**Math 3001** **Due Wed Oct 19**

**Assignment #5**

1. Given the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, suppose that there exists a number $N$ such that $a_n = b_n$ for all $n > N$. Prove that $\sum_{n=1}^{\infty} a_n$ is convergent iff $\sum_{n=1}^{\infty} b_n$ is convergent.

   Is it true that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$?

   **Solution**

   We have

$$\begin{aligned}\sum_{i=1}^{\infty} a_n &= \sum_{i=1}^{N} a_n + \sum_{i=N+1}^{\infty} a_n \\ &= \sum_{i=1}^{N} a_n + \sum_{i=N+1}^{\infty} b_n\end{aligned}$$

   So clearly, $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent iff $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent.

   Also, it is not true that $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$ if $\displaystyle\sum_{n=1}^{N} a_n \neq \sum_{n=1}^{N} b_n$. $\square$

2. Determine whether or not the series $\sum_{n=1}^{\infty}(\sqrt{n+1}+\sqrt{n})^{-1}$ is convergent? Justify your answer.

   **Solution**

   We have $\sum_{n=1}^{\infty}(\sqrt{n+1}+\sqrt{n})^{-1} = \sum_{n=1}^{\infty} \sqrt{n+1}-\sqrt{n}$. Let $S_n$ be the $n^{th}$ partial sum of the series. Then we have

$$S_n = (\sqrt{2}-1)+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+\cdots+(\sqrt{n+1}-\sqrt{n}) = \sqrt{n+1}-1,$$

   which diverges as $n \to \infty$. Thus $\sum_{n=1}^{\infty}(\sqrt{n+1}+\sqrt{n})^{-1}$ diverges.

3. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $y_n = x_n - x_{n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} y_n$ is convergent iff the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent.

   If the series $\sum_{n=1}^{\infty} y_n$ is convergent, what is the sum?

   **Solution**

   Let $S_n$ be the $n^{th}$ partial sum of $\displaystyle\sum_{i=1}^{\infty} y_n$. Then

$$S_n = (x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) + \cdots + (x_n - x_{n+1}) = x_1 - x_{n+1}.$$

Thus we have $\lim_{n\to\infty} S_n = x_1 - \lim_{n\to\infty} x_{n+1}$. Since $\sum_{i=1}^{\infty} y_n$ converges iff $\lim_{n\to\infty} S_n$ exists, we can see that $\sum_{i=1}^{\infty} y_n$ converges iff $\{x_n\}_{n=1}^{\infty}$ converges.

Also, if $\sum_{i=1}^{\infty} y_n$ is convergent, the sum, $S$, is $S = x_1 - x$,

where $x = \lim_{n\to\infty} x_{n+1}$. $\square$

4. Find an example to show that the convergence of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ not necessarily imply convergence of $\sum_{n=1}^{\infty} a_n b_n$.

   **Solution**
   Let $a_n = \dfrac{(-1)^n}{\sqrt{n}} = b_n$. Then $\sum a_n$ and $\sum b_n$ converges, but $\sum a_n b_n = \sum \dfrac{1}{n}$ diverges.

5. Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges and $\{b_n\}_{n=1}^{\infty}$ is a bounded sequence then $\sum_{n=1}^{\infty} a_n b_n$ converges.

   **Solution**
   Since $\{b_n\}_{n=1}^{\infty}$ is bounded, we have $b_n \leq K$ for all $n$ and for some $K \in \mathbb{R}$. Also, since $\sum_{n=1}^{\infty} |a_n|$ converges, by Cauchy Criterion, given $\epsilon > 0$, there exist $N$ such that for all $n > m > N$ we have

   $$|S_n - S_m| = ||a_n| - |a_{n-1}| + \cdots + |a_{m+1}|| < \frac{\epsilon}{K},$$

   where $S_n$ is the $n^{th}$ partial sum of $\sum_{n=1}^{\infty} |a_n|$. To show $\sum_{n=1}^{\infty} a_n b_n$ converges, we will use Cauchy Criterion again. For same $N$, and for all $m, n > N$, we have

