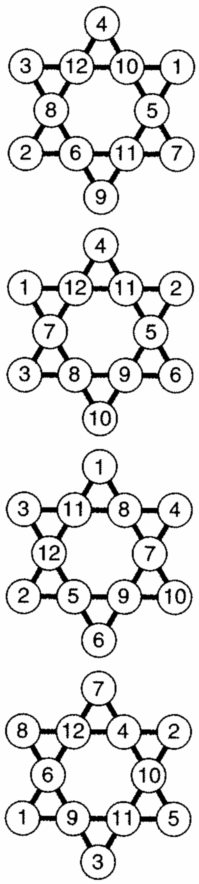
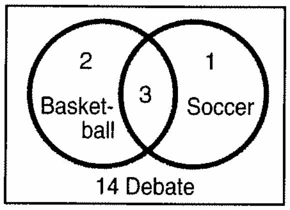
Answers to Calendar

This month's problems 1–17 were submitted by Richard Evans, Plymouth State College, Plymouth, NH 03264. Problems 18–23 have been provided by Tony Trono, Burlington High School, Burlington, VT 05401. Problems 24–31 were submitted by Rudd A. Crawford's Mathematics 211: Introduction to Mathematics Education class, Oberlin College, Oberlin, OH 44074-1095. These seven problems were created by Dana Sandvoss, class of 1991.

1 Four possible solutions are shown.



2 In the Venn diagram pictured, we see that three players were on the basketball team and the soccer team, leaving two players on the basketball team not on the soccer team and one player on the soccer team not on the basketball team. Thus a total of six players are on those two teams, leaving fourteen players to be on the debate team.

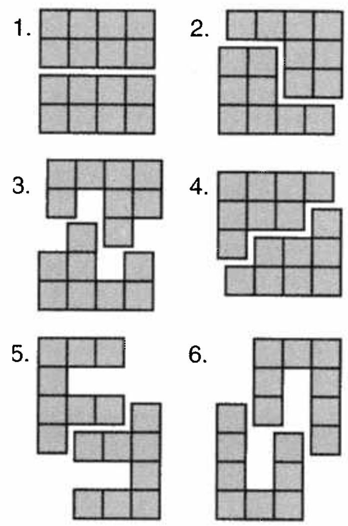


3 It remained 3 1/4 degrees. Magnified images are similar in shape and do not change angle sizes.

4 You would have to flip the 6 card and the G card. The 6 card needs to be flipped, since it has an even number on its side and you must check to see if it has a vowel on the other side. The G card needs to be flipped; since it is not a vowel it cannot have an even number on the other side or it would disprove the statement in question (that is, be the contrapositive of the original statement).

5 5. Represent the whole numbers by $x, x + 1, x + 2, x + 3$, and $x + 4$. Their sum is $5x + 10$. The largest whole number that divides those numbers for certain is 5, since $5x + 10 = 5(x + 2)$.

6 Six possible ways are pictured.



7 Pumpwarts are five-digit odd numbers whose digits alternate odd and even, sum to 17, and never repeat. Therefore, the pumpwarts are 72341, 70523, and 52703.

8 Work backward:

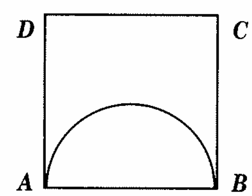
Transaction	Left after Paying Ima	Left after Doubling	Started With
Third	\$ 0	\$120	\$ 60
Second	\$ 60	\$180	\$ 90
First	\$ 90	\$210	\$105

9 39 percent, 0 percent, 61 percent. If the third vertex lies on the semicircle with $\overline{AB}$ as diameter in the square's interior, it is a right triangle. If the vertex lies inside the semicircle, the triangle will be obtuse. And if the vertex lies outside the semicircle but inside the square, the triangle will be acute. A comparison of areas yields the probability of each kind of triangle.

Right triangle. The area of the arc is 0, thus the probability is zero. This may seem to be somewhat of a paradox. For further information and exploration, refer to *Geometric Probability* (NCTM 1988), by the Department of Mathematics and Computer Science of the North Carolina School of Science and Mathematics.

