Leonhard Euler Tercentennary — 2007

Dr. Margo Kondratieva and Andrew Stewart

Overview

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- Introduction
- Timeline

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- Polyhedral Problem

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- Basel Problem

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- Conclusion

Introduction

Some numbers about Euler

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- 76 years

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- 76 years
- 13 children

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- 76 years
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- 886 articles

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- $72 \cdot 600$

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- $72 \cdot 600 = 43200$ pages

Introduction

Opera Omnia

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- Pure Math (29 volumes)

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- Mechanics and Astronomy (31 volumes)

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- Pure Math (29 volumes)
- Mechanics and Astronomy (31 volumes)
- Physics and Miscellaneous (12 volumes)

Introduction

In Math

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- Analysis: 60%

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- Algebra: 7%

Introduction

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- Other: 3%

Introduction

Some of Euler's Topics

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- The base of the natural number, $2.719281828459\ldots$, and the symbol e to denote $2.719281828459\ldots$

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- Euler's Identity:

$$e^{i\pi} + 1 = 0$$

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- Euler's Identity:

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(Note that this last formula contains $e, i, \pi, 0$, and 1, which are five of the most important symbols in mathematics.)

Timeline

1707: Euler was born.

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Duties: Scientific consultant to the government, prepares maps, advises Russian Navy, tests designs for engines

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1727: Moves to St. Petersburg, Russia
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1736: First book — Mechanica
1741: Moves to Berlin

Timeline

1748: Introductio in Analysin

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1755: Books on Differential Calculus

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1755: Letters to Princess(over 300 in total)

Timeline

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1766: Returns to St. Petersburg

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1772: Treatise on the motion of the moon
1773: Katharina died

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1776: Remarries (his wife's sister)

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1783: Died

Timeline

Gottfried Leibniz (1646-1716)

Timeline

Jacob Bernoulli (1654-1705)

Timeline

Johann Bernoulli (1667-1748)

Timeline

Christian Goldbach (1690-1764)

Timeline

Daniel Bernoulli (1700-1782)

Timeline

Leonhard Euler

Timeline

Peter the Great (1672)-(1725)

Timeline

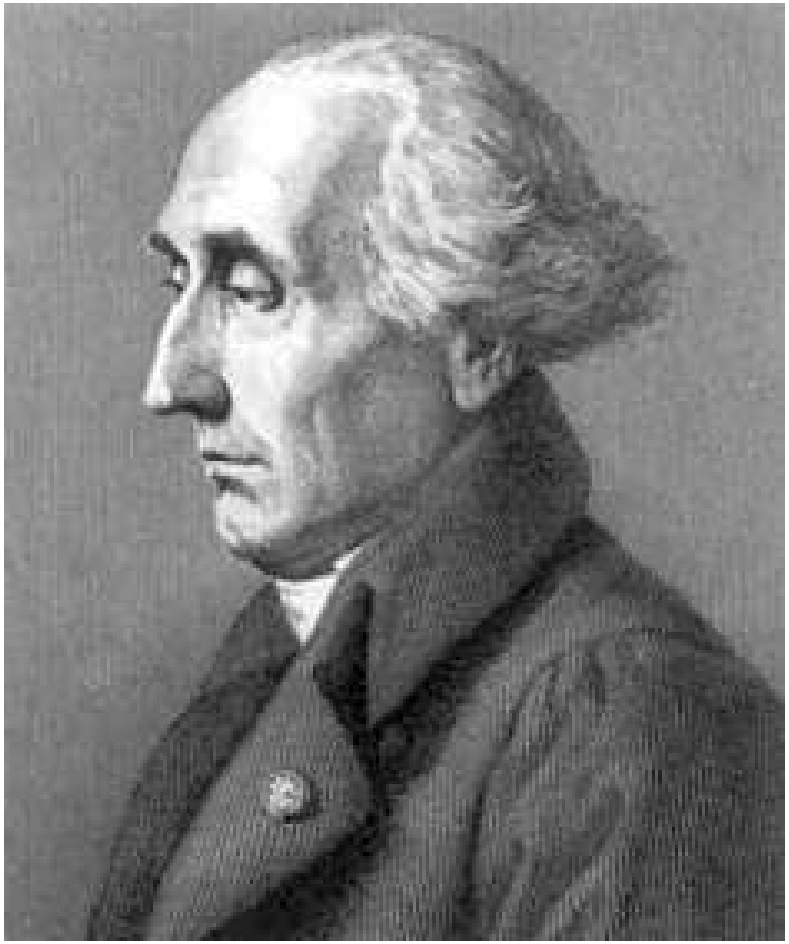
Catherine the Great (1729-1796)

