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Scalar polynomial curvature invariants in Lorentzian manifolds

A. Introduction

Scalar curvature invariants are scalars constructed from the Riemann tensor and its covariant derivatives.

We define the set of all scalar invariants (considered as a function of the metric and its derivatives) on $(\mathcal{M}, g)$ by

$$\mathcal{I} \equiv \{R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}, R_{\mu\nu\alpha\beta;\gamma}R^{\mu\nu\alpha\beta;\gamma}, R_{\mu\nu\alpha\beta;\gamma\delta}R^{\mu\nu\alpha\beta;\gamma\delta}, \dots\}\,.$$

Scalar curvature invariants can be used to study the inequivalence of metrics and curvature singularities. Some spaces can be completely characterized by their scalar curvature invariants [Lorentzian $\mathcal{I}$-non-degenerate spaces]. In particular, scalar curvature invariants have been well studied due to their potential use in general relativity.

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1. Introduction.
2. $\mathcal{I}$-non-degenerate Theorem: proof, details.
3. VSI. Classification
4. b Scalars
5. Applications
6. Other

I. $\mathcal{I}$-NON-DEGENERATE THEOREM

By utilyzing an appropriate set of projection operators, we discuss when an (arbitrary-dimensional) Lorentzian manifold can be completely characterized by the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives. [The theorem is also true in 3D, and it is also likely true in arbitrary dimenions.]

Definition I.1. A (one-parameter) *metric deformation* $\hat{g}_\tau, \tau \in [0, \epsilon)$ of a spacetime $(\mathcal{M}, g)$ is a family of smooth metrics on $\mathcal{M}$ such that

1. $\hat{g}_\tau$ is continuous in τ,
2. $\hat{g}_0 = g$,
3. $\hat{g}_\tau$ for $\tau > 0$ is not diffeomorphic to g.

Definition I.2. Given a spacetime $(\mathcal{M}, g)$ with a set of invariants $\mathcal{I}$, then if there does not exist a metric deformation of g with the same set of invariants as g, then we will call the set of invariants *non-degenerate*. The spacetime metric g will be called *$\mathcal{I}$-non-degenerate*.

Hence a Lorentzian metric that is $\mathcal{I}$-non-degenerate is locally characterized uniquely by its invariants. We now have the following theorem :

Theorem I.3. *Given a spacetime metric, either*

1. *the metric is $\mathcal{I}$-non-degenerate or*
2. *the metric is a degenerateKundt metric.*

A. Four Dimensions

The 4D Lorentzian manifolds are characterized algebraically by their Petrov and Segre types or, equivalently, in terms of their and Riemann types. By determining an appropriate set of projection operators from the Riemann tensor and its covariant derivatives, we are able to determine (on a case-by-case, depending on the algebraic type, and using a boost weight decomposition) when a spacetime metric is $\mathcal{I}$-non-degenerate.

It was proven that *if a 4D spacetime metric is locally of Ricci type I or Weyl type I (i.e., algebraically general) the metric is $\mathcal{I}$-non-degenerate* [CHP]. This indicates that in general the spacetime metric is $\mathcal{I}$-non-degenerate and thus the metric is locally determined by its curvature invariants.

For the algebraically special cases the Riemann tensor itself does not give enough information to provide us with all the required projection operators, and it is also necessary to consider the covariant derivatives. In terms of the boost weight decomposition, for an algebraically special metric (which has a Riemann tensor with zero positive boost weight components) which is not Kundt, by taking covariant derivatives of the Riemann tensor positive boost weight components are acquired and a set of higher derivative projection operators are obtained. It was shown that *if the 4D spacetime metric is algebraically special, but ∇R, $\nabla^{(2)}R$, $\nabla^{(3)}R$, or $\nabla^{(4)}R$ is of type I or more general, the metric is $\mathcal{I}$-non-degenerate.*

B. Kundt

The remaining metrics which *do not* acquire a positive boost weight component when taking covariant derivatives have a very special curvature structure. Indeed, we proved the important result that *for a spacetime metric, either the metric is $\mathcal{I}$-non-degenerate, or the metric is a Kundt metric.* This is a striking result because it tells us that metrics not determined by their curvature invariants must be of Kundt form. The Kundt class is defined by those metrics admitting a null vector that is geodesic, expansion-free, shear-free and twist-free. These Kundt metrics therefore correspond to degenerate metrics in the sense that many such spacetimes can have identical invariants. This exceptional property of the the degenerate Kundt metrics essentially follows from the fact that they *do not define a unique timelike curvature operator*.