$$\begin{aligned}
|a_n b_n - a_m b_m| &= |a_n b_n + a_{n-1}b_{n-1} + \cdots + a_{m+1}b_{m+1}| \\
&\leq |a_n b_n| + |a_{n-1}b_{n-1}| + \cdots |a_{m+1}b_{m+1}| \\
&\leq K\left(|a_n| + |a_{n-1}| + \cdots |a_{m+1}|\right) \\
&\leq K\left(||a_n| + |a_{n-1}| + \cdots |a_{m+1}||\right) \\
&< K\tfrac{\epsilon}{K} = \epsilon.
\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} a_n b_n$ converges.□

6. a)Show by example that *grouping of terms* may change a divergent series to convergent.

   b)Is (a) possible for a divergent series with all nonnegative terms?

   c)Is it possible to change the sum of a convergent series by grouping of terms?

   **Solutions**

   (a) Consider series $\sum_{n=0}^{\infty}(-1)^n$. This series is divergent, but we can group the terms to make it convergent. We can take a string

   $$1 - 1 + 1 - 1 + 1 - 1 \cdots$$

   and group them thus

   $$(1-1) + (1-1) + (1-1) + \cdots = 0 + 0 + 0 + \cdots$$

   (b) No

   (c) No

7. Show that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} + \cdots$$

is divergent. Why doesn't this contradict the Alternating Series Test?

**Solution**

Consider the $2n^{th}$ partial sum, $S_{2n}$. Then we have

$$\begin{aligned} S_{2n} &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + \tfrac{1}{2n-1} - \tfrac{1}{2^n} \\ &= 1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots + \tfrac{1}{2n-1} - \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots + \tfrac{1}{2^n}\right) \\ &= \sum_{i=0}^{n} \frac{1}{2i+1} - \frac{1}{2}\sum_{i=0}^{n} \frac{1}{2^n} \end{aligned}$$

Since $\displaystyle\sum_{i=0}^{n} \frac{1}{2i+1}$ diverges as $n \to \infty$ and $\sum_{i=0}^{n} \frac{1}{2^n}$ converges as $n \to \infty$, we have $S_{2n}$ diverges as $n \to \infty$. Therefore, the series diverges.

Note that this does not contradict the Alternating series test since the absolute value of the terms are not monotonically decreasing.

8. Prove that if a series is conditionally convergent, then the series of its negative terms is divergent.

**Solution**

Let $\sum a_n$ be a conditionally convergent series. Let $q_n = \dfrac{|a_n| - a_n}{2}$. Then the set $\{-q_n\}$ contains all the negative terms, but no positive terms, of $\sum a_n$. Assume that $\sum q_n$ converges. Since $|a_n| = 2q_n + a_n$ and $\sum a_n$ converges as well, we have $\sum |a_n| = 2\sum q_n + \sum a_n$ converges, which is a contradiction. Therefore, $\sum q_n$ diverges. $\square$

9. Suppose that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent series, and $s$ is a real number.

   a)Explain why there exists a *rearrangement* of $\sum_{n=1}^{\infty} a_n$ that converges conditionally to $s$.

   b)Is there a rearrangement of $\sum_{n=1}^{\infty} a_n$ that diverges?

**Solutions**

(a) The idea is to take first the positive terms until the sum exceeds $s$ (which is possible since the series with positive terms diverges). Then, we take the negative terms until we are below $s$ (which happens since the series of negative terms diverges). Then we go on adding positive terms until $s$ is again exceeded, and so on. In this way, we obtain a rearranged series that converges to $s$.

(b) Yes

10. EXTRA POINTS

Let $a_n > 0$ for all $n \geq 1$. Let $b_n = (a_1 + a_2 + \cdots + a_n)/n$. Is the series $\sum_{n=1}^{\infty} b_n$ converegnt or diveregnt? Explain.

**Solution**

Divergent by direct comparison. We have $b_n \geq \dfrac{a_1}{n}$ for all $n$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{a_1}{n}$ diverges, we have $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.