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Editors
Fou-Lai Lin, Feng-Jui Hsieh
Gila Hanna, Michael de Villiers

The Department of Mathematics, National Taiwan Normal University
Taipei, Taiwan

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PRESENTATIONS

GEOMETRICAL SOPHISMS AND UNDERSTANDING OF MATHEMATICAL PROOFS

Margo Kondratieva

Memorial University of Newfoundland, Canada.

By "geometrical sophism", we refer to a paradoxical conclusion which results from an impossible figure. We give an example of students learning the art of deduction through the analysis of geometrical sophisms. This approach can be useful in teaching students to understand the nature and purpose of proofs. Searching for a flaw, they focus on the essence and details of the proving process rather than on the truthfulness of a statement as perceived from either its substantive mathematical meaning or its adequacy to a physical model.

INTRODUCTION

How often do our students talk about space perception, possible and impossible drawings, illusions, fallacies, truth and, finally, produce their own justifications and proofs in the context of geometry? Most teachers would say, not often. Why not, especially if defending the truth is so attractive for youth and so essential for mathematics? In [Kondratieva, 2007] we observed that paradoxes are potentially useful for teaching mathematics due to their engaging power and the effect of surprise. A learner's puzzlement with an unexpected conclusion can efficiently drive the process of accommodating new information. However, a learner may not be mature enough to benefit from the exposition of paradoxes. Here we explore this delicate matter in the context of teaching proofs, one of the most important but difficult concepts in mathematical instruction.

While teaching an undergraduate university-level course that included elements of formal logic, deductions and types of proofs, I noticed that the students often perceived a statement and its formal proof as two almost independent items. A statement was perceived as an idea which they did or did not trust based on examples, analogies, intuitive explanations and figures. At the same time, the study of proofs had very little effect on the way the students understood and used the statements. Many students were able to repeat the proofs and even compose their own proofs in similar situations, but they did so more as a mechanical ritual of symbolic manipulation than a response to an intrinsic call for justification. At this point I introduced paradoxes, aiming to show that a form by itself does not guarantee the correctness of a deduction.

VISUAL REASONING, MENTAL MODELS, AND PROOFS

Generally, learners trust pictures and like short explanations based on them. This is common not only for younger ages. Adults operating in verbal-logical mode employ visual representations for their thoughts. Since ancient times the achievements of mathematicians have often been largely due to their ability to combine logic and imagination, to visualize their proofs. Children's geometrical perception of the world develops very early, rudimentary at first, but essential for their orientation in space. While geometrical intuition and children's achievements in geometry are often independent of their overall mathematical training, the development of geometric skills in children increases their overall mathematical and logical abilities. It seems to contribