Math 3001 **Due Fri Sept 30** **Assignment #3**

1. Consider piece-wise constant function on $[0,1]$ defined by formula
$$F(x) = \frac{1}{2^n}, \quad \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, \quad n = 0, 1, 2, ... \quad F(0) = 0.$$
Explain why this function is integrable and find the integral $\int_0^1 F(x)dx$.

2. Show that if $f(x) \le g(x)$ for all $x \in [a,b]$, and are integrable functions, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

3. Let f be integrable on $[a,b]$ and suppose that $m \le f(x) \le M$ for all $x \in [a,b]$. Show that $m(b-a) \le \int_a^b f dx \le M(b-a)$.

4. Prove the mean value theorem for integrals: *It f is continuous on $[a,b]$ then there exists $c \in [a,b]$ such that*
$$f(c) = \frac{1}{b-a}\int_a^b f dx$$

5. Let f and g be continuous on $[a,b]$, and suppose that $\int_a^b f\,dx = \int_a^b g\,dx$. Prove that there exists a point $c \in [a,b]$ such that $f(c) = g(c)$.

6. Let f be integrable function on $[a,b]$ and let k be any real number. Show that kf is also integrable and that $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.

   Hint: consider cases $k = 0$, $k > 0$, $k < 0$.

7. Show if $a < c < b$ then for any integrable function f
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

8. Is it true or false that *it $|f|$ is integrable on $[a,b]$ then f must be integrable on $[a,b]$* ?

9. **Extra Points Problem**

   a) Consider function $f(x) = \sin(1/x)$ for $0 < x \le 1$ and $f(0) = 0$. Is that function integrable on $[0,1]$? Explain.

   b) Let f be bounded on $[a,b]$ and integrable on $[c,b]$ for any $c \in (a,b)$. Is such a function integrable on $[a,b]$?