Math 3001 **Due Fri Sept 16** **Assignment #1**

1. Find the limit as $n \to \infty$ of the sequence given by its general term, or explain why the limit does not exist.

   (a) $\dfrac{n^3+n^2+1}{1-n^2-n^3}$

   (b) $\left(\dfrac{2+n}{n}\right)^{3n}$

   (c) $\cos(\pi n)$

   (d) $\left(\dfrac{n}{3+n}\right)^{n}$

   (e) $3n\sin\left(\dfrac{2}{n}\right)$

   Solutions:

   (a) Clearly $\lim\limits_{n\to\infty} a_n = -1$.

   (b) Let $y = \lim\limits_{n\to\infty}\left(\dfrac{2+n}{n}\right)^{3n}$. Then

   $\ln y = \lim\limits_{n\to\infty} 3n\ln\left(1+\dfrac{2}{n}\right) = 3\lim\limits_{n\to\infty}\dfrac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} = 3(2)$, by L'Hôpital's rule. Thus $y = e^6$.

   (c) This sequence diverges since $\cos(\pi n)$ oscillates between 1 and -1.

   (d) By method similar to (b), we have $\lim\limits_{n\to\infty}\left(\dfrac{n}{3+n}\right)^{n} = e^{-3}$.

   (e) $3n\sin\left(\dfrac{2}{n}\right) = 6\dfrac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}}$. Hence $\lim\limits_{n\to\infty} 6\dfrac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}} = 6(1) = 6$.

2. Prove that each of the following sequences (given recursively) is convergent, and find the limit.

   (a) $a_{n+1} = (6+a_n)^{1/2}$, $a_1 = \sqrt{6}$

   (b) $a_{n+1} = \dfrac{a_n}{2} + \dfrac{x}{a_n}$, where $x > 0$ and $a_1 > 0$ are arbitrary real numbers.

Solutions:

(a) We prove by induction on n that $a_n < a_{n+1}\ \forall\, n$. Since $a_1 = \sqrt{6} < \sqrt{6+\sqrt{6}} = a_2$, we know that it is true for $n = 1$. So assume true for $n = k$. Then $a_{k+1} = \sqrt{6+a_k} < \sqrt{6+a_{k+1}} = a_{k+2}$. Since it is true for $n = k+1$ if true for $n = k$, we can conclude that $a_n < a_{n+1}\ \forall\, n$. Thus the sequence is monotonically increasing.

We can also prove by induction on n that $a_n \leq 3\ \forall\, n$. It is clearly true for $n = 1$. So assume true for $n = k$. Then $a_{k+1} = \sqrt{6+a_k} < \sqrt{6+3} = 3$. Since it is true for $n = k+1$ if true for $n = k$, we can conclude that $a_n \leq 3\,\forall\, n$. Therefore, the sequence is bounded. So we can conclude that the sequence converges.

Let $L = \lim\limits_{n\to\infty} a_n$. Then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{6+a_n}$. This gives us
$L = \sqrt{6+L} \Rightarrow L^2 - L - 6 = 0 \Rightarrow L = 3$, since $L > 0$. $\square$

(b) Let $f(x) = x^2 - 2a$, where $a > 0$. If we graphed this function, then the vertex of the parabola would be at the point $(0, -2a)$. Hence, one of the x-intercepts of the graph is positive, i.e. $f(x') = 0$ for some $x' > 0$. We will use Newton's Method to approximate x'. Choose $x_1 > 0$ as an approximation. The $n+1$ approximation is given by

$$\begin{aligned} x_{n+1} &= x_n - \tfrac{f(x_n)}{f'(x_n)} \\ &= x_n - \tfrac{x_n^2 - 2a}{2x_n} \\ &= \tfrac{x_n}{2} + \tfrac{a}{x_n} \end{aligned}$$

From the shape of the graph it is easily seen that the method will eventually converge to x'. Thus we can conclude that the sequence defined above converges.
We now find the limit of the sequence. Let $L = \lim\limits_{n\to\infty} a_n$. Then we have $L = \frac{L}{2} + \frac{x}{L} \Rightarrow L = \pm\sqrt{2x} \rightarrow L = \sqrt{2x}$ since $L > 0$. $\square$

3. Show that if $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$ then the sequence $(a_n)_{n=1}^{\infty}$ is convergent to L.

Is it always true that if two given subsequences of a sequence have the same finite limit then the sequence converges?

