Math 3001 **Due Wed Nov 16**

Assignment #8

1. Prove the statement

A) Let $f_n(x)$ be continuous on $[a,b]$ and the series $\sum_{n=1}^{\infty} f_n(x)$ converge uniformly to $f(x)$ on $[a,b]$. Then $\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$.

Proof. Let $S_n(x) = \displaystyle\sum_{i=1}^{n} f_i(x)$. Then we have

$$S_n(x) \xrightarrow{\text{uniform}} f(x) = \sum_{n=1}^{\infty} f_n(x) \text{ on } [a,b].$$

Since $f_n(x)$ is continuous on $[a,b]$ for all n we have $S_n(x)$ and $f(x)$ is continuous on $[a,b]$ for all n. Hence, $S_n(x)$ and $f(x)$ is integrable on $[a,b]$ for all n. Thus, by uniform convergence of sequences, we have

$$\begin{aligned} \int_a^b f(x) &= \lim_{n\to\infty} \int_a^b S_n(x) \\ &= \lim_{n\to\infty} \int_a^b \sum_{i=1}^{n} f_i(x) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \int_a^b f_i(x) \\ &= \sum_{n=1}^{\infty} \int_a^b f_n(x) \end{aligned}$$

□

B) Let $\sum_{n=1}^{\infty} f_n(x)$ be a series of functions that converges to $f(x)$ on $[a,b]$. Suppose that $f_n'(x)$ exists and is continuous on $[a,b]$ for all $n = 0, 1, ...,$ and the series $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to $g(x)$ on $[a,b]$. Then $g(x) = f'(x)$ on $[a,b]$.

Proof. Let $S_n(x) = \displaystyle\sum_{i=1}^{n} f_i(x)$. Then we have

$$S_n(x) \xrightarrow{\text{uniform}} f(x) = \sum_{n=1}^{\infty} f_n(x) \text{ on } [a,b].$$

We are also given that $S_n'(x)$ exists and is continuous on $[a,b]$ for all n. Thus we have

$$S_n'(x) \xrightarrow{\text{uniform}} g(x) = \sum_{n=1}^{\infty} f_n'(x) \text{ on } [a,b].$$

And by uniform convergence of sequences we have $f'(x) = g(x)$. $\square$

2. (a) Show that

$$\int_0^x \ln(1+t)\,dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n+1}}{n(n+1)}, \quad |x| < 1.$$

Solution:
Let $t, x \in (-1,1)$. Consider the following equations:

$$\begin{aligned}
\frac{1}{1-t} &= 1 + t + t^2 + t^3 + \cdots \\
\Rightarrow \frac{1}{1+t} &= 1 - t + t^2 - t^3 + \cdots \\
\Rightarrow \int \frac{1}{1+t}\,dt &= \int 1\,dt - \int t\,dt + \int t^2\,dt - \int t^3\,dt + \cdots \\
\Rightarrow \ln(1+t) &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \\
\Rightarrow \int_0^x \ln(1+t)\,dt &= \int_0^x t\,dt - \int_0^x \frac{t^2}{2}\,dt + \int_0^x \frac{t^3}{3}\,dt - \int_0^x \frac{t^4}{4}\,dt + \cdots \\
&= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n+1}}{n(n+1)}, \quad |x| < 1.
\end{aligned}$$

(b) Does (a) hold for $|x| = 1$?

Solution:
When $x = 1$ we have the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$ which converges by A.S.T.

When $x = -1$ we have the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ which converges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. So (a) holds for $|x| = 1$.

(c) Show that

$$\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \cdots = \ln 4 - 1.$$

Solution:
We have, from (a), that

$$\begin{aligned} \int_0^x \ln(1+t)\,dt &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n(n+1)} \\ \Rightarrow (1+x)\ln(1+x) - (1+x) + 1 &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n(n+1)} \end{aligned}$$

So when we let $x = 1$ we have

$$\ln 4 - 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}.$$

3. Find the Taylor series at $x_0 = 0$ for

(a) $x \sin(3x^2)$;

Solution:
We have

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ \Rightarrow \sin(3x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n+1}}{(2n+1)!} \\ \Rightarrow x\sin(3x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{4n+3}}{(2n+1)!} \end{aligned}$$

(b) $\int_0^x e^{-t^2}\,dt$.

Solution:
We have

$$\begin{aligned} e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ \Rightarrow e^{-t^2} &= \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} \\ \Rightarrow \int_0^x e^{-t^2}\,dt &= \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{n!}\,dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \end{aligned}$$

4. Consider the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}.$$

(a) Find the radius of convergence;

Solution:
Let $a_n = \dfrac{1}{4^n (n!)^2}$. Then we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{4(n+1)^2} = 0$$

Therefore, $R = \infty$.

(b) show that $y(x) = J_0(x)$ is a solution of the differential equation

$$xy'' + y' + xy = 0.$$

Solution:
We have

$$\begin{aligned} xy'' + y' + xy &= \sum_{n=0}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n-1}}{4^n (n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{4^n (n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{4^n (n!)^2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 4n^2 x^{2n-1}}{4^n (n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{4^{n-1}((n-1)!)^2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 4n^2 x^{2n-1}}{4^n (n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4n^2 x^{2n-1}}{4^n (n!)^2} = 0 \end{aligned}$$

5. Find the function given by the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+2}.$$

Solution:
We know that

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Thus, we have

$$\begin{aligned} \frac{1}{2}\ln(1+x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+2} \\ \Rightarrow \frac{\ln(1+x^2)-1}{2} &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+2} \end{aligned}$$

6. EXTRA POINTS

Find the limit for each $x \in \mathbf{R}$.

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \cos\left(\frac{kx}{n}\right)}{n}$$

Is the convergence uniform on $\mathbf{R}$?

Solution:
Consider the function $f(t) = \dfrac{\cos(t)}{x}$ on interval $[0, x]$, where $x \in \mathbb{R}$.

Let $P = \{ x_0, x_1, \ldots, x_n \}$ be a partition of the interval $[0, x]$. Then

$\Delta x_k = \dfrac{x}{n}$ and $c_k = \dfrac{xk}{n}$, where c_k is the right endpoint of the k^{th}

subinterval of P. Thus,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{\cos(\frac{kx}{n})}{x} \right) \frac{x}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\cos(\frac{kx}{n})}{n} \\
&= \int_0^x \frac{\cos(t)}{x}\, dt = \frac{\sin(x)}{x}
\end{aligned}$$

This convergence is not uniform on $\mathbb{R}$ since $\dfrac{\sum_{k=1}^{n} \cos\left(\frac{kx}{n}\right)}{n}$ is continuous for all n, k, and x, but $\dfrac{\sin(x)}{x}$ is discontinuous at $x = 0$.