Math 3001 **Due Fri Oct 7**

Assignment #4

1. Let f, g be integrable on $[a,b]$. Introduce notations

$$||f|| = (\int_a^b f^2(x)dx)^{1/2}, \quad (f,g) = \int_a^b f(x)g(x)dx$$

a) Prove Cauchy-Swartz inequality $|(f,g)| \leq ||f|| \cdot ||g||$.

Solution

Let $\lambda \in \mathbb{R}$. Consider

$$\begin{aligned} 0 \leq \int_a^b (f+\lambda g)^2\,dx &= \int_a^b f^2\,dx + 2\lambda \int_a^b fg\,dx + \lambda^2 \int_a^b g^2\,dx \\ &= ||f||^2 + 2\lambda(f,g) + \lambda^2 ||g||^2 \end{aligned}$$

If we consider the last line as a function in λ then we know that the discriminant, $b^2 - 4ac \leq 0$. Thus,

$$\begin{aligned} [2(f,g)]^2 - 4||f||^2 \cdot ||g||^2 &\leq 0 \\ \Rightarrow 4\left[(f,g)\right]^2 &\leq 4||f||^2 \cdot ||g||^2 \\ |(f,g)| &\leq ||f|| \cdot ||g|| \quad \square \end{aligned}$$

b) Show that Cauchy-Swartz inequality implies the *triangle inequality*

$$||f+g|| \leq ||f|| + ||g||.$$

Solution

$$\begin{aligned} ||f+g||^2 &= \int_a^b (f+g)^2\,dx \\ &= \int_a^b f^2\,dx + 2\int_a^b fg\,dx + \int_a^b g^2\,dx \\ &= ||f||^2 + 2(f,g) + ||g||^2 \\ &\leq ||f||^2 + 2||f|| \cdot ||g|| + ||g||^2 \\ &= (||f|| + ||g||)^2 \\ \Rightarrow ||f+g|| &\leq ||f|| + ||g|| \quad \square \end{aligned}$$

2. Find the second derivative $F''(x)$

   a) $F(x) = \int_0^{\sin x} \cos(t^2)dt$

   Solution

$$F'(x) = \cos(\sin^2(x))\cos(x)$$
$$F''(x) = -\sin(x)\cos(\sin^2(x)) - 2\sin(x)\cos^2(x)\sin(\sin^2(x))$$

   b) $F(x) = \int_{-x}^{x^2} \sqrt{1+t^2}dt$

   Solution

$$F(x) = -\int_c^{-x} \sqrt{1+t^2}\,dt + \int_c^{x^2} \sqrt{1+t^2}\,dt$$
$$F'(x) = \sqrt{1+x^2} + 2x\sqrt{1+x^4}$$
$$F''(x) = \frac{x}{\sqrt{1+x^2}} + 2\sqrt{1+x^4} + \frac{4x^4}{\sqrt{1+x^4}}$$

   c) $F(x) = \int_0^x xe^{t^2}dt$

   Solution

$$F(x) = x\int_0^x e^{t^2}\,dt$$
$$F'(x) = \int_0^x e^{t^2}\,dt + xe^{x^2}$$
$$F''(x) = 2e^{x^2}(1+x^2)$$

3. Evaluate $\lim_{x\to 0}(x^{-1}\int_0^x \sqrt{9+t^2}dt)$

   Solution

$$\lim_{x\to 0}\frac{\int_0^x \sqrt{9+t^2}}{x} \overset{L'H}{=} \lim_{x\to 0}\sqrt{9+x^2} = 3$$

4. Let f be continuous on $[a,b]$. Suppose $\int_a^x f(t)dt = \int_x^b f(t)dt$ for all $x \in [a,b]$. Find function f.

Solution
We have $\int_a^x f(t)dt = -\int_b^x f(t)dt$ for all $x \in [a,b]$. Therefore, when we differentiate both sides we get

$$f(x) = -f(x) \Rightarrow f(x) = 0.$$

5. Let f be continuous on $[0,\infty)$. Let $f(x) \neq 0$ for $x > 0$ and $f^2(x) = 2\int_0^x f(t)dt$. Find function f.

   Solution
   Differentiate both sides to get

   $$2f(x)f'(x) = 2f(x) \Rightarrow f'(x) = 1 \Rightarrow f(x) = x + c \text{ for some } c$$

   We now solve for c. We have

   $$f^2(x) = x^2+2cx+c^2 = 2\int_0^x (t+c)\,dt = \left[t^2 + 2ct\right]_0^x = x^2+2cx \Rightarrow c = 0.$$

   Therefore, $f(x) = x$.

6. Let $I_n = \int_0^\infty x^{-n}dx$. For which real values n the integral I_n is convergent?

   Hint: consider separately $\int_0^1 x^{-n}dx$ and $\int_1^\infty x^{-n}dx$

   Solution
   We have to consider the three cases $n = 1, n < 1$, and $n > 1$.

   If $n < 1$, then $\int_1^\infty x^{-n}\,dx = \lim_{t\to\infty} [x^{1-n}/(1-n)]_1^t = \infty$. Therefore, $\int_0^\infty x^{-n}\,dx$ diverges.