1. Degenerate Kundt

Indeed, the Kundt metrics are the *only metrics not determined by their curvature invariants* (in the sense described above). In fact, we can be somewhat more precise since only the subclass of *aligned algebraically special Riemann type-II* and *aligned algebraically special nabla-Riemann type-II* Kundt spacetimes or *degenerate* Kundt spacetimes [CHP] have these exceptional properties. We note that the important constant curvature invariants (CSI) and vanishing scalar invariant (VSI) spacetimes are degenerate Kundt spacetimes.

2. Discussion

Intuitively, if we have a TL direction defined, a TL curvature operater can be defined. So the special cases are when there are 2 null directions but no unique TL one. Suppose ℓ is a null direction, we can construct the associated kinematical 'scalars' θ, σ and ω. We can then construct the gradients of these scalars, and ask if any of these gradients are TL. In this way we can determine conditions on spacetime for it not to admit TL direction. In particular, a degenerate Kundt spacetime (in which $\theta = \sigma = \omega = 0$ and no $\nabla^{(k)}R$ has any positive boost-weight terms) is not $\mathcal{I}$-non-degenerate (since no negative boost-weight terms can appear in any scalar polynomial invariant).

II. SPACETIMES WITH VANISHING SCALAR CURVATURE INVARIANTS

The class of 4D Lorentzian VSI spacetimes. All curvature invariants of all orders vanish if and only if the following two conditions are satisfied:

(A) The spacetime possesses a non-diverging shear-free, geodesic null congruence.

(B) Relative to the above null congruence, all curvature scalars with non-negative boost-weight vanish.

[An alternative characterization of VSI spacetimes are that they are of Petrov type III, N or O, all eigenvalues of the Ricci tensor are zero (the Ricci tensor is consequently of Plebański-Petrov type (PP-type) N or O, or alternatively, of Segre type $\{(31)\}$, $\{(211)\}$ or $\{(1111)\}$) and the common multiple null eigenvector of the Weyl and Ricci tensors is geodesic, shear-free, non-expanding.]

Perhaps the best known class of spacetimes with vanishing curvature invariants are the pp-waves (or plane-fronted gravitational waves with parallel rays), which are characterized as Ricci-flat (vacuum) type N spacetimes that admit a covariantly constant null vector field. VSI solutions need not be plane waves (which have a 5D isometry group acting on 3D null orbits), and are not necessarily vacuum solutions.

For CSI metrics, all curvature invariants of all orders are constant. A corollary of the $\mathcal{I}$-non-degenerate theorem is that in 4D, *a CSI spacetime then either the spacetime is locally homogeneous or a subclass of the Kundt spacetimes.*

1. *A Class of Exact Classical Solutions to String Theory*

The pp-wave spacetimes have a number of important physical applications. pp-wave spacetimes are exact vacuum solutions to string theory to all order in α', the scale set by the string tension. It was shown that all of the VSI spacetimes are classical solutions of the string equations to all orders in σ-model perturbation theory; the proof consists of showing that all higher order correction terms vanish.

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We can generalize these results to higher dimensions. In particular,

A. VSI Spacetimes in Higher Dimensions

All curvature invariants of all orders vanish in an N-dimensional Lorentzian spacetime if and only if there exists an aligned non-expanding ($S_{ij} = 0$), non-twisting ($A_{ij} = 0$), geodesic null direction ℓ^a.

1. *Generalized Kundt spacetimes*

By assuming that the higher order (differential) curvature invariants vanish we are consequently led to study spacetimes which admit a geodesic, shear-free, divergence-free, irrotational null congruence $l = \partial_v$, and hence belong to the "generalized Kundt" class in which the metric can be written

$$\mathrm{d}s^2 = -2\mathrm{d}u[H\mathrm{d}u + \mathrm{d}v + W_i \mathrm{d}x^i] + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \ ,$$

where $i = 1 \ldots N$, and the metric functions

$$H = H(u, v, x^i), \quad W_i = W_i(u, v, x^i), \quad g_{ij} = g_{ij}(u, x^i)$$

satisfy the remaining vanishing invariant conditions and the Einstein field equations. We may use the remaining coordinate freedom to simplify g_{ij} (and, if $g_{ij} = \delta_{ij}$, H and W_i).

III. CLASSIFICATION OF THE WEYL TENSOR IN HIGHER DIMENSIONS

The algebraic classification of the Weyl tensor in higher dimensional Lorentzian manifolds by means of the existence of aligned null vectors of various orders of alignment. Further classification is obtained by specifying the alignment type and utilizing the notion of reducibility. The classification reduces to the classical Petrov classification in 4D.