Timeline

Frederick the Great (1712-1786)

Timeline

Joseph Louis Lagrange(1736-1813)



Timeline

St. Petersburg Academy of Science(1724)

Euler's Formula for Polyhedra

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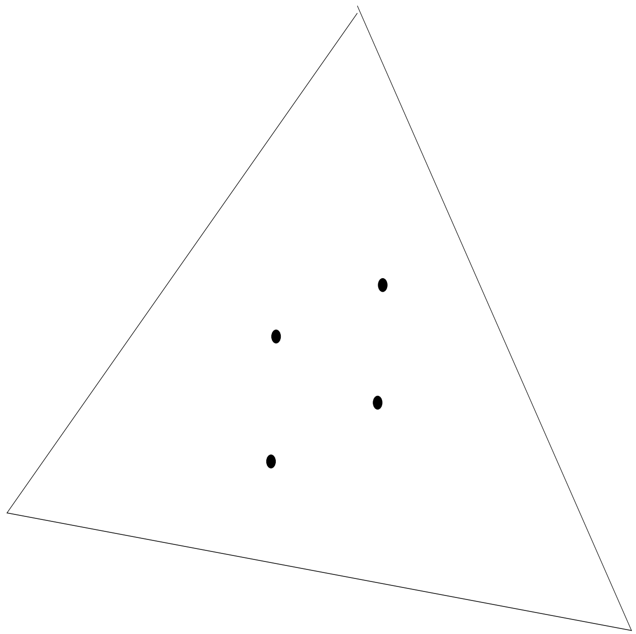
- Let V be the number of vertices.
- Let E be the number of edges.
- Let F be the number of faces.

Then we have

$$V - E + F = 2.$$

Euler's Formula for Polyhedra

Auxilary Problem: Suppose we have n points inside a triangle that are connected to form triangles (triangulation). How many triangles are there?



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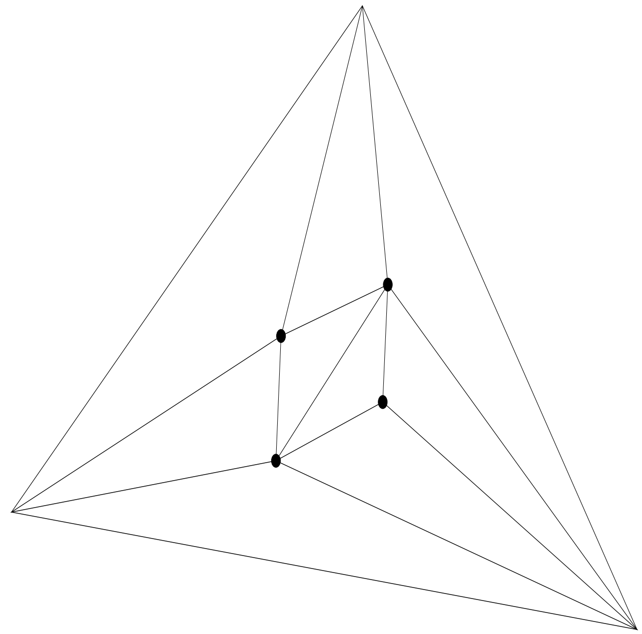

Euler's Formula for Polyhedra

Suppose there are N interior triangles. We will find N.

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a

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- Sum the angles of the interior triangles in two ways.

---

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Euler's Formula for Polyhedra

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- Sum the angles of the interior triangles in two ways.
- First way: N triangles; sum of angles is[a] $N\pi$.

[a]Remember that π radians is the same as 180°.

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- Second way: Interior points + exterior triangle;

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Thus, if we have n interior points, we will have $2n+1$ triangles.

