Math 3001 **Due Fri Sept 23**

Assignment #2

1. Find the area and the perimeter of the fractal called Koch snowflake.

   Solution:
   Let s denote the length of a side of the triangle and let A_n denote the area of the figure after the n^{th} iteration. Then

$$
\begin{aligned}
A_0 &= \tfrac{\sqrt{3}}{4}s^2 \\
A_1 &= A_0 + 3\tfrac{\sqrt{3}}{4}\left(\tfrac{s}{3}\right)^2 \\
A_2 &= A_1 + 12\tfrac{\sqrt{3}}{4}\left(\tfrac{s}{9}\right)^2 \\
A_3 &= A_2 + 48\tfrac{\sqrt{3}}{4}\left(\tfrac{s}{27}\right)^2 \\
\Rightarrow A_n &= A_0(1 + 3(\tfrac{s}{3})^2 + 12(\tfrac{1}{9})^2 + \cdots) \\
&= A_0\left(1 + \tfrac{1}{3} + \tfrac{1}{3}\left(\tfrac{4}{9}\right) + \tfrac{1}{3}\left(\tfrac{16}{81}\right) + \cdots\right) \\
&= A_0\left(1 + \tfrac{1}{3}\sum_{k=0}^{n}\left(\frac{4}{9}\right)^k\right) \\
&= \tfrac{8}{5}A_0
\end{aligned}
$$

   Let P_n denote the perimeter of the figure after the n^{th} iteration.

$$
\begin{aligned}
P_0 &= 3s \\
P_1 &= 3\left(\tfrac{4}{3}\right)s \\
P_2 &= 3\left(\tfrac{16}{9}\right)s \\
\Rightarrow P_n &= 3\left(\tfrac{4}{3}\right)^n s
\end{aligned}
$$

   Since $\frac{4}{3} > 1$, $P_n \to \infty$. $\square$

2. Let f be a bounded function on $[a, b]$. Show that $L(f, P) \leq U(f, Q)$ for any two partitions P and Q of the segment.

Solution:
Let P and Q be partitions of $[a, b]$. Define $R = P \cup Q$. Then R is a partition of $[a, b]$ and a refinement of P and Q. Thus,

$$L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, Q).$$

Thus, $L(f, P) \leq U(f, Q)$. $\square$

3. Let $f(x) = x^3$. Consider partition $P_n = (0,\ 1/n,\ 2/n, ...n/n)$ of $[0, 1]$. Find $L(f, P_n)$, $U(f, P_n)$, $L(f)$, $U(f)$, and $\int_0^1 f(x)dx$.

   Solution:
   Let $d_i = x_i - x_{i-1}$. Then $d_i = 1/n$ for all i. Let
   $m_i = \inf\{\, f(x) \mid x \in [x_{i-1}, x_i]\,\}$ and $M_i = \sup\{\, f(x) \mid x \in [x_{i-1}, x_i]\,\}$.
   Then $m_i = \left(\frac{i-1}{n}\right)^3$, $M_i = \left(\frac{i}{n}\right)^3$ for all i. So

   $$L(f, P_n) = \sum_{i=1}^{n} m_i d_i = \frac{1}{n^4} \sum_{i=0}^{n-1} i^3 = \frac{(n-1)^2}{4n^2}$$

   Similarly, we have

   $$U(f, P_n) = \sum_{i=1}^{n} M_i d_i = \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \frac{(n+1)^2}{4n^2}$$

   Now we have $\lim_{n\to\infty} L(f, P_n) = L(f) = 1/4$. Also, $\lim_{n\to\infty} U(f, P_n) = U(f) = 1/4$. Therefore, since $L(f) = U(f) = 1/4$, we have
   $\int_0^1 f(x)\, dx = 1/4.$$\square$

4. Give an example of a function which is not integrable on $[a, b]$, but f^2 is integrable on $[a, b]$.

   Example:
   Consider the function

   $$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

   Then f is not integrable, but $f^2 = 1$ is integrable.$\square$

5. Let f be continuous and non-negative on $[a, b]$. Show that if $L(f) = 0$ then $f(x) = 0$ for all $x \in [a, b]$.

   Solution: Since f is continuous, f is integrable. Thus, $L(f) = 0 = U(f)$. Hence $\int_a^b f(x)\,dx = 0$. Since f is non-negative on $[a, b]$, if $f(x) > 0$ for some $x \in [a, b]$, then $\int_a^b f(x)\,dx > 0$, which is a contradiction. Therefore, $f(x) = 0$ for all $x \in [a, b]$.$\Box$

6. Let S be a finite set of points on $[a, b]$. Let f be bounded and $f(x) = 0$ for all x outside from S. Show that f is integrable and $\int_a^b f(x)dx = 0$.