We shall consider a null frame (with ℓ, n null with $\ell^a \ell_a = n^a n_a = 0, \ell^a n_a = 1$, m^i real and spacelike with $m_i{}^a m_{ja} = \delta_{ij}$; all other products vanish) in an N-dimensional Lorentz-signature space(time), so that $g_{ab} = 2l_{(a} n_{b)} + \delta_{jk} m^j_a m^k_b$. Indices a, b, c range from 0 to $N-1$, and space-like indices i, j, k also indicate a null-frame, but vary from 2 to $N-1$ only. The frame is covariant relative to the group of linear Lorentz transformations consisting of *null rotations* about ℓ and n and boosts and spins.

Let $T_{a_1 \dots a_p}$ be a rank p tensor. For a fixed list of indices $A_1, ..., A_p$, we call the corresponding $T_{A_1 \dots A_p}$ a null-frame scalar. These scalars transform under a boost ($\bar{\ell} = \lambda \ell$, $\bar{n} = \lambda^{-1} n$, where $\lambda \neq 0$) according to

$$\hat{T}_{A_1 \dots A_p} = \lambda^b T_{A_1 \dots A_p}, \quad b = b_{A_1} + \dots + b_{A_p} \tag{2}$$

[where $b_0 = 1, \quad b_i = 0, \quad b_1 = -1$]. We call the above b the boost-weight of the scalar. We define the *boost order* of the tensor T to be the boost weight of its leading term.

We introduce the notation $T_{\{pqrs\}} \equiv \frac{1}{2}(T_{[ab][cd]} + T_{[cd][ab]})$: We can decompose the Weyl tensor and sort the components of the Weyl tensor by boost weight

$$\begin{aligned} C_{abcd} = {} & \overbrace{4C_{0i0j} n_{\{a} m^i{}_b n_c m^j{}_{d\}}}^{2} + \overbrace{8C_{010i} n_{\{a} \ell_b n_c m^i{}_{d\}} + 4C_{0ijk} n_{\{a} m^i{}_b m^j{}_c m^k{}_{d\}}}^{1} + \\ & \left\{ \begin{array}{l} 4C_{0101} n_{\{a} \ell_b n_c \ell_{d\}} + 4C_{01ij} n_{\{a} \ell_b m^i{}_c m^j{}_{d\}} + \\ 8C_{0i1j} n_{\{a} m^i{}_b \ell_c m^j{}_{d\}} + C_{ijkl} m^i_{\{a} m^j{}_b m^k{}_c m^l{}_{d\}} \end{array} \right\}^{0} + \\ & \overbrace{8C_{101i} \ell_{\{a} n_b \ell_c m^i{}_{d\}} + 4C_{1_{ijk}} \ell_{\{a} m^i{}_b m^j{}_c m^k{}_{d\}}}^{-1} + \overbrace{4C_{1i1j} \ell_{\{a} m^i{}_b \ell_c m^j{}_{d\}}}^{-2} . \end{aligned} \tag{3}$$

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IV. BASIS FOR SCALAR CURVATURE INVARIANTS

For the most part, scalar curvature invariants have been studied by considering the scalar curvature invariants formed from the Riemann tensor R_{abcd} only (contractions involving products of the undifferentiated Riemann tensor only, the so called *algebraic invariants*) in 4D Lorentzian spacetimes. This leads to the natural problem of finding a basis for the scalar curvature invariants formed from the Riemann tensor (up to some order of covariant differentiation). Much work has gone into the problem of constructing a basis in the 4D Lorentzian algebraic case. In this case one can form 14 functionally independent scalar curvature invariants. The smallest set that contains a maximal set of algebraically independent scalars consists of 17 polynomials. All of these sets were shown to be deficient for various reasons. A set of algebraic invariants was presented by CM, consisting of 16 curvature invariants, that contains invariants of lowest possible degree and contain s a minimal set for any Petrov type and for any specific choice of Ricci tensor type in the perfect fluid and Einstein-Maxwell cases. In general, the expressions relating invariants to the basis members of an independent set can be very complicated, and can be singular in certain algebraic cases.

There are differing notions of what is meant by a basis. The number $N(n,p)$ of algebraically independent quantities formed from the first p derivatives of the Riemann tensor in dimension n is known and corresponds to the independent components of the Riemann tensor. Hence, in principal a basis could be these $N(n,p)$ scalars as all other curvature invariants are functions of these scalars. However, the expressions relating these invariants can be very complicated, involving roots of high order, and they may therefore be singular in certain algebraic cases. For actual classification using invariants, a different type of basis is needed. The common solution to this problem is to seek a basis of scalars such that all other scalars are polynomials in this basis. This is the approach that has been widely used in the study of scalar curvature invariants in 4D Lorentzian signature.

A. Bounds on p

In Lorentzian spacetimes we seek a minimal set of algebraically independent scalar curvature invariants formed by the contraction of the Riemann tensor and its first p covariant derivatives. We must determine the bound on p.

