Two approximate symmetry frameworks for nonlinear DEs with a small parameter. Comparisons, relations, approximate solutions

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Symmetry, Invariants, and their Applications

A celebration of Peter Olver’s 70th birthday

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Outline

1. Perturbed DEs
2. Baikov-Gazizov-Ibragimov approximate symmetries
3. Fushchich-Shtelen approximate symmetries
4. BGI vs. FS: a computational comparison
5. Approximate solutions from approximate symmetries
6. Discussion
7. Appendix: some models with small parameters
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Perturbed DEs

Regular perturbation of an ODE

- Original (unperturbed) ODE:
  \[ y^{(n)}(x) = f_0[y] \equiv f_0(x, y(x), \ldots, y^{(n-1)}(x)) \]

- Small parameter: \( \epsilon \) of an ODE: the leading derivative(s) does not change.

- Perturbed ODE:
  \[ y^{(n)}(x) = f_0[y] + \epsilon f_1[y] + o(\epsilon). \]

ODE systems, PDEs, PDE systems

- Same idea: a regular perturbation where \( O(\epsilon) \) terms do not break the structure of the solved form; DEs still have the same leading derivatives.

Difficulty:

- Analytical structure of the unperturbed model may be lost under perturbation.
A simple example: lost point symmetries

Consider a nonlinear wave-type equation on $u = u(x, t)$:

\[ u_{tt} = u_x u_{xx} \]  

(1)

Exact point symmetry generator:

\[ X^0 = \xi^1_0(x, t, u) \frac{\partial}{\partial x} + \xi^2_0(x, t, u) \frac{\partial}{\partial t} + \eta_0(x, t, u) \frac{\partial}{\partial u} \]

Final result: the PDE (1) admits six point symmetries given by

\[ X_1^0 = \frac{\partial}{\partial u}, \quad X_2^0 = t \frac{\partial}{\partial u}, \quad X_3^0 = \frac{\partial}{\partial t}, \quad X_4^0 = \frac{\partial}{\partial x}, \]

\[ X_5^0 = u \frac{\partial}{\partial u} - \frac{t}{2} \frac{\partial}{\partial t}, \quad X_6^0 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, \]

(2)

corresponding to three translations ($X_1^0$, $X_3^0$ and $X_4^0$), the Galilei group ($X_2^0$), and two scalings ($X_5^0$ and $X_6^0$).
A simple example: lost point symmetries

A perturbed PDE:

\[ u_{tt} + \epsilon u u_t = u_x u_{xx} \]  \hspace{1cm} (3)

- The perturbation term distorts symmetry structure. Exact point symmetries of (3) are generated by

\[ Y_1 = X^0_3 = \frac{\partial}{\partial t}, \]
\[ Y_2 = X^0_4 = \frac{\partial}{\partial x}, \]
\[ Y_3 = \frac{4}{3} X^0_5 - \frac{1}{3} X^0_6 = -t \frac{\partial}{\partial t} - \frac{x}{3} \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, \]  \hspace{1cm} (4)

a three-dimensional subalgebra of the six-dimensional Lie algebra of point symmetries (2).

- Where did the other three go? Stable vs. unstable symmetries.

- Can unstable symmetries re-appear as, in some sense, “approximate” symmetries of the perturbed PDE (3)?
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For simplicity, consider a single PDE on \( u = u(x) = (x_1, \ldots, x_n) \).

Original (unperturbed) PDE:

\[
F_0[u] = 0 \tag{5}
\]

Perturbed DE (regular perturbation):

\[
F_0[u] + \epsilon F_1[u] = 0 \tag{6}
\]

BGI approximate point symmetry generator:

\[
X = X^0 + \epsilon X^1 = \left( \xi^i_0(x, u) + \epsilon \xi^i_1(x, u) \right) \frac{\partial}{\partial x^i} + \left( \eta_0(x, u) + \epsilon \eta_1(x, u) \right) \frac{\partial}{\partial u}
\]

Easier to work with characteristic forms which generalize to local symmetries:

\[
\hat{X} = \hat{X}^0 + \epsilon \hat{X}^1 = (\zeta_0[u] + \epsilon \zeta_1[u]) \frac{\partial}{\partial u}
\]

Determining equations:

\[
(\hat{X}^0 \infty + \epsilon \hat{X}^1 \infty)(F_0[u] + \epsilon F_1[u])|_{F_0[u] + \epsilon F_1[u] = o(\epsilon)} = o(\epsilon)
\]
BGI approximate symmetries

- $O(\epsilon^0)$ part of the determining equations:
  \[
  \hat{X}^0 F_0[u] \bigg|_{F_0[u]=0} = 0
  \]

- Every BGI approximate point symmetry of the perturbed equation corresponds to an exact point symmetry $\hat{X}^0$ of the unperturbed equation.

