Math 3001 **Due Fri Nov 25**

Assignment #9

1. Prove Abel's theorem:

   Let $\sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R = 1$, and let $\sum_{n=0}^{\infty} a_n$ be convergent. Then

$$\lim_{x \to 1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n.$$

   Proof. If $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R = 1$, and $\displaystyle\sum_{n=0}^{\infty} a_n$ converges then $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$. Thus, $f(x)$ is continuous on $[0, 1]$. Therefore,

$$\lim_{x \to 1^-} f(x) = f(1) = \sum_{n=0}^{\infty} a_n$$

   □

2. Use the statement obtained in Problem 1 and Cauchy's theorem about multiplication of two absolutely convergent series to show that

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \left(\sum_{n=0}^{\infty} c_n \right), \qquad c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0,$$

   if all there series converge (not necessarily absolutely).

   P.S. This statement was published by Abel in 1826.

   Proof. Let $A(x) = \sum a_n x^n$, $B(x) = \sum b_n x^n$, and $C(x) = \sum c_n x^n$. Since $\sum a_n$, $\sum b_n$, and $\sum c_n$ are all convergent, the series $A(x)$, $B(x)$, and $C(x)$ are all absolutely convergent for $|x| < 1$. So, by Cauchy's Theorem, we have

$$A(x) \cdot B(x) = C(x) \text{ on } (0, 1).$$

   Hence, by Abel's Theorem,

$$\lim_{x \to 1^-} A(x) \cdot \lim_{x \to 1^-} B(x) = \lim_{x \to 1^-} C(x).$$

Therefore, we have

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \left(\sum_{n=0}^{\infty} c_n\right), \quad c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0,$$

□

3. A) For which values of $x \in\mathbf{R}$ the sequence

$$S_n = |\sum_{k=1}^{n} \cos(kx)|$$

is bounded?

Solution:

We use the fact that $2\cos a \sin b = \sin(a+b) - \sin(a-b)$. So

$$\begin{aligned}
2\sin(x/2)\sum_{k=1}^{n}\cos(kx) &= \sum_{k=1}^{n}\Big(\sin\big((k+1/2)x\big) - \sin\big((k-1/2)x\big)\Big) \\
&= \sin\left(\frac{2n+1}{2}x\right) - \sin\left(\frac{x}{2}\right) \text{ (telescoping series)} \\
&= 2\cos\left(\frac{n+1}{2}x\right)\sin\left(\frac{nx}{2}\right), \\
&\quad \text{since } \sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)
\end{aligned}$$

Thus we have

$$\left|\sum_{k=1}^{n}\cos(kx)\right| < \frac{1}{|\sin(x/2)|}, \text{ where } x \neq 2m\pi,\ m \in \mathbb{Z}$$

B) Let sequence $\{f_n\}_{n=1}^{\infty}$ be monotone and $\lim_{n\to\infty} f_n = 0$. For which values of $x \in\mathbf{R}$ the trigonometric series $\sum_{n=1}^{\infty} f_n \cos(nx)$ converges?

Solution:

By Dirichlet's Theorem, if f_n is monotone and convergent to 0 and $\left|\sum_{n=1}^{N} a_n\right| < K$ for all N, then $\sum_{n=1}^{\infty} a_n f_n$ converges. In $3a$, we have shown that $|\sum \cos(nx)|$ is bounded for $x \neq 2m\pi$ for any $m \in \mathbb{Z}$. Thus, $\sum_{n=1}^{\infty} f_n \cos(nx)$ converges for all $x \neq 2m\pi$.

4. A) Explain why series

$$\sum_{n=1}^{\infty} \frac{-(-1)^n \sin(nx)}{n}$$

converges uniformly on $(\pi + \delta, \pi - \delta)$ for any $0 < \delta < \pi$.

Solution:

By Dirichlet-Abel-Hardy Theorem, we know that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ converges uniformly on $[\delta, 2\pi - \delta]$ since $|\sum \sin(nx)| < \csc(x/2)$. Also, since $\csc(x/2)$ is unbounded as $x \to 0^+$ or $x \to 2\pi^-$, we need $\delta > 0$ for uniformly bounded. Now if we change $x \to x + \pi$ then we get

$$\begin{aligned} \sin(nx) &\to \sin(nx + n\pi) \\ &= \sin(nx)\cos(n\pi) + \sin(n\pi)\cos(nx) \\ &= (-1)^n \sin(nx) \end{aligned}$$

Therefore, we have

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \text{ converges on } (\delta, 2\pi - \delta) \xrightarrow{x \to x + \pi} \sum_{n=1}^{\infty} (-1)^n \frac{\sin(nx)}{n} \text{ converges on } (-\pi + \delta, \pi - \delta).$$

This implies that $\displaystyle\sum_{n=1}^{\infty} -(-1)^n \frac{\sin(nx)}{n}$ converges on $(-\pi + \delta, \pi - \delta)$ as well.

B) Show that the series in (A) is the Fourier series for function $F(x) = x/2$ on $(-\pi, \pi)$.

Solution:

The Fourier series of a function is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Since $f(x) = \dfrac{x}{2}$ is an odd function we have $a_0 = a_n = 0$. So we now need to find b_n.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x\sin(nx)}{2}\,dx \\ &= \frac{1}{2\pi}\left(\frac{-x\cos(nx)}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos(nx)\,dx \right) \\ &= \frac{(-1)^{n+1}}{n} \end{aligned}$$

Therefore, we have

$$f(x) = \frac{x}{2} = \sum_{n=1}^{\infty} \frac{-(-1)^n \sin(nx)}{n}$$

C) Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sin(5\pi n/4)}{n}$$

Solution:
Since $x = \dfrac{5\pi}{4} \notin (-\pi, \pi)$ and $\sin x$ has a period of 2π, we need to evaluate the series at $x = \dfrac{-3\pi}{4} = \dfrac{5\pi}{4} - 2\pi$. Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(5\pi n/4)}{n} &= \sum_{n=1}^{\infty} \frac{(-1)^n \sin(-3\pi n/4)}{n} \\ &= \frac{-x}{2}\Big|^{x=\frac{-3\pi}{4}} = \frac{3\pi}{8} \end{aligned}$$

5. Give an example of a function for which corresponded to it Fourier series has only finite number of terms.

   Solution:
   Some examples: any constant function, $\sin(x)$, $\cos(x)$, etc.

6. EXTRA POINTS We have seen that the function defined as $F(x) = e^{-x^{-2}}$ for $x \neq 0$ and $F(x) = 0$ for $x = 0$ is not equal to its Taylor series centered at zero for all $x \neq 0$.

   Consider now Taylor series centered at $a \neq 0$ for this function. Can you use it to evaluate $F(0)$?