The bound $q = p + 1$ is used in the Cartan-Karlhede algorithm for determining the equivalence of spacetimes. The bound on the algorithm (by considering the 'worst-case scenario' for the number of steps required for the algorithm to finish) is

$$q = N_0 + n + 1 \tag{6}$$

where N_0 is the dimension of the isotropy group of the Riemann tensor and n is the dimension of the spacetime.

1. Bounds in 4D

Cartan's argument for the bound on q gives a bound of $n(n+1)/2 = 10$. It is, however, impossible in the 4D Lorentzian case to have $p > 7$ (except for the constant curvature case). To see this suppose that $p > 7$. In this case, in the Karlhede algorithm we must find at most 10 functions on the 10 dimensional frame bundle (6 Lorentz group parameters and 4 independent functions of spacetime). Now, since we need 8 or 9 steps to terminate the algorithm, then at most two of the parameters were already fixed at the beginning. Then the undetermined part of the Lorentz group would be of dimension at least 4, but there is no choice of Weyl and Ricci curvature invariant under the 4D subgroup of the Lorentz group. Then the curvature would be invariant under the whole Lorentz group, which implies that the spacetime is of constant curvature. Furthermore, in 4D we know that if $p \geq 6$ then the spacetime is degenerate Kundt. Therefore, for $\mathcal{I}$-non-degenerate spacetimes, we have $p \leq 5$.

2. Bounds in 3D

In 3D spacetimes, the Weyl tensor vanishes, and the canonical frame of the Karlhede algorithm is aligned with principal directions of the Ricci tensor rather than the Weyl tensor. All spacetimes have $N_0 \leq 1$, hence we have $p \leq 4$ (the same as the 3D Riemannian case, which has no 2D isotropy group). [There is a 2D isotropy group of the Lorentz group spanned by a boost and a null rotation; however, the resulting spacetimes must be degenerate Kundt.] The bound cannot be improved any further in 3D.

The number of algebraically independent scalars constructible from the Riemann tensor and its covariant derivatives up to order p is given by:

$$N(n,p) = \frac{n[n+1][(n+\bar{p})!]}{2n!\bar{p}!} - \frac{(n+\bar{p}+1)!}{(n-1)!(\bar{p}+1)!} + n, \tag{7}$$

where $\bar{p} = p+2$ ($\bar{p}$ is the number of derivatives of the metric), with $\bar{p} \geq 2$, except for $N(2,2) = 1$.

In 3D, we note that $N(3,0) = 3, N(3,1) = 18, N(3,2) = 45, N(3,3) = 87$, $N(3,4) = 147$, and $N(3,5) = 228$. We also note that for all values of p, $N(n,p)$ given by equation (7) is equal to the number for independent components up to the pth derivative of the Riemann tensor given in minus three (we can deduct three because of the coordinate conditions that can be imposed).

B. Cartan Invariants in 3D

Since $\mathcal{I}$-non-degenerate spacetimes can be characterized by their scalar curvature invariants alone, the full machinery of the Cartan equivalence method is only necessary for the classification of the degenerate Kundt spacetimes.

It is conjectured that for $\mathcal{I}$-non-degenerate spacetimes all Cartan invariants are determined (up to possible discrete complex transformations) by scalar polynomial curvature invariants (as is the case in the Riemannian case).

The equivalence problem in 3D has been studied in a number of spacetimes.

C. Cartan-Karlhede algorithm

Set $i = 0$, $q = 0$. then:

1. Calculate the set I_i (derivatives of the curvature up to the i-th order).
2. Fix the frame, as much as possible, by putting the elements of I_i into canonical forms.
3. Find the frame freedom given by the isotropy group H_i of transformations which leave invariant the normal form of I_i.
4. Find the number t_i of functionally independent functions of spacetime coordinates in the elements of I_i, brought into the normal forms.
5. If the isotropy group H_q is the same as H_{q-1} and the number of functionally independent functions t_q is equal to t_{q-1} , then let $q = p+1$ and stop. If not, set $i = i+1$ and repeat the algorithm.

V. APPLICATIONS

The dynamical content of GR is fully expressed by the EFE. Nevertheless, even in a purely classical (i.e., non-quantum) context, it is convenient and useful for many purposes to have Lagrangian and Hamiltonian formulations of GR. We note that in GR the EFE are more fundemental than the action, since the boundary terms are added precisely to cancel the surface terms and exactly produce the EFE of GR.

The formulation of a quantum theory of gravitation requires that GR be expressed in a Lagrangian or Hamiltonian form. A Lagrangian formulation of a field theory is "spacetime covariant." A Hamiltonian (ADM) formulation necessarily requires a global breakup of the spacetime into space and time. The action is a global object and is well-defined when the global topology is fixed as $R \times S^3$.