Solutions:

(a) Since the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ both converge to L, we know that for all $\epsilon > 0$, $\exists\, N_1$ such that $\forall\, n > N_1$ we have $|a_{2n} - L| < \epsilon$. Also, $\exists\, N_2$ such that $\forall\, n > N_2$ we have $|a_{2n+1} - L| < \epsilon$. Now let $N = \max\{2N_1,\, 2N_2 + 1\}$. Now for $n > N$, we have $n = 2k$ or $n = 2k+1$, for some $k \in \mathbb{N}$. If $n = 2k$, we have $|a_{2k} - L| < \epsilon$ since $k > N_1$. If $n = 2k+1$, we have $|a_{2k+1} - L| < \epsilon$ since $k > N_2$. Thus, $|a_n - L| < \epsilon$ for all $n > N$. Therefore, the sequence $\{a_n\}$ converges to L.$\square$

(b) This statement is false. Consider the sequence $\{a_n\} = \{1, 1, 2, 1, 1, 2, \ldots\}$. Then the subsequences $\{a_{3n-2}\}$ and $\{a_{3n-1}\}$ converge to 1, but the sequence diverges.

4. A sequence is said to be contractive if there is such a number $1 > k > 0$ that

$$|a_{n+2} - a_{n+1}| < k|a_{n+1} - a_n|.$$

Prove that every contractive sequence is a Cauchy sequence.

Solution:
First, we have

$$|a_{n+2} - a_{n+1}| < k|a_{n+1} - a_n| < k^2|a_n - a_{n-1}| < \cdots < k^n|a_2 - a_1|.$$

So we can choose N such that $k^N < \frac{\epsilon}{(m-n)|a_2-a_1|}$. Then for all $m > n > N$ we have

$$\begin{aligned}
|a_m - a_n| &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\
&< k^{m-2}|a_2 - a_1| + k^{m-1}|a_2 - a_1| + \cdots + k^{n-1}|a_2 - a_1| \\
&\leq k^N|a_2 - a_1| + k^N|a_2 - a_1| + \cdots + k^N|a_2 - a_1| \\
&= k^N(m-n)|a_2 - a_1| < \epsilon.
\end{aligned}$$

Hence the sequence is Cauchy.□

5. Show that limsup $(a_n + b_n) \leq$ limsup a_n + limsup b_n for any bounded sequences.

Solution:

We show two ways to prove this statement:

(a) For all $n \in \mathbb{N}$, we have $a_k \leq \sup\{a_j \mid j \geq n\}, \forall\, k \geq n$. Similarly, $b_k \leq \sup\{b_j \mid j \geq n\}, \forall\, k \geq n$. Thus,

$$a_k + b_k \leq \sup\{a_j \mid j \geq n\} + \sup\{b_j \mid j \geq n\}, \forall\, k \geq n.$$

Thus $\sup\{a_j \mid j \geq n\} + \sup\{b_j \mid j \geq n\}$ is an upper bound for the set $\{a_k + b_k \mid k \geq n\}$. So

$$\sup\{a_k + b_k \mid k \geq n\} \leq \sup\{a_j \mid j \geq n\} + \sup\{b_j \mid j \geq n\}$$

$$\Rightarrow \lim_{n\to\infty} \sup\{a_k + b_k \mid k \geq n\} \leq \lim_{n\to\infty} (\sup\{a_j \mid j \geq n\} + \sup\{b_j \mid j \geq n\})$$

$$\Rightarrow \text{limsup}\,(a_n + b_n) \leq \text{limsup}\, a_n + \text{limsup}\, b_n. \square$$

(b) Let limsup $a_n = A$ and limsup $b_n = B$. Then for all $\epsilon > 0$, $\exists\, N_1$ such that for all $n > N_1$ we have $a_n \leq A + \epsilon$. Similarly, $\exists\, N_2$ such that for all $n > N_2$ we have $b_n \leq B + \epsilon$. Now let $N = \max\{N_1, N_2\}$. Then for all $n > N$ we have $a_n + b_n \leq A + B + 2\epsilon$. Since this is true for all ϵ, the result follows.□

6. True or False?

(a) a bounded sequence is convergent if and only if it is monotone
(b) every sequence has a convergent subsequence
(c) every cluster point is a limit point
(d) every limit point is a cluster point
(e) every bounded and divergent sequence has at least two cluster points
(f) limit superior is always greater that limit inferior
(g) every contractive sequence of real numbers converges

Solutions:

(a) False – The sequence $\{\frac{(-1)^n}{n}\}$ is bounded and convergent, but not monotone.
(b) False – The sequence $\{n\}$ is a counterexample.
(c) False – Consider the sequence $\{(-1)^n\}$. Then 1 is a cluster point but not a limit point.
(d) True
(e) True – if the sequence is bounded and divergent, then it must oscillate between at least two different values as $n \to \infty$.
(f) False –they may be equal
(g) True – from question 4

7. **Extra Points Problem**

Give an example of a sequence which has a countably infinite collection of cluster points.

Solution:

One possible sequence is $\{a_n\} = \{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots\}$.