- Opposite is not true (previous example).

- $O(\epsilon)$ part of the determining equations:
  \[
  \hat{X}^1 F_0[u] \bigg|_{F_0[u]=0} = H[u],
  \]

  where $H[u]$ is the $O(\epsilon)$ part of the expression

  \[
  -\hat{X}^0 (F_0[u] + \epsilon F_1[u]) \bigg|_{F_0[u]+\epsilon F_1[u]=o(\epsilon)}.
  \]

- These are extra conditions on $\hat{X}^0$ that can make some symmetries unstable.
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FS approximate symmetries

- Original (unperturbed) DE:
  \[ F_0[u] = 0 \]  
  (7)

- Perturbed DE (regular perturbation):
  \[ F_0[u] + \epsilon F_1[u] = 0 \]  
  (8)

- Seek solution as a regular perturbation
  \[ u(x) = v(x) + \epsilon w(x) + o(\epsilon). \]

Split (8) into \(O(1)\) and \(O(\epsilon)\) parts (FS system):

\[
G_1[v, w] \equiv F_0[v] = 0, \\
G_2[v, w] \equiv F_0[v] w + F_0[v] w_i + F_0[v] w_{ij} + \ldots + F_0[v] w_{i_1 i_2 \ldots i_k} + F_1[v] = 0
\]  
(9)

- Find usual point/local symmetries of (9): infinitesimal generators

\[
\hat{Z} = \zeta_0[v, w] \frac{\partial}{\partial v} + \zeta_1[u, w] \frac{\partial}{\partial w}
\]

- Determining equations are again restrictive on \(\zeta_0\); can lead to stable or unstable local symmetries of (7), now in the FS sense.
BGI vs. FS approximate symmetry forms

- **Point:**
  \[
  X = \left( \xi_0^i(x, u) + \epsilon \xi_1^i(x, u) \right) \frac{\partial}{\partial x^i} + (\eta_0(x, u) + \epsilon \eta_1(x, u)) \frac{\partial}{\partial u}
  \]
  \[
  Z = \lambda^i(x, v, w) \frac{\partial}{\partial x^i} + \phi_1(x, v, w) \frac{\partial}{\partial v} + \phi_2(x, v, w) \frac{\partial}{\partial w}
  \]

- **General local, characteristic form:**
  \[
  \hat{X} = \hat{X}^0 + \epsilon \hat{X}^1 = (\zeta_0[u] + \epsilon \zeta_1[u]) \frac{\partial}{\partial u}
  \]
  \[
  \hat{Z} = \zeta_0[v, w] \frac{\partial}{\partial v} + \zeta_1[u, w] \frac{\partial}{\partial w}
  \]

Which is more general?

In both BGI and FS approximate symmetries, the following kinds arise.

- **Directly inherited from the original PDE:** \( \zeta_0 = \zeta_0[u], \zeta_1 = 0 \)
- **Genuine approximate:** \( \zeta_0, \zeta_1 \neq 0 \)
- **“Trivial” approximate:** \( \zeta_0 = 0 \) ("trivial" = "‘always appear’")
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BGI vs. FS: a computational comparison

- Unperturbed and perturbed PDEs, \( u = u(x, t) \):
  \[
  \begin{align*}
  u_{tt} &= u_x u_{xx}, \\
  u_{tt} + \epsilon F_1(u, u_t) &= u_x u_{xx}
  \end{align*}
  \]

- Six point symmetries of the unperturbed PDE:
  \[
  \begin{align*}
  X_1^0 &= \frac{\partial}{\partial u}, & X_2^0 &= t \frac{\partial}{\partial u}, & X_3^0 &= \frac{\partial}{\partial t}, & X_4^0 &= \frac{\partial}{\partial x}, \\
  X_5^0 &= u \frac{\partial}{\partial u} - \frac{t}{2} \frac{\partial}{\partial t}, & X_6^0 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u},
  \end{align*}
  \]

- Which of these are stable (as point symmetries) when \( F_1 \neq 0 \), in BGI and/or FS frameworks?