To determine the EL equations, we need to know not only the topology, but also the appropriate boundary conditions. Regarding the boundary conditions, the action certainly makes more sense when S^3 is compact. The surface integral is more complicated in open universes, in which boundary terms enter in a more fundemental way (and are different for each type of spacetime), and there are problems with boundary conditions at infinity (which might be timelike or null).

Thus the action in GR (as used in approaches to quantum gravity), an integral over the manifold plus an integral over the boundary, is a global object and is only well defined when the topology is fixed, there is a prefered (global) timelike vector, and hence a global $1+3$ split of spacetime. A global topology $R \times S^3$ is necessary.

1. *Invariants*

Therefore, in current quantum gravity there exists a unique time. A Lorentzian spacetime with global topology $R \times S^3$ is $\mathcal{I}$-non-degenerate and thus completely classified by its set of scalar polynomial curvature invariants. In this case all gravitational degrees of freedom are curvature invariants. For example, in many theories of fundamental physics there are geometric classical corrections to GR. Different polynomial curvature invariants are required to compute different loop-orders of renormalization of the Einstein-Hilbert action. In specific quantum models such as supergravity there are particular allowed local counterterms.

However, the global split into three spatial dimensions and one time dimension seems to be contrary to the whole spirit of GR (Hawking). In order to do canonical quantization additional spacetime structure is needed; only attempt to quantize the subset of spacetimes with global topology $R \times S^3$. Also it is questionable as to whether modern theories of quantum gravity (in which fields evolve on a fixed background) respect Einstein's geometric interpretion of gravitational physics.

It might be advantageous to consider quantum gravity in a more general context. It might be expected that quantum gravity should allow all possible topologies of spacetime, and it is precisely these other topologies that may give more interesting effects (Hawking).

For example (from the $\mathcal{I}$-non-degenerate theorem), a Lorentzian degenerate Kundt spacetime is not completely classified by its set of scalar polynomial curvature invariants. The Kundt class of spacetimes have important geometrical information that is not contained in the scalar invariants and, in principle, the Einstein-Hilbert action may require geometric corrections that are not scalar invariants. In particular, the physical fundamental properties that do not depend only on scalar invariants may lead to interesting and novel physics in models of quantum gravity or string theory.

It is perhaps within string theory that the Kundt spacetimes may play a fundamental role. A Lorentzian manifold admitting an indecomposable but non-irreducible holonomy representation (i.e., with a one-dimensional invariant lightlike subspace) is a degenerate Kundt spacetime, which contains the VSI and (non locally homogeneous) CSI subclasses (in which all of the scalar invariants are zero or constant, respectively) as special cases.

Solutions of the classical FE for which the counter terms required to regularize quantum fluctuations vanish are of importance because they offer insights into the behaviour of the full quantum theory of gravity (regardless of what the exact form of this theory might be). A classical metric is called universal if the quantum correction is a multiple of the metric, and consequently such metrics can be interpreted as having vanishing quantum corrections to all loop orders and are automatically solutions to the quantum theory. In particular, VSI and CSI spacetimes are exact solutions in string theory to all perturbative orders in the string tension scale.

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A. Other work

Arbitrary signature. Neutral signature in 4D.

The averaging problem in cosmology is of considerable importance for the correct interpretation of cosmological data. A rigorous mathematical definition of averaging in a cosmological model is necessary. In general, a cosmological spacetime is completely characterized by its scalar curvature invariants, and a particular spacetime averaging scheme based entirely in terms of scalar curvature invariants has been proposed.

1 Introduction

For time-dependent black holes (BHs) apparent horizons (AHs), which are defined as the zero-set of the vanishing expansion of a null geodesic congruence normal to a trapped surface with spherical topology, are utilized. A related concept is marginally outer trapped surfaces (MOTSs) which are two-dimensional (2D) surfaces for which the expansion of the outgoing null vector normal to the surfaces vanishes. Assuming a smooth time evolution for the MOTSs, the 2D surfaces can be combined to construct a three-dimensional (3D) surface known as a marginally trapped tube (MTTs). If the MTT is foliated by MOTSs and the expansion of the ingoing null vector normal to the surface is negative then it is called a dynamical horizon (DH).

Unlike the event horizon, the AH and MTTs are quasilocal, and they are intrinsically foliation-dependent, and hence observer dependent so that different observers may observe different MTTs, and trapping horizons and DHs can be non-unique. A DH is particularly well-suited to analyze realistic dynamical processes involving BH such as BH growth and coalescence. This motivates the idea of using MTTs as viable replacements for the event horizon of BH.