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\begin{aligned}
V &= 3+n \\
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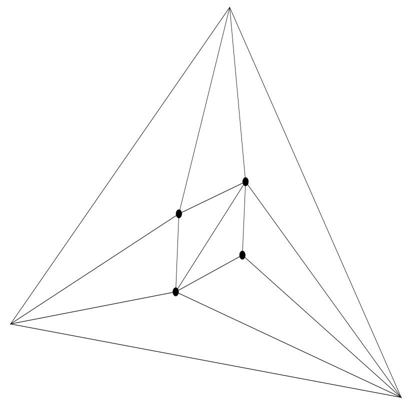
E &= \frac{3F}{2} = 3(n+1)
\end{aligned}
$$



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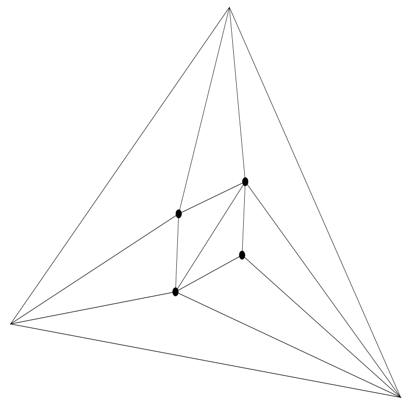

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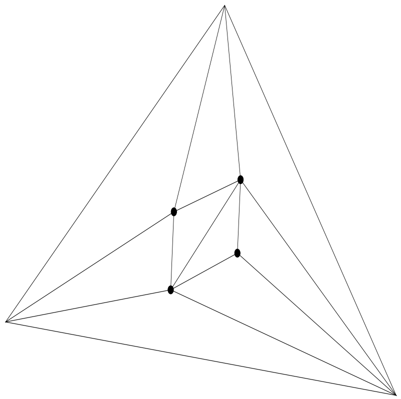

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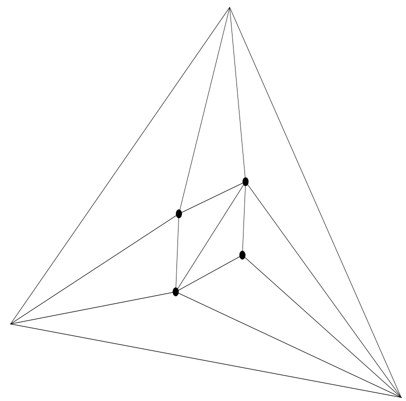

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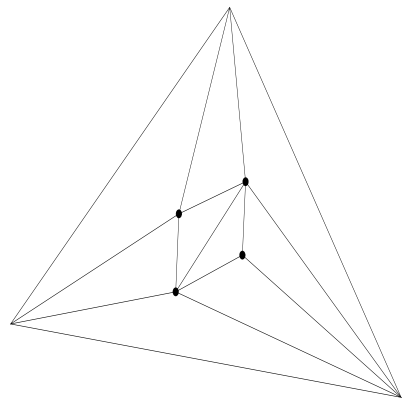

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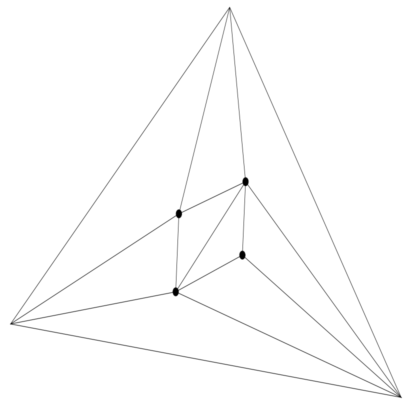

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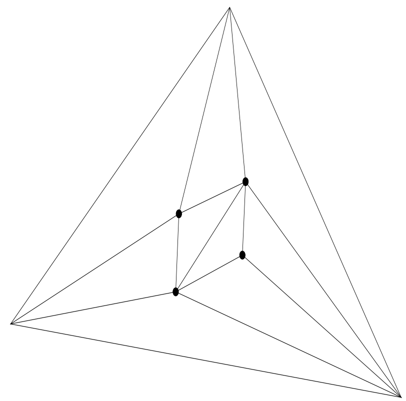

$$\begin{aligned} V &= 3 + n \\ F &= N + 1 = 2(n+1) \\ E &= \frac{3F}{2} = 3(n+1) \end{aligned}$$

Then we see that

$$V - E + F = (3+n) - (3n+3) + (2n+2) = 2,$$

which is Euler's formula!