- Classify with respect to inequivalent forms of \( F_1(u, u_t) \).
## BGI vs. FS: a computational comparison

| $\hat{X}_1^0$ | $F_1 = Q_1(u_t) + a_1 uu_t + a_2 u,$  
$= \frac{\partial}{\partial u}$  
$\hat{X}_1 = \left(1 - \epsilon \left(\frac{a_1}{10} t(tu_t + 4u) + \frac{a_2}{2} t^2\right)\right) \frac{\partial}{\partial u}$ |  
- $F_1 = e^{a_3 v} Q_4(v_t) + a_2 v_t + a_1,$  
$\hat{Z}_1 = \frac{\partial}{\partial v} + a_3 \left(\frac{a_2}{10} t(tv_t + 4v) + w + \frac{a_1}{2} t^2\right) \frac{\partial}{\partial w}$  
- $F_1 = Q_4(v_t) + a_1 vv_t + a_2 v,$  
$\hat{Z}_1 = \frac{\partial}{\partial v} - \left(\frac{a_1}{10} t(tv_t + 4v) + \frac{a_2}{2} t^2\right) \frac{\partial}{\partial w}$ |
|---|---|
| $\hat{X}_2^0$ | $F_1 = a_1 u_t^2 + a_2 uu_t + a_3 u + a_4,$  
$= t \frac{\partial}{\partial u}$  
$\hat{X}_2 = \left(t - \epsilon \left(\frac{a_1}{5} t(tu_t + 4u) + \frac{1}{6} t^2(a_3 t + 3a_2)\right)\right) \frac{\partial}{\partial u}$ |  
- $F_1 = a_1 u_t^2 + a_2 uu_t + a_3 v + a_4,$  
$\hat{Z}_2 = t \frac{\partial}{\partial v} - \left(\frac{a_1}{5} t(tv_t + 4v) + \frac{1}{6} t^2(a_3 t + 3a_2)\right) \frac{\partial}{\partial w}$  
- $F_1 = a_3 e^{a_4 v_t} + a_2 v_t + a_1,$  
$\hat{Z}_2 = t \frac{\partial}{\partial v} + \left(\frac{a_2 a_4}{10} t(tv_t + 4v) + \frac{a_1 a_4 - a_2}{2} t^2 + a_4 w\right) \frac{\partial}{\partial w}$ |
| $\hat{X}_3^0$ | $F_1 = F_1(u, u_t),$  
$= u_t \frac{\partial}{\partial u}$  
$\hat{X}_3 = u_t \frac{\partial}{\partial u}$ |  
$F_1 = F_1(v, v_t),$  
$\hat{Z}_3 = v_t \frac{\partial}{\partial v} + w_t \frac{\partial}{\partial w}$ |
### BGI vs. FS: a computational comparison

<table>
<thead>
<tr>
<th>( \hat{X}_4^0 )</th>
<th>( F_1 = F_1(u, u_t) ), ( \hat{X}_4 = u_x \frac{\partial}{\partial u} )</th>
<th>( F_1 = F_1(v, v_t) ), ( \hat{Z}_4 = v_x \frac{\partial}{\partial v} + w_x \frac{\partial}{\partial w} )</th>
</tr>
</thead>
</table>
| \( u \left[ u_x \frac{\partial}{\partial u} + \frac{tu_t}{2} \right] \) | \( F_1 = u^2 Q_2 \left( u_t / u^{3/2} \right) + a_2 u_t + a_1 \) | \( F_1 = v^{a_3} Q_5 \left( v_t / v^{3/2} \right) + a_2 v_t + a_1, \)
| \( \hat{X}_5 = \left( u + \frac{tv_t}{2} + \epsilon \left( a_1 t^2 + \frac{a_2}{20} t(tu_t + 4u) \right) \right) \frac{\partial}{\partial u} \) | \( \hat{Z}_5 = \left( v + \frac{tv_t}{2} \right) \frac{\partial}{\partial v} + \left( (a_3 - 1)w + \frac{tw_t}{2} \right) \frac{\partial}{\partial w} \) \( + \frac{a_2}{20} (2a_3 - 3)t(tu_t + 4v) + \frac{a_1 a_3 t^2}{2} \)
| \( \hat{X}_6 = \left( u - xu_x - tu_t \right) \frac{\partial}{\partial u} \) | \( \hat{Z}_6 = (v - xv_x - tw_t) \frac{\partial}{\partial v} \) \( + \left( (a_3 + 2)w - xw_x - tw_t \right) \frac{\partial}{\partial w} \) \( + \frac{a_2 a_3}{10} t(tu_t + 4v) + \frac{a_1 a_3 t^2}{2} \) \( \frac{\partial}{\partial w} \) |
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A nonlinear wave equation

- A class of nonlinear wave equations with a small parameter:

\[ u_{tt} = (1 + \epsilon Q(u_x))u_{xx} \]

- FS system: \( v_{tt} = v_{xx}, \ w_{tt} = w_{xx} + Q(v_x)v_{xx} \).