1.1 The Geometric Horizon Conjectures

An alternative more geometrical approach to identify the boundary of a BH is provided by a *geometric horizon* (GH), in which the GH is identified by surfaces in the spacetime on which the curvature tensor or its covariant derivatives are algebraically special, in that positive boost weight (b.w.) terms in an appropriate null frame are zero. That is, (*Conjecture part I) for a BH spacetime arising in the dynamical collapse or merger of real BH, the geometry is typically of general algebraic type away from the GH, but is more algebraically special on the GH.*

Using recent results in invariant theory, such GHs can be identified by the alignment type **II** or **D** discriminant conditions in terms of scalar polynomial (curvature) invariants ($SPIs$), which are not dependent on spacetime foliations. Therefore, a particular set of SPIs vanish on the GH due to the fact that on the horizon the curvature tensor and its covariant derivatives must be of type **II**/**D** relative to the alignment classification.

Comments: In 4D, for the Weyl tensor algebraically general means type **I**. We note that SPIs may not specify the GH completely in the sense that the invariants may vanish at particular points such as the fixed points of an isometry or along an axis of symmetry. Unlike AHs, we emphasise that a GH does not depend on a chosen foliation in the spacetime.

The necessary real conditions for the Weyl tensor to be of type **II/D** are known. These 2 real conditions are equivalent to the real and imaginary parts of the complex syzygy $\mathcal{D} \equiv I^3 - 27J^2 = 0$ in terms of the complex Weyl tensor, where the Ψ_i are components of the complex-valued Weyl spinor in the NP formalism, with I and J defined as

$$I = \Psi_4\Psi_0 - 4\Psi_3\Psi_1 + 3\Psi_2^2, \tag{1}$$

$$J = det \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}. \tag{2}$$

We are primarily interested in applications in 4D, and particularly in numerical computations. In physically relevant problems with dynamical evolution, such as asymmetric collapse and BH coalescences, the horizon might not be unique, or may not be defined by the specified invariants at all, and the conjectures may have to be modified accordingly: *(Conjecture part II) If the whole spacetime is zeroth-order algebraically special (and on the horizon the spacetime is thus also algebraically special) and if the whole spacetime has an algebraically general first order covariant derivative of the Riemann tensor, then on the GH this will be algebraically special and we can identify this surface using SPIs.*

We will discuss some examples later.

2 Algebraically preferred null frame

In 4D, we may always choose the null frame relative to which either the Weyl tensor or the Ricci tensor is of algebraic Petrov type **I**; we shall refer to this an an *algebraically preferred null frame* (APNF). The GH is then identified by a surface on which the spacetime is more algebraically special; i.e., where positive boost weight (b.w.) components vanish on this surface. If the spacetime is of type **II** or **D**, the covariant derivative of the Weyl or Ricci tensor can be used instead to define the GH, as one of these tensors will generally be of type **I**, and we can identify this surface using SPIs.

In principle, the GH could be identified by surfaces of vanishing SPIs but, unfortunately, the relevant SPIs are difficult to calculate. However, the algebraic nature of a GH means that the APNF plays a central role: indeed, a null frame approach provides a more direct approach to verify the type **II**/**D** property of the horizon, and generally, the associated Cartan invariants are less computationally expensive to compute. The Newman-Penrose (NP) formalism is also useful for computing the expansion of the outcoming null frame vector of the invariant coframe, which is related to (minus) the real part of the NP spin coefficient, ρ (relative to the APNF).

2.1 Examples

There are many examples and analytical results that support the GH conjectures. For example, in 4D it was demonstrated that on the non-expanding weak isolated horizon (WIH) the Ricci and Weyl tensors are of type **II/D** and that the covariant derivatives of the Riemann tensor on a WIH are of type **II** on the GH. There are examples of dynamical BH solutions that admit GHs.

In particular, in many cases it has been shown that relative to an APNF the GH is identified by the vanishing of the extended Cartan scalar (NP spin coefficient) ρ. For stationary spacetimes with stationary horizons (such as the Kerr-Newman-NUT-anti de Sitter metric) the spacetime is everywhere of Weyl and Ricci type **D**, and the APNF is adopted so that both these algebraic conditions are manifest, and the location of the GH is then obtained by the surface on which the covariant derivative of the Weyl and Ricci tensors are of type **II**, which is equivalent to ρ vanishing. In the case of spherically symmetric dynamical BHs, the Weyl tensor is of type **D**, and for vacuum solutions or known exact solutions such as Vaidya or LTB dust solutions (in which the horizons are known and are WIHs) the Ricci tensor is simple and does not help to identify the GH. So we choose the APNF adapted to Weyl type **D**, in which case the type **II** GH surface of the covariant derivative of the Weyl tensor is identified by $\rho = 0$. In the case of the non-spherically symmetric quasi-spherical Weyl type **D** Szekeres dust spacetimes, the APNF is adapted to the Ricci tensor which is of type **I**. The covariant derivative of the Weyl tensor is then considered, and the GH is shown to be identified by $\rho = 0$.