Euler’s Formula for Polyhedra

We have proven Euler’s Formula for polyhedra with only triangular faces. We can prove it in general, but we must recall a useful fact.

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Theorem: *For n greater than 3, the sum of the interior angles of a polygon with n sides will be*

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For example, for a triangle, $n=3$, and the sum of the angles is $(3-1)\pi = \pi$.

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For example, for a triangle, $n=3$, and the sum of the angles is $(3-1)\pi=\pi$.
For a square, $n=4$, and the sum of the angles is $(4-2)\pi=2\pi$.

Euler's Formula for Polyhedra

Now assume that we can project any polyhedron onto the plane so that none of the projections of the edges cross. We will prove Euler's Formula for the general case.

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- Let there be m polygons on the projection.

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- Let the exterior polygon have n_0 edges.
- Let there be m polygons on the projection. (This means that there were $m+1$ faces on the polyhedron, where the extra face is the exterior polygon in the projection.)

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Now assume that we can project any polyhedron onto the plane so that none of the projections of the edges cross. We will prove Euler's Formula for the general case.

- Let the exterior polygon have n_0 edges.
- Let there be m polygons on the projection.
- Let n_i be the number of edges of the ith polygon.

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- Let the exterior polygon have n_0 edges.
- Let there be m polygons on the projection.
- Let n_i be the number of edges of the ith polygon.
- Then we have

$$n_1 + n_2 + \ldots + n_m$$

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- Let the exterior polygon have n_0 edges.
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- Let n_i be the number of edges of the ith polygon.
- Then we have

$$n_1 + n_2 + \ldots + n_m + n_0 = 2E.$$

Euler's Formula for Polyhedra

Again, we sum the interior angles of interior polygons of the projection in two ways:

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- Interior polygons:

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$$(n_1 - 2)\pi + (n_2 - 2)\pi + \ldots + (n_m - 2)\pi$$

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Again, we sum the interior angles of interior polygons of the projection in two ways:

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$$(n_1 - 2)\pi + (n_2 - 2)\pi + \ldots + (n_m - 2)\pi$$

$$= (n_1 + n_2 + \ldots + n_m)\pi - \overbrace{(2 + 2 + \ldots + 2)}^{m}\pi$$

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&= (n_1 + n_2 + \ldots + n_m)\pi - \overbrace{(2 + 2 + \ldots + 2)}^{m}\pi \\
&+(n_0 - 2)\pi - (n_0 - 2)\pi
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&+(n_0 - 2)\pi - (n_0 - 2)\pi \\
&= (n_1 + n_2 + \ldots + n_m + n_0)\pi - 2(m+1)\pi - (n_0 - 2)\pi
\end{aligned}
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&= (n_1 + n_2 + \ldots + n_m)\pi - \overbrace{(2 + 2 + \ldots + 2)}^{m}\pi \\
&+(n_0 - 2)\pi - (n_0 - 2)\pi \\
&= (n_1 + n_2 + \ldots + n_m + n_0)\pi - 2(m+1)\pi - (n_0 - 2)\pi \\
&= 2E\pi - 2F\pi - (n_0 - 2)\pi
\end{aligned}
$$

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- Interior points + exterior polygon:

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- Interior polygons:

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- Interior points + exterior polygon:

$$(V - n_0)2\pi$$

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Again, we sum the interior angles of interior polygons of the projection in two ways:

- Interior polygons:

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- Interior points + exterior polygon:

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$$\begin{aligned} &2E\pi - 2F\pi - (n_0 - 2)\pi = 2V\pi - (n_0 - 2)\pi - 4\pi \\ &\Rightarrow E - V + F = 2. \end{aligned}$$

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First we give a little explanation of the term *infinite series*. If we ignore all the terms after the Nth term, we obtain

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We call this a *partial sum*. If the partial sums tend to a certain value S as N becomes large, then we say the infinite series has a value S.

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It is easy to see that as N grows, this partial sums go to infinity.

