Math 3202 **Answers** **Assignment #3**

1. Given position vector $\vec{r}(t) = (e^t, e^t \sin t, e^t \cos t)$, find tangential and normal components of the acceleration vector at $(1, 0, 1)$.

   Answer: From Homework 2, Problem 6b we know $t = 0$ as well as vectors $\vec{T}(0)$ and $\vec{N}(0)$. The acceleration vector is $\vec{a}(t) = (e^t, 2e^2 \cos t, -2e^t \sin t)$.

   Thus $\vec{a}(0) = (1, 2, 0)$ and its representation in terms of unit tangent vector $\vec{T}(0)$ and unit normal vector $\vec{N}(0)$ is $\vec{a}(0) = \sqrt{3}\vec{T}(0) + \sqrt{2}\vec{N}(0)$. Thus tangential part of acceleration is vector $(1, 1, 1)$, and normal part is $(0, 1, -1)$.

2. The position function of the spaceship is

$$\vec{r}(t) = (1 + t, 8 + t^2, 28 + t^3)$$

   and the coordinates of the space station are $(0, 0, 0)$. At what moment of time t should the captain turn off the engines in order to coast into the station?

   Answer: If the captain turns off the engines at time t_0 then the trajectory will be straight line in the direction of velocity $\vec{v}(t_0)$, it other words $\vec{r}(k) = \vec{r}(t_0) + k\vec{v}(t_0)$, where $k > 0$. Solving $\vec{r}(t_0) + k\vec{v}(t_0) = \vec{0}$ for t_0 and k we have $t = 2$ and $k = -3$. But $k = -3$ means that the spaceship can only get to the station if it reverses the direction of its motion along the same line which is not possible (engines are off) so there is no way it coasts into the station.

3. Answer questions 1 – 5 for each of the following functions (a) –(f).

   1. Sketch level curves $f(x, y) = k$, $k = 0, 1, 2$ (if such a curve exists).
   2. Name and sketch the surface given by $z = f(x, y)$.
   3. Find partial derivatives f_x and f_y.
   4. Write equation of the tangent plane at point (x_0, y_0, z_0), where x_0, y_0, z_0 are given below.
   5. Find gradient vector at point (x_0, y_0) and sketch it at the same pictire as the level curves.

   (a) $f(x, y) = x^2 - \frac{y^2}{4}$, $x_0 = 1$, $y_0 = 2$, $z_0 = 0$;
   Answers
   1. $k = 0$ gives two crossing lines $y = \pm 2x$;
   $k = 1$ gives hyperbola $x^2 - y^2/4 = 1$ with vertices at $(\pm 1, 0)$ and asymptotes $y = \pm 2x$.
   $k = 2$ gives hyperbola $x^2 - y^2/4 = 2$ with vertices at $(\pm\sqrt{2}, 0)$ and asymptotes $y = \pm 2x$.
   2. This is Hyperbolic Paraboloid (Saddle).
   3. $f_x = 2x$, $f_y = -y/2$.
   4. Tangent plane at point $(1; 2; 0)$ is $z = 2x - y$.
   5. Gradient vector at point $(1; 2)$ is $\vec{w} = (2; -1)$. Note that it is orthogonal to the level curve $k = 0$.

   (b) $f(x, y) = x^2 + \frac{y^2}{4}$, $x_0 = 1$, $y_0 = 2$, $z_0 = 2$;
   Answers
   1. $k = 0$ gives one point $y = x = 0$;
   $k = 1$ gives ellipse $x^2 + y^2/4 = 1$ with vertices at $(\pm 1, 0)$ and $(0; \pm 2)$.
   $k = 2$ gives ellipse $x^2 + y^2/4 = 2$ with vertices at $(\pm\sqrt{2}, 0)$ and $(0; \pm\sqrt{8})$.