- Power nonlinearity \( Q(v_x) = v_x^s \ (s \neq -1): \) genuine approximate FS symmetry

\[ Z = v \frac{\partial}{\partial v} + (s + 1)w \frac{\partial}{\partial w} \]

- “Large” and “small” solution components are scaled differently

- An approximately invariant solution:

\[ \frac{dt}{0} = \frac{dx}{0} = \frac{dv}{v} = \frac{dw}{(s + 1)w} \]

Take \( v = g(x \pm t) \). Then \( w = g^{s+1} \phi \), and

\[ g^{s+1} (\phi_{tt} - \phi_{xx}) + 2(s + 1)g^s g' (\pm \phi_t - \phi_x) - (g')^s g'' = 0 \]

- \( \phi = \phi(x, t) \), and so the approximate solution \( u = v + \epsilon w \), can be found explicitly.
Motivation: biological and artificial elastic materials with families of aligned fibers

Anisotropic elastodynamics
Composite reinforced by collagen fibers arranged in helical structures

Helically arranged fiber-reinforced medial layers

Bundles of collagen fibrils
External elastic lamina
Elastic lamina
Elastic fibrils
Collagen fibrils
Smooth muscle cell
Internal elastic lamina
Endothelial cell
1D nonlinear waves in fiber-reinforced solids
1D nonlinear waves in fiber-reinforced solids

- Fully nonlinear Eulerian shear displacements \( u(x, t) \) (not small) in terms of material coordinate \( X \)

- Viscoelastic dynamics [A.S. and Ganghoffer (2015)]

\[
\begin{align*}
    u_{tt} &= \left( \alpha + 3\beta u_x^2 \right) u_{xx} \\
    &= \alpha u_{xx} + 3\beta u_x u_{xx} \\
    &\quad + \eta u_x \left( 2 \left( 4u_x^2 + 1 \right) u_{xx} u_{tx} + 2u_x^2 + 1 \right) u_x u_{txx} \\
    &\quad + \zeta u_x^3 \left( 4 \left( 6u_x^2 + 1 \right) u_{xx} u_{tx} + 4u_x^2 + 1 \right) u_x u_{txx}
\end{align*}
\]

- Possible small parameters: \( \beta, \eta, \zeta \)

- Hyperelastic simplification:

\[
    u_{tt} = \left( \alpha + 3\beta u_x^2 \right) u_{xx} \rightarrow \quad u_{tt} = (1 + \epsilon u_x^2) u_{xx}
\]

- Member of the previous wave equation family with power nonlinearity \( u^s, s = 2 \)
- Produces “breaking waves:” finite-time singularity formation
Numerical solution, $\epsilon = 0.5$: 

![Plot of $\hat{G}(\hat{x}, \hat{t})$ vs $\hat{x}$]
One can show that the PDE $u_{tt} = (1 + \epsilon u_x^2)u_{xx}$ can be reduced to the first-order characteristic form

$$u_t = \pm \frac{1}{2\sqrt{\epsilon}} \left( \sqrt{\epsilon} u_x \sqrt{1 + \epsilon (u_x)^2} + \ln \left( \sqrt{\epsilon} u_x + \sqrt{1 + \epsilon (u_x)^2} \right) \right)$$

on the characteristic curves

$$\frac{dx}{dt} = \pm \sqrt{1 + \epsilon (u_x)^2}.$$
1D nonlinear waves in fiber-reinforced solids

- Compare behaviour with the approximate solution
Wave breaking $\sim$ formation of an extra inflection point on the approximate solution.

Left: $u_{xx}$ for the numerical solution; right: same for the approximate solution.
- Estimate wave breaking times [Tarayrah, Pitzel, A.S.]
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Approximate symmetries can be found using BGI and FS frameworks ("somewhat" related), and are useful.

Approximate conservation laws can arise in a similar manner.

PDE models with multiple small parameters?