2.2 Kastor-Traschen spacetimes

The axisymmetric evolution of a 4D non-vacuum two-equal-mass BH Kastor-Traschen (KT) spacetime has also been investigated. The Weyl tensor is of type **I**. These spacetimes are exact closed universe solutions to the Einstein-Maxwell equations with a cosmological constant representing an arbitrary number of charge-equal-to-mass BHs. When the sum of the two BH masses does not exceed a critical mass M_C, the BHs coalesce and form a larger BH. If coalescence does not occur, the collision will presumably produce a naked singularity. We are particularly interested in the numerical investigation of axisymmetric 2 KT BH spacetimes.

3 BH mergers

The problem of numerically simulating the axisymmetric head-on collision of two unequal mass BHs has been recently considered, leading to the usual "pair of pants" description in which a spacetime foliation could have four MOTS, with an AH as the outer-most of these MOTS. In particular, it was found numerically that the MOTS associated with the final BH merges with the two (initially disjoint) surfaces associated with the two initial BHs. This produces a "connected sequence of MOTSs" which interpolates between the initial and final state throughout the non-linear BBH merger process. Furthermore, the computation was tracked up to and beyond the merger point. Lastly, directly following the merger, it was found that MOTS formed which contained self-intersections.

REFS: D. Pook-Kolb, O. Birnholtz, B. Krishnan and E. Schnetter, 2019, *Phys. Rev. D.* **100**, 084044 & *Phys. Rev. D*, **99** 064005 & 2019, *Phys. Rev. Lett.*, **123**, 171102; see also 2018, *Phys. Rev.*, **D97** 084028 & 2006, *Phys. Rev.*, **D74** 024028 2003, & *Phys. Rev.*, **D67** 024018 (various authors).

In an alternative approach to the head-on collision of two unequal mass BHs, the algebraic properties of the Weyl tensor through the merger of two non-spinning BHs was studied numerically. The vanishing of the complex scalar invariant $\mathcal{D} \equiv I^3 - 27J^2$ characterizes a smooth foliation independent surface (GH) associated with the BH. In the particular simulation, the level sets of $\mathrm{Re}(\mathcal{D})$ (since $\mathrm{Im}(\mathcal{D}) = 0$) were investigated. At late times when the spacetime is essentially of type **D** everywhere, $\mathcal{D} = 0$, the GH is conjectured to be located by the fact that the covariant derivative of the Weyl tensor is of type **II/D** there. In the APNF frame there is numerical evidence that these surfaces are characterized by the condition that the spin coefficient ρ is zero.

3.0.1 Two merging equal mass non-spinning BHs

The ultimate goal is the study of curvature invariants in a *non-axisymmetric* binary BH merger. Recently, we have studied numerically the GH conjecture by tracing the level zero set of the magnitude of the complex scalar polynomial invariant, $|\mathcal{D}|$, through a quasi-circular merger of two non-spinning, equal mass BHs (by approximating the level–0 sets of $\mathcal{D}$). The numerical results presented provide evidence that a (unique) smooth GH can be identified throughout all stages of the binary BH merger. To study this more comprehensively in future we need to compute the covariant derivative of the Weyl tensor and the spin coefficient ρ in an APNF (ab initio, since the null frame used in previous work is not an APNF).

In this simulation, which has not been presented elsewhere, the Einstein toolkit infrastructure was used utilizing Brill-Lindquist initial data with BH positions and momenta set up to satisfy the "QC-0" initial condition. In the actual simulations, the real and imaginary parts of I and J are calculated using the Cartan (Weyl spinor) invariants, $\{\mathbf{\Psi}_i\}_{i=0}^{5}$, and the calculations are carried out using the orthonormal fiducial tetrad.