Obtuse. The area of the semicircle is $\pi(AB)^2/8$. The area of the square is $(AB)^2$. Their ratio is $\pi/8 \approx 0.39$, which is the probability of an obtuse triangle.

Acute. The area outside the circle but in the square is $(AB)^2 - \pi(AB)^2/8$, or $(AB)^2(1 - \pi/8)$, or about $0.61(AB)^2$. The ratio of that quantity to the area of the square is 0.61, the probability of an acute triangle.



(Continued on page 47)

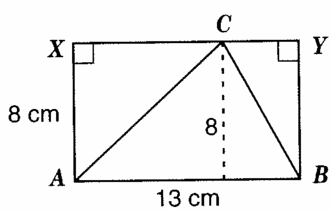
The Editorial Panel of the Mathematics Teacher *is now considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the calendar during the 1993–94 academic year. Please write to the editorial coordinator, 1906 Association Drive, Reston, VA 22091, for guidelines.*

Two other sources of problems in calendar form are available from NCTM: "Calendars for the Calculating" (a set of nine monthly calendars that originally appeared from September 1983 to May 1984, order number 344, \$5.75) and "A Year of Mathematics" (one annual calendar that originally appeared in September 1982, order number 311, \$2.50; set of five, order number 312, \$5.00). Individual members receive a 20 percent discount off these prices. Write to NCTM, Department P, for the catalog of educational materials, which includes Exploratory Problems in Mathematics.—*Ed.*

10 Y. The Y terms appear in the first, third, sixth, tenth, and so on positions. These terms are triangular numbers, and the nth triangular number is given by the formula $n(n+1)/2$. Setting $276 = n(n+1)/2$, we get $n^2 + n - 552 = 0$. Solving for n yields $n = 23$. Thus, the 276th term is a triangular number and the term must be Y.

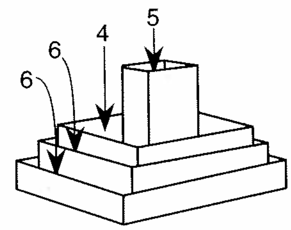
11 Since eight logs were burned, each person's share was 2 ⅔ logs. Larry bought 5 logs, so he had 2 ⅓ logs more than his share. Mo bought 3 logs, so he bought ⅓ more than his share. Together they bought 8/3 extra logs. Therefore, ⅓ of a log is equivalent to \$1. Thus, Larry should get \$7 and Mo should get \$1.

12 1:1. The area of triangle $ABC = 1/2\,(8)(13)$ cm^2, or 52 cm^2. This quantity is half the area of the rectangle (8.13 cm^2). Thus, the sum of the areas of the two smaller triangles equals 52 cm^2 also. Consequently, the ratio of their area is one to one.



13 22. The sum of the first n integers is $1 + 2 + 3 + \cdots + n = n(n+1)/2$. Since the denominator of the average is 13, the divisor for the average must have been a multiple of 13. Thus, the largest number on the chalkboard should be congruent to 1 mod 13, or a member of the set $\{1, 14, 27, 40, \ldots\}$. The sum of the first fourteen integers is $14(15)/2 = 105$, and $13\ 9/13 = 178/13$, which means that $n = 14$ is too small. If $n = 27$, the sum would have been $27(28)/2 = 378$ and the class's average would have been 378/26. Since $178/13 = 356/26$, then $378 - 356 = 22$ and the number erased was 22.

14 Many solutions to this problem are possible. The idea is to put boxes inside of one another, since of course the sum of four odd numbers is even. For example, place five marbles in the small box, place that inside the second smallest box along with



four marbles, place that inside of the next box with six marbles, and place that inside the largest box along with the last six marbles. Thus each box contains five, nine, fifteen, and twenty-one marbles, respectively.