2. This is Elliptic Paraboloid.
3. $f_x = 2x$, $f_y = y/2$.
4. Tangent plane at point $(1; 2; 2)$ is $z = 2x + y - 2$.
5. Gradient vector at point $(1; 2)$ is $\vec{w} = (2; 1)$. Note that it is orthogonal to the level curve $k = 2$.

(c) $f(x, y) = \sqrt{2 + x^2 + \frac{y^2}{4}}$, $x_0 = 1$, $y_0 = 2$, $z_0 = 2$;
Answers
1. $k = 0$ gives no points
$k = 1$ gives no points
$k = 2$ gives ellipse $x^2 + y^2/4 = 2$ with vertices at $(\pm\sqrt{2}, 0)$ and $(0; \pm\sqrt{8})$.
2. This is the upper part of Hyperboloid of two sheets.
3. $f_x = \dfrac{x}{\sqrt{2 + x^2 + y^2/4}}$, $f_y = \dfrac{y}{4\sqrt{2 + x^2 + y^2/4}}$.
4. Tangent plane at point $(1; 2; 2)$ is $z = \frac{1}{2}x + \frac{1}{4}y + 1$.
5. Gradient vector at point $(1; 2)$ is $\vec{w} = (1/2; 1/4)$. Note that it is orthogonal to the level curve $k = 2$.

(d) $f(x, y) = \sqrt{x^2 + \frac{y^2}{4}}$, $x_0 = 1$, $y_0 = 0$, $z_0 = 1$;
Answers
1. $k = 0$ gives one point $x = y = 0$
$k = 1$ gives ellipse $x^2 + y^2/4 = 1$ with vertices at $(\pm 1, 0)$ and $(0; \pm 2)$.
$k = 2$ gives ellipse $x^2 + y^2/4 = 4$ with vertices at $(\pm 2, 0)$ and $(0; \pm 4)$.
2. This is the upper part of Cone.
3. $f_x = \dfrac{x}{\sqrt{x^2 + y^2/4}}$, $f_y = \dfrac{y}{4\sqrt{x^2 + y^2/4}}$.
4. Tangent plane at point $(1; 0; 1)$ is $z = x$.
5. Gradient vector at point $(1; 0)$ is $\vec{w} = (1; 0)$. Note that it is orthogonal to the level curve $k = 1$.

(e) $f(x, y) = \sqrt{3 - x^2 - \frac{y^2}{4}}$, $x_0 = 1$, $y_0 = 2$, $z_0 = 1$;
Answers
1. $k = 0$ gives ellipse $x^2 + y^2/4 = 3$
$k = 1$ gives ellipse $x^2 + y^2/4 = 2$
$k = 2$ gives no points.
2. This is the upper part of Ellipsoid.
3. $f_x = -\dfrac{x}{\sqrt{3 - x^2 - y^2/4}}$, $f_y = -\dfrac{y}{4\sqrt{3 - x^2 - y^2/4}}$.
4. Tangent plane at point $(1; 2; 1)$ is $z = -x - y/2 + 3$.
5. Gradient vector at point $(1; 2)$ is $\vec{w} = (-1; -1/2)$. Note that it is orthogonal to the level curve $k = 1$.

(f) $f(x,y) = \sqrt{x^2 + \frac{y^2}{4} - 1}$, $x_0 = 2$, $y_0 = 2$, $z_0 = 2$;

Answers

1. $k = 0$ gives ellipse $x^2 + y^2/4 = 1$
$k = 1$ gives ellipse $x^2 + y^2/4 = 2$
$k = 2$ gives ellipse $x^2 + y^2/4 = 5$
2. This is the upper part of Hyperboloid of one sheet.
3. $f_x = \dfrac{x}{\sqrt{x^2 + y^2/4 - 1}}$, $f_y = \dfrac{y}{4\sqrt{x^2 + y^2/4 - 1}}$.
4. Tangent plane at point $(2; 2; 2)$ is $z = x + y/4 - 1/2$.
5. Gradient vector at point $(2; 2)$ is $\vec{w} = (1; 1/4)$. Note that it is orthogonal to the level curve $k = 2$.