   If $n > 1$, then $\int_0^1 x^{-n}\,dx = \lim_{c\to 0^+} [x^{1-n}/(1-n)]_c^1 = \infty$. Therefore, $\int_0^\infty x^{-n}\,dx$ diverges.

   If $n = 1$, then we have $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} [\ln x]_1^t = \infty$.

   Therefore, $\int_0^\infty x^{-n}\,dx$ diverges for all n.□

7. Is the following argument correct? Explain. $\int_{-L}^L \sin x dx = 0$ for any $L \geq 0$. Thus $\int_{-\infty}^\infty \sin x dx = 0$.

   Solution
   False. Consider $\int_0^\infty \sin(x)\,dx$. Then $\lim_{t\to\infty}\int_0^t \sin(x)\,dx = \lim_{t\to\infty} [-\cos(x)]_0^t$, which diverges since $\cos(t)$ oscillates between $-1, 1$ as $t \to \infty$. Therefore, $\int_{-\infty}^\infty \sin(x)\,dx$ diverges.

8. Prove the following statement:

   Let f be continuous on $[a, b]$ and g be continuous on $[c, d]$, where $f([a, b]) \subset [c, d]$. Then the composition $g \circ f$ is integrable on $[a, b]$.

   Solution
   Let $x \in [a, b]$. Then f is continuous at x. Since $f(x) \in [c, d]$ and g is continuous on $[c, d]$, we have g is continuous at $f(x)$. Therefore, $g \circ f$ is continuous at x. Since this is true for all $x \in [a, b]$, $g \circ f$ is continuous for all $x \in [a, b]$, hence it is integrable on $[a, b]$. $\square$

9. **Extra Points Problem**

   Prove the following statement:

   Let f be integrable on $[a, b]$ and g be continuous on $[c, d]$, where $f([a, b]) \subset [c, d]$. Then the composition $g \circ f$ is integrable on $[a, b]$.

   Proof: Given any $\epsilon > 0$, let $K = \sup\{|g(t)| : t \in [c, d]\}$ and choose $\epsilon' > 0$ such that $\epsilon'(b - a + 2K) < \epsilon$. Since g is continuous on $[c, d]$, it is uniformly continuous on $[c, d]$. Thus there exists a $\delta > 0$ such that $\delta < \epsilon'$ and such that $|g(s) - g(t)| < \epsilon'$ whenever $|s - t| < \delta$ and $s, t \in [c, d]$. Since f is integrable on $[a, b]$, there exists a partition $P = \{x_0, x_2 \ldots, x_n\}$ of $[a, b]$ such that

   $$U(f, P) - L(f, P) < \delta^2.$$

   We claim that for this partition we also have

   $$U(g \circ f, P) - L(g \circ f, P) = \sum_{i=1}^{n} [M_i(g \circ f) - m_i(g \circ f)] \Delta x_i < \epsilon.$$

   To show this, we separate the set of indices of the partition$P$ into two disjoint sets.

   $$A = \{i : M_i(f) - m_i(f) < \delta\} \text{ and } B = \{i : M_i(f) - m_i(f) \geq \delta\}.$$

   Then if $i \in A$ and $x, y \in [x_{i-1}, x_i]$, we have

   $$|f(x) - f(y)| \leq M_i(f) - m_i(f) < \delta,$$

   so that $|g \circ f(x) - g \circ f(y)| < \epsilon'$. But then $M_i(g \circ f) - m_i(g \circ f) \leq \epsilon'$. It follows that

   $$\sum_{i \in A} [M_i(g \circ f) - m_i(g \circ f)] \Delta x_i \leq \epsilon' \sum_{i \in A} \Delta x_i \leq \epsilon'(b - a).$$

On the other hand, if $i \in B$, then $[M_i(f) - m_i(f)]/\delta \geq 1$, so that

$$\begin{aligned}
\sum_{i \in B} \Delta x_i &\leq \frac{1}{\delta} \sum_{i \in B} [M_i(f) - m_i(f)] \Delta x_i \\
&\leq \frac{1}{\delta} [U(f, P) - L(f, P)] < \delta < \epsilon'.
\end{aligned}$$

Thus since $M_i(g \circ f) - m_i(g \circ f) \leq 2K$ for all i, we have

$$\sum_{i \in B} [M_i(g \circ f) - m_i(g \circ f)] \Delta x_i \leq 2K \sum_{i \in B} \Delta x_i < 2K\epsilon'.$$

Now when we combine all the indices we obtain

$$\begin{aligned}
U(g \circ f, P) - L(g \circ f, P) &= \textstyle\sum_{i \in A} [M_i(g \circ f) - m_i(g \circ f)] \Delta x_i \\
&+ \textstyle\sum_{i \in B} [M_i(g \circ f) - m_i(g \circ f)] \Delta x_i \\
&\leq \epsilon'(b - a) + 2K\epsilon' = \epsilon'(b - a + 2K) < \epsilon. \square
\end{aligned}$$