Math 3001 **Due Wed Nov 16** **Assignment #8**

1. Prove the statement

   A) Let $f_n(x)$ be continuous on $[a,b]$ and the series $\sum_{n=1}^{\infty} f_n(x)$ converge uniformly to $f(x)$ on $[a,b]$. Then $\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$.

   B) Let $\sum_{n=1}^{\infty} f_n(x)$ be a series of functions that converges to $f(x)$ on $[a,b]$. Suppose that $f_n'(x)$ exists and is continuous on $[a,b]$ for all $n = 0, 1, ...$, and the series $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to $g(x)$ on $[a,b]$. Then $g(x) = f'(x)$ on $[a,b]$.

2. (a) Show that

   $$\int_0^x \ln(1+t)\,dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n+1}}{n(n+1)}, \quad |x| < 1.$$

   (b) Does (a) hold for $|x| = 1$?

   (c) Show that

   $$\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} + \cdots = \ln 4 - 1.$$

3. Find the Taylor series at $x_0 = 0$ for

   (a) $x\sin(3x^2)$;

   (b) $\int_0^x e^{-t^2}\,dt$.

4. Consider the Bessel function

   $$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}.$$

   (a) Find the radius of convergence;

   (b) show that $y(x) = J_0(x)$ is a solution of the differential equation

   $$xy'' + y' + xy = 0.$$

5. Find the function given by the series

   $$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+2}.$$

6. EXTRA POINTS

   Find the limit for each $x \in \mathbf{R}$.

   $$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \cos\left(\frac{kx}{n}\right)}{n}$$

   Is the convergence uniform on $\mathbf{R}$?