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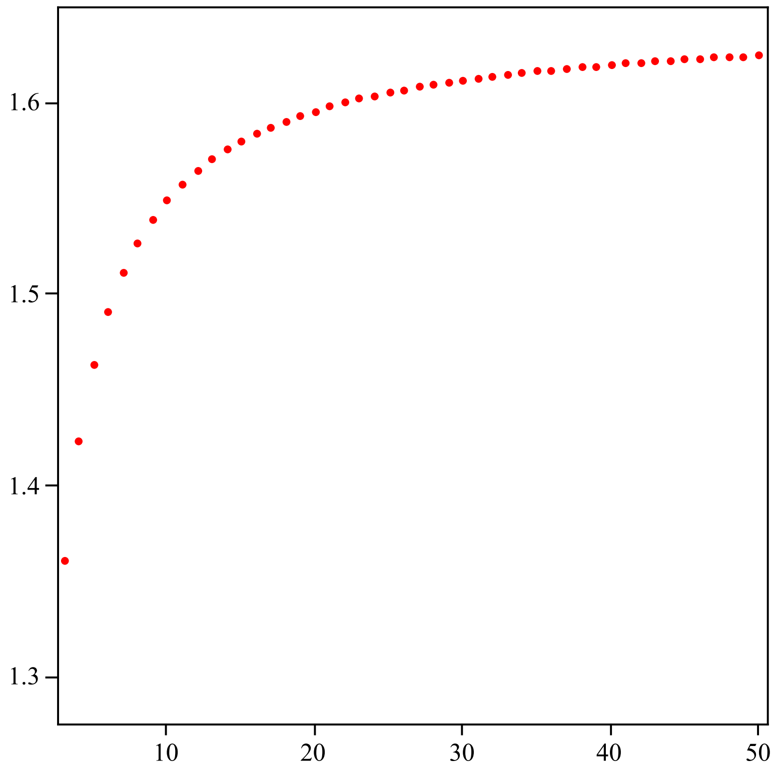
But if we look at the graph of the partial sums of our series, we obtain the following picture.



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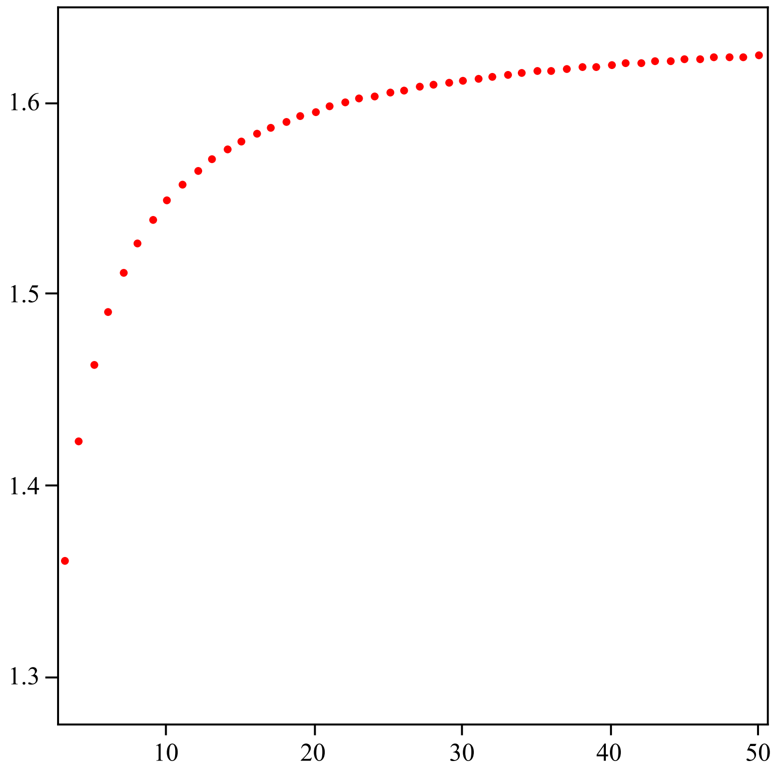

From this picture, we suspect that the partial sums tend to some number which is a little more than 1.6.

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- Then $(p + q) = -b$.

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- Therefore, the roots of $f(x)$ are

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(Instead of writing the factor $(n\pi - x)$, we write the factor $(1 - \frac{x}{n\pi})$, which corresponds to the same root, $n\pi$.)

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$$\Rightarrow \quad 1 - \tfrac{1}{4} \quad -\frac{1}{9} - \ldots = \frac{\pi^2}{6}.$$

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and the circumference of a circle whose diameter is 1 is, of course, the number π!

Conclusion