   Solution: Let $g(x)$ be a function on $[a, b]$ such that $g(x) = 0$ for all $x \in [a, b]$. Then g is $R - D$ integrable on $[a, b]$. Now f is obtained by altering the values of g at a finite number of points (those in S). Hence, by Theorem 5.2.7 in the text, f is integrable on $[a, b]$. Also, $\int_a^b f = \int_a^b g = 0$. $\Box$

7. Define $F : [0, 1] \to R$ by $F(x) = x$, if x is rational, and $F(x) = 0$ if x is irrational.

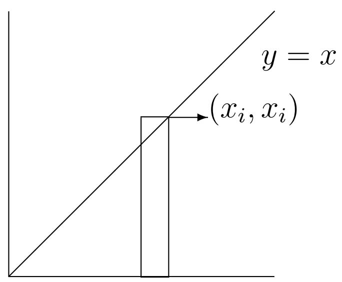
   a) Show that $U(f, P) > 1/2$ for any partition P.

   b) Show that $\lim_{n\to\infty} U(f, P_n) = 1/2$ for $P_n = (0,\ 1/n,\ 2/n, ...n/n)$.

   c) Is the function integrable?

   Solutions:

   (a) Let P be a partition of $[0, 1]$ and let $d_i = x_i - x_{i-1}$. Now we notice that $M_i = x_i$ on each subinterval $[x_{i-1}, x_i]$. So we use the value $M_i d_i$ to approximate the area beneath the graph of F on the subinterval $[x_{i-1}, x_i]$. As you can see from the picture below, this area is greater than the area beneath the graph of $y = x$ on the same interval $[x_{i-1}, x_i]$. This is true for all subintervals of the partition P. Thus, $U(F, P) > \int_0^1 x\,dx = 1/2$.



Another way to prove that $U(f,P) > 1/2$ for all partitions P of $[0,1]$ is as follows: Let P be a partition of the interval $[0,1]$. Then

$$U(f,P) = \sum_{i=1}^{n} x_i d_i > \sum_{i=1}^{n} \frac{x_i + x_{i-1}}{2}\, d_i = \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) = 1/2$$

(b) Let $d_i = x_i - x_{i-1}$. Then $d_i = 1/n$ for all i. Let $M_i = \sup\{\, F(x) \mid x \in [x_{i-1}, x_i]\,\}$. Then $M_i = i/n$ for all i. So

$$U(f,P_n) = \sum_{i=1}^{n} M_i d_i = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{n(n+1)}{2n^2}.$$

Therefore, $\lim_{n\to\infty} U(f,P_n) = 1/2$.

(c) No. Let P be a partition of $[0,1]$. Then each subinterval contains irrational numbers. Hence, $m_i = 0$ for each i. Thus $L(F) = 0 \neq 1/2 = U(F)$. Therefore, F is not integrable.

8. **Extra Points Problem**

a) For which functions, if any, $|\int_a^b f(x)dx| = \int_a^b |f(x)|dx$?

b) For which functions, if any, $|\int_a^b f(x)dx| > \int_a^b |f(x)|dx$?

c) For which functions, if any, $|\int_a^b f(x)dx| < \int_a^b |f(x)|dx$?

Solutions:

(a) This is true for all functions $f(x)$, which are integrable on $[a,b]$, where $f(x) \geq 0$ or $f(x) \leq 0$ for all $x \in [a,b]$.

(b) This is false for all functions $f(x)$ which are integrable on $[a,b]$.

(c) This is true for all functions $f(x)$ which are integrable on $[a,b]$ and which admit both positive and negative values on the interval $[a,b]$.