15 Thirty-four. If you draw the gloves first, you would need twenty-one draws to be sure you have a pair of gloves, since you need to be concerned with left or right hand. It then would take twenty-two more draws from the socks box to be sure you get a pair of socks the same color, for a total of forty-three draws. If you draw from the socks box first, you need only three draws to get a pair of some color. It then would take thirty-one draws from the glove box to be sure you have a pair of gloves the same color as the socks, for a total of thirty-four draws.

16 Work backward: Since he now has sixteen in the first pile, he must have taken eight from the third pile because he doubled what he had in the first pile. Therefore, the third pile must have had twenty-four in it. Since this amount was double the amount it originally started with, it must have started with twelve. So Scott took twelve from the second pile to make twenty-four. Therefore, the second pile must have had twenty-eight before the removal. This result came about from doubling the original amount, so the second pile must have had fourteen in it. Thus Scott took fourteen from the first pile to start the process. When this amount is added to the eight left just before the end, twenty-two remain for the first pile.

First Pile	Second Pile	Third Pile
16	16	16
8	16	24
8	28	12
22	14	12

17 14142. The sum of $1 + 2 + 3 + \cdots + n = n(n+1)/2$. The smallest number that exceeds the capacity of the calculator is 100 000 000, so setting it equal to $n(n+1)/2$ yields $n(n+1)/2 = 100\,000\,000$ or, $n^2 + n - 200\,000\,000 = 0$; and using the quadratic formula, we have

$$n = \frac{-1 \pm \sqrt{1 + 800\,000\,000}}{2},$$

$$\frac{-1 + 28284}{2} < \frac{-1 \pm \sqrt{800\,000\,001}}{2} < \frac{-1 \pm 28285}{2},$$

and $14\,141.5 < n < 14\,142$. Thus, the value of n that makes the calculator exceed its eight-digit limit is 14 142.

18

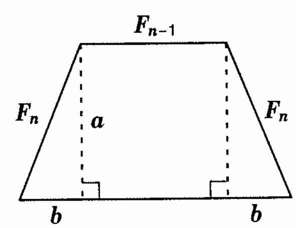
$$b = \frac{F_{n+1} - F_{n-1}}{2} = \frac{F_n}{2}$$

$$a = \frac{F_n\sqrt{3}}{2}$$

$$A = \frac{F_n\sqrt{3}}{4}\left(F_{n-1} + F_{n+1}\right)$$

$$A = \frac{\sqrt{3}}{4}\left(F_{n-1}\cdot F_n + F_n \cdot F_{n+1}\right)$$

$$A = \frac{F_{2n}\sqrt{3}}{4}$$



19

$$\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \cfrac{1}{\cfrac{F_n}{F_{n-1}}}$$

If

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = x,$$

then

$$x = 1 + \frac{1}{x},$$

or $x^2 - x - 1 = 0$; and $x = \phi$, since

$$\frac{F_{n+1}}{F_n} > 1.$$

20 In $\triangle ABC$:

$$(AC)^2 + 1 = \phi^2 = 1 + \phi$$
$$AC = \sqrt{\phi}$$

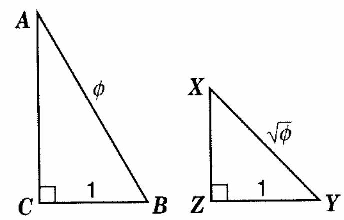
In $\triangle XYZ$:

$$(XZ)^2 + 1 = \phi$$
$$XZ = \sqrt{-\phi_1}$$

$$\sin \angle A = \frac{1}{\phi}$$

$$\sin \angle Y = \sqrt{\frac{-\phi_1}{\phi}} = \sqrt{\frac{1}{\phi^2}} = \frac{1}{\phi}$$

$$\therefore \triangle ABC \sim \triangle YZX$$



21 *a*) To find k, let $a = b = 1$, $c = 2$, $d = 3$, and $25 = 9 + (2k)^2$ and $k = 2$.