Anisotropic dynamic viscoelasticity, shear waves

\[ u_{tt} = (\alpha + 3\beta u_x^2) u_{xx} \]
\[ + \eta u_x \left( 2(4u_x^2 + 1)u_{xx} u_{tx} + (2u_x^2 + 1)u_x G_{txx} \right) \]
\[ + \zeta u_x^3 \left( 4(6u_x^2 + 1)u_{xx} u_{tx} + (4u_x^2 + 1)u_x u_{txx} \right) \]

Serre-Su-Gardner-Green-Naghdi shallow water equations

\[ u_t + \epsilon uu_x^* + \eta_x = \frac{\delta^2}{3h} \left( h^3 \left( u_{xt} + \epsilon uu_{xx} - \epsilon u_x^2 \right) \right)_x, \]
\[ h_t + \epsilon (hu)_x = 0 \]
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Shallow water models

\[ \frac{\delta}{\lambda} = \frac{h_0}{\lambda} : \text{dispersion parameter}; \quad \varepsilon = \frac{A}{h_0} : \text{amplitude parameter} \]

Weakly nonlinear dispersionless equations, characterized by \( \delta^2 \ll \varepsilon \ll 1. \)
Example: shallow water equations

\[ u_t^* + \eta_x^* + \varepsilon u_x^* u_t^* = 0, \]
\[ h_t^* + (h^* u^*)_x^* = 0 \]
Shallow water models

\[ \delta = \frac{h_0}{\lambda} : \text{dispersion parameter;} \quad \varepsilon = \frac{A}{h_0} : \text{amplitude parameter} \]

Weakly nonlinear and weakly dispersive equations: Boussinesq regime \( \delta^2 \sim \varepsilon \ll 1 \).

Examples: the Boussinesq equation

\[
\eta_{t^*t^*} = \eta_{x^*x^*} + \frac{\varepsilon}{2} \left( ((\eta^*)^2)_{x^*x^*} + 2((u_0^*)^2)_{x^*x^*} \right) + \frac{\delta^2}{3} \eta_{x^*x^*x^*x^*}
\]

The KdV equation

\[
\eta_{t^*} + \eta_{x^*} + \frac{3}{2} \varepsilon \eta^* \eta_{x^*} + \frac{\delta^2}{6} \eta_{x^*x^*x^*} = 0.
\]
Shallow water models

- \( \delta = \frac{h_0}{\lambda} \): dispersion parameter; \( \varepsilon = \frac{A}{h_0} \): amplitude parameter

- Strongly nonlinear weakly dispersive models: \( \delta^2 \ll 1, \varepsilon = O(1) \).
  Example: the Su-Gardner equations

\[
\begin{align*}
  u_{t^*}^* + \varepsilon u^* u_{x^*}^* + \eta_{x^*}^* &= \frac{\delta^2}{3h^*} \left((h^*)^3 \left(u_{x^* t^*}^* + \varepsilon u_{x*x^*}^* - \varepsilon (u_{x^*}^*)^2)\right)_{x^*}, \\
  h_{t^*}^* + \varepsilon (h^* u^*)_{x^*} &= 0
\end{align*}
\]
Physical dimensionless vs. canonical forms

- KdV canonical: \( u_t + 6uu_x + u_{xxx} = 0 \)

- BBM canonical: \( u_t + u_x + uu_x - u_{xxt} = 0 \)

- KdV physical dimensionless:
  \[
  \eta^{*}_{t} + \eta^{*}_{x} + \frac{3}{2} \varepsilon \eta^{*}_{x} \eta^{*}_{x} + \frac{\delta^2}{6} \eta^{*}_{x}x_{x}x_{*} = 0
  \]

- BBM physical dimensionless:
  \[
  \eta^{*}_{t} + \eta^{*}_{x} + \frac{3}{2} \varepsilon \eta^{*}_{x} \eta^{*}_{x} - \frac{\delta^2}{6} \eta^{*}_{x}x_{x}t_{*} = 0
  \]

- Both in the Boussinesq regime \( \delta^2 \sim \varepsilon \)

- Same order of approximation \( o(\varepsilon^2) \), very different analytical properties.
Arnold’s principle

If a model bears a name, it is not the name of the person who discovered it.

Examples:

- Korteweg-de Vries $\rightarrow$ Boussinesq (25 years earlier)
- Su-Gardner (Green-Naghdi) $\rightarrow$ Serre (13 and 21 years earlier)
- Camassa-Holm $\rightarrow$ Fokas and Fuchssteiner (12 years earlier)
- ...

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Approximate symmetries

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Some references

