_1 R_2 = 0\}$.

Example: continued

On $\Sigma^k = \{R_1 = 0\}^{(k-1)} \subset J^k$ the function

$$Q_2 = z_1 y_2 - y_1 z_2$$

is a conditional relative differential invariant. On $\Sigma^k \setminus \{Q_2 = 0\}$ the following are conditional absolute differential invariant and invariant derivation:

$$J_4 = \frac{4Q_2 D_x^2(Q_2) - 7D_x(Q_2) + 4Q_2^4}{Q_2^3}, \quad \nabla_\Sigma = \frac{D_x(J_4)}{Q_2} D_x.$$

On $\Pi^k = \{R_2 = 0\}^{(k-2)} \subset J^k$ the following are conditional absolute differential invariant and invariant derivation:

$$K_3 = \frac{(R_1 y_3 - \frac{3}{2} D_x(R_1)) y_2^2}{(y_1^2 z_2 - y_1 y_2 z_1 - z_2)^3}, \qquad \nabla_\Pi = \frac{R_1 y_3 - \frac{3}{2} D_x(R_1)}{y_2 R_1^2} D_x.$$

Example: concluded

Theorem

Let $\mathcal{E} = \{F = 0, G = 0\} \subset J^k$ be a $\mathfrak{g}$-invariant determined ODE system of orders k and $l \leq k$. Then, either both F and G can be expressed through rational absolute differential invariants generated by I_2, I_{3a} and ∇ or the system takes one of the following forms:

- *$G = R_1$ and $\deg F \geq 2$: either $F = Q_2$ or F is a function of the conditional absolute invariants, which are generated by ∇_Σ and J_4.*
- *$G = R_2$ and F is a function of the conditional absolute invariants, which are generated by ∇_Π and K_3.*
- *$G = y_2$ and $F = z_2$.*

Note that this is a global statement, so the involved functions are polynomial (with cleared denominators/localizations). Other ODEs with essentially point symmetry have similar descriptions.

Concluding remarks

Several interesting aspects arose during this study:

- Algebraic actions, classification global or local (on the level of germs), algebraic or analytic: sensitive to setup
- Change of coordinates results in singular orbits possible escape to/return from infinity (G-compactification)
- Differential constraint may obstruct eventual freeness of prolonged Lie group actions (cf. Ovsiannikov, Adams-Olver)
- Higher dimensional generalization of ODEs with essentially point symmetry classification is algorithmically possible
- Justifies classification of ODEs/systems wrt fiber-preserving transformations

Some references

[1] S. Lie, *Classification und Integration von gewöhnlichen differentialgleichungen zwischen xy, die eine Gruppe von Transformationen gestatten* I-IV, Math. Ann. **32** (1888); also in: Gesammelte Abhandlungen, Vol. 5, B.G. Teubner, Leipzig, pp. 240-310, 362-427, 432-448 (1924).

[2] A. Tresse, *Sur les invariants differentiels des groupes continus de transformations*, Acta Math. **18**, 1–88 (1894).

[3] L. Ovsiannikov, *Group properties of differential equations*, Novosibirsk (1962); *Group analysis of differential equations*, Nauka, Moscow (1978); Academic Press, New York (1982).

[4] A. Kumpera, *Invariants differentiels d'un pseudogroupe de Lie* I-II, J. Differential Geometry **10**, 289–345, 347–416 (1975).

[5] P. Olver, *Differential Invariants and Invariant Differential Equations*, Lie Groups Appl. **1**, 177–192 (1994); *Equivalence, invariants, and symmetry*, Cambridge University Press (1995).

References

[6] L. Hsu, N. Kamran, *Classification of second-order ordinary differential equations admitting Lie groups of fibre-preserving point symmetries*, Proc. London Math. Soc. **58**, 387–416 (1989).

[7] M. Fels, P. Olver, *On relative invariants*, Math. Ann. **308**, 701–732 (1997).

[8] B. Kruglikov, V. Lychagin, *Global Lie-Tresse theorem*, Selecta Math. **22**, 1357-1411 (2016).

[9] B. Kruglikov, *Symmetries of second order ODEs*, J. Math. Anal. Appl. **461**, 591–594 (2018).

[10] B. Kruglikov, E. Schneider, *ODEs whose symmetry groups are not fiber-preserving*, arXiv:2202.10331 & submitted (2022); *Relative differential invariants*, talks by BK at Sirius-Sochi-2019, at BIRS-Banff-2021 and a joint paper in preparation (2022).

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Thanks for your attention.

Happy birthday, Peter!