b) Proof:

$$a^2 = a^2$$
$$(a + b - b)^2 = a^2$$
$$(c - b)^2d^2 = a^2d^2$$
$$(c - b)^2(c + b)^2 = a^2d^2$$
$$(c^2 - b^2)^2 = a^2d^2$$
$$(c^2 + b^2)^2 = a^2d^2 + (2bc)^2$$
$$(c^2 + b^2 + ab - ab)^2 = (ad)^2 + (2bc)^2$$
$$(c^2 + bc - ab)^2 = (ad)^2 + (2bc)^2$$
$$(cd - ab)^2 = (ad)^2 + (2bc)^2$$

22 To prove

$$P(n) = \frac{F_{n+2}}{2^n},$$

use proof by induction. For $n = 1$, the sample space is $\{H, T\}$ and $P(1) = 1$. For $n = 2$, the sample space is $\{HH, HT, TH, TT\}$ and $P(2) = 3/4$. Assume the formula true for k tosses, then $P(k) = F_{k+2}/2^k$. If a tail shows on the next toss then consecutive heads will not show F_{k+2} times. If a head shows on that toss, then consecutive heads will not show F_{k+1} times. Thus $P(k+1) = (F_{k+1} + F_{k+2})/2^{k+1} = F_{k+3}/2^{k+1}$ and the theorem is proved.

(Continued on page 55)

ANSWERS TO CALENDAR—*Continued from page 47*

23 A 1 can be added to a string of 1's in

$$\binom{n}{0}$$

ways. One 2 and the remaining string of 1s can be grouped in

$$\binom{n-1}{1}$$

ways. Two 2s and the remaining string of 1s can be grouped in

$$\binom{n-2}{2}$$

ways. But

$$\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots$$

$$=\sum_{k=0}^{n}\binom{n-k}{k} \text{ for } n-k \geq k$$

$$=F_{n+1}.$$

24 A, B, and E have blue eyes, and C and D have green eyes.

Here's why: If E's statement were true, then everyone would have green eyes and they would all have to speak the truth. In this situation they'd all say the same thing that E said, which they didn't. So E's statement is false, and E must have blue eyes.

Next consider B's statement. If B were telling the truth, then one person would have green eyes (B) and all other statements would be false (they would all have blue eyes). But then C's statement would be true and C would have to have green eyes. This requirement raises a contradiction, so B's statement must be false also, and B must have blue eyes.

Since both E and B have blue eyes, we can conclude that A's statement is false and that A has blue eyes.

If C lied, then D's eyes must be blue, and then they would all have blue eyes, which would make B's statement true; but we already know that B lied. Therefore C must have spoken the truth and has green eyes.

Since C's statement is true, then C actually sees one person with green eyes and three people with blue eyes, so D's eyes must be green. (Adapted from *Puzzles for Pleasure* by E. R. Emmett [White Plains, N.Y.: Emerson Books, 1972])

25 Let w be the number of cows, and let h be the number of chickens. Then we can set up two equations like this:

$$4w + 2h = \text{\# legs}$$
$$w + h = \text{\# heads}$$

We are also given the information that

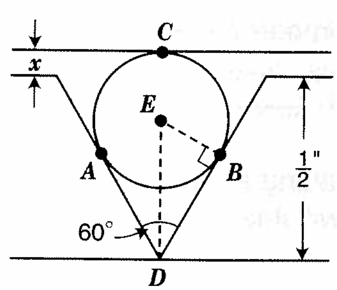
$$\text{\# legs} = 2(\text{\# heads}) + 14.$$

So we combine all this information to get

$$4w + 2h = 2(w + h) + 14,$$
$$4w + 2h = 2w + 2h + 14,$$
$$4w = 2w + 14,$$
$$2w = 14,$$
$$w = 7.$$

So the problem involves seven cows. The number of chickens is indeterminate; thus any number of chickens will work. (Adapted from *The Contest Problem Book I*, compiled by Charles T. Salkind [Washington, D.C.: Mathematical Association of America, 1961])