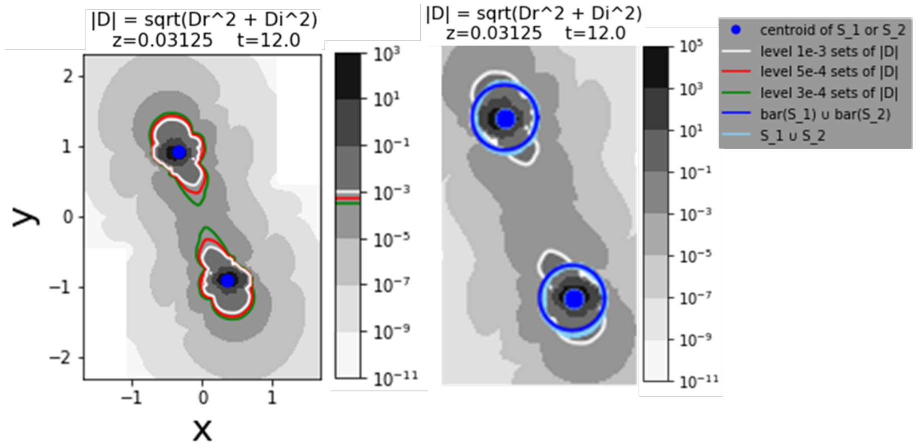

Figure 1: Plots of $|D| = \sqrt{x^2 + y^2}$ at time $t = 12$ from the quasi-circular orbit of two merging, equal mass and non spinning BHs as functions of x and y for fixed $z = 0.03125$. The right panel 2 has the same horizontal and vertical scale as the left panel 1 but the plot is taken at magnified resolution.

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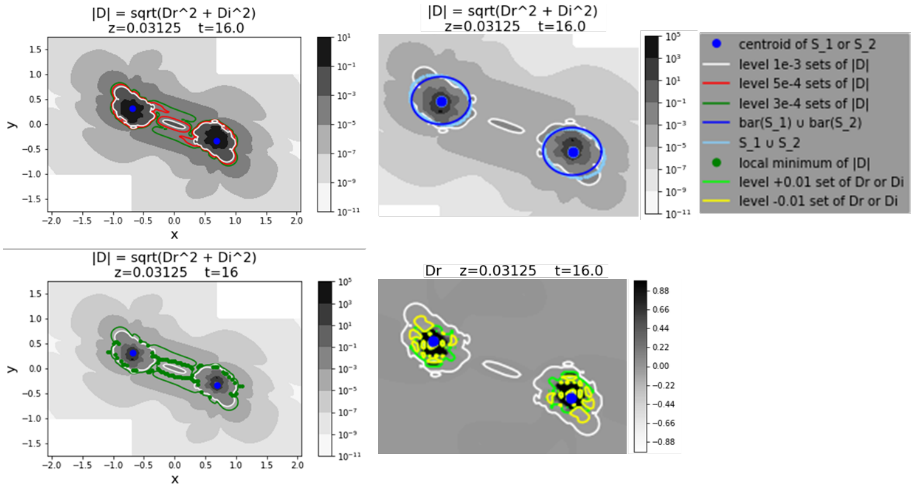

Figure 2: Upper left, lower left (panel 3) and upper right panels: plots of $|\mathcal{D}| = \sqrt{x^2 + y^2}$ at time $t = 16$ from quasi-circular orbit of two merging, equal mass and non spinning BHs as functions of x and y for fixed $z = 0.03125$. Lower right panel 4: Plot of $\mathcal{D}_r$ at time $t = 16$ from the quasi-circular BBH merger. In this figure and in Figure 4, all 4 plots are taken to be at the same scale but the lower and upper right plots are at increased resolution.

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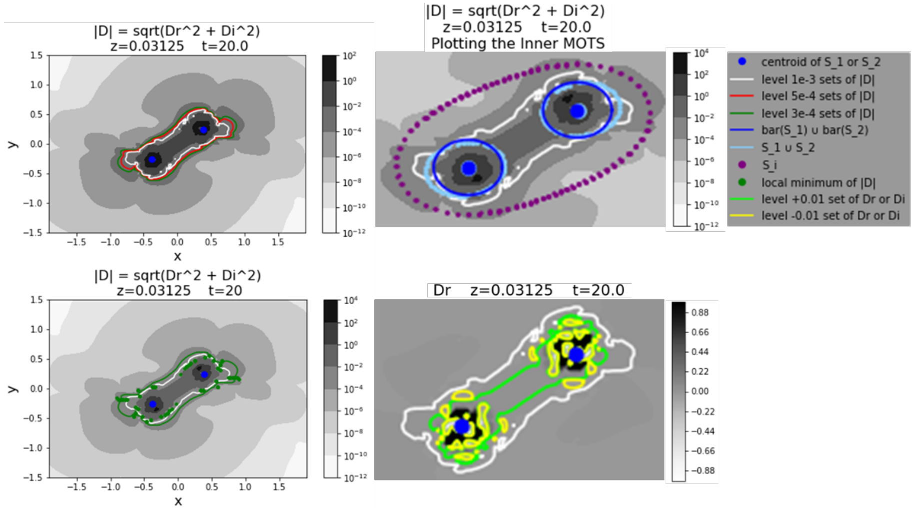

Figure 3: Upper left, lower left and upper right panels: plots of $|\mathcal{D}| = \sqrt{x^2 + y^2}$ at time $t = 20$ from the quasi-circular orbit of two merging, equal mass and non-spinning BHs as functions of x and y for fixed $z = 0.03125$. Lower right panel 4: Plot of $\mathcal{D}_r$ at time $t = 16$.