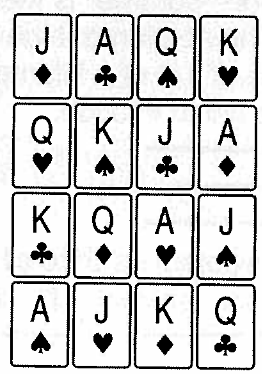
26 If we draw the radius of the circle to point B ($\overline{EB}$, where E is the center of the circle), it will be perpendicular to $\overline{BD}$. Then we complete a right triangle by drawing $\overline{DE}$. This ray bisects the 60-degree angle, so the triangle that was just drawn is a 30°-60°-90° triangle. The ratio of $\overline{EB}$ to $\overline{ED}$ is 1:2, and we know that $\overline{EB}$ is the radius; so $\overline{ED}$ is twice the radius, or simply the diameter of the circle, namely 3/8 inches.



Next we see that the distance from C to D is going to be the diameter of the circle ($\overline{ED}$) plus the radius ($\overline{EC}$), and this quantity will be equal to x + 1/2 inches. $3/8 + 3/16 = x + 1/2$; $x = 3/8 + 3/16 - 1/2$. Therefore, $x = 1/16$ inch. (Adapted from *The Contest Problem Book I*, compiled by Charles T. Salkind [Washington, D.C.: Mathematical Association of America, 1961])

27 Let N stand for the number we are seeking. $N = 10a_9 + 9 = 9a_8 + 8 = 8a_7 + 7 = \cdots = 2a_1 + 1$. From this equation we can see that $N + 1 = 10a_9 + 10 = 9a_8 + 9 = 8a_7 + 8 = \cdots = 2a_1 + 2$ and $N + 1 = 10(a_9 + 1) = 9(a_8 + 1) = 8(a_7 + 1) = \cdots = 2(a_1 + 1)$. Then it can be seen that $N + 1$ has factors 2, 3, 4, . . . , 9, 10. The least common multiple of all of these factors is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. So $N = 2519$. (Adapted from *The Contest Problem Book I*, compiled by Charles T. Salkind [Washington, D.C.: Mathematical Association of America, 1961])

28 Pictured is one of the many solutions to this puzzle. Notice that in this solution the two main diagonals also follow the rules for rows and columns. (This outcome doesn't always occur.) (Adapted from *The Book of Modern Puzzles* by Gerald L. Kaufman [New York: Dover Publications, 1954])



29 Use initials to denote the people.

First day: A and C work.

Second day: The conditions are met for B and E because C and E worked and D and B didn't on opening day. Continue in this manner.

Third day: A and D work.

Fourth day: C and E work.

Fifth day: B and D work.

Sixth day: A and C work.

The sixth day is the same as the first, and since this occurrence is the first repetition, the cycle must be AC, BE, AD, CE, BD, AC, and so on. This cycle will repeat every five days indefinitely. Divide 100 by 5 to get 20, and since this operation leaves no remainder, the 100th day must be the same as the 5th day. Readers can go through the same process to find that the 383d day will be the same as the 3d, so B and D will work on the 100th day and A and D will work on the 383d day. (Adapted from *Puzzles for Pleasure* by E. R. Emmett [White Plains, N.Y.: Emerson Books, 1972])

30 The probability of getting a detention ball (D) will be

$$\frac{\text{\# of D balls}}{\text{Total \# of balls}} = \frac{5}{20} = \frac{1}{4}.$$

The probability of choosing a skip-school ball (S) is

$$\frac{\text{\# of S balls}}{\text{Total \# of balls}} = \frac{5}{20} = \frac{1}{4}.$$

The probability of taking a ball with nothing on it is

$$\frac{\text{\# of nothing balls}}{\text{Total \# of balls}} = \frac{10}{20} = \frac{1}{2}.$$

The probabilities change as people ahead of you take balls out, but you can't calculate them unless you know what everyone ahead of you has drawn. If you do know, then you can recalculate in terms of what remains in the bowl.

31 (Adapted from *The Book of Modern Puzzles* by Gerald L. Kaufman [New York: Dover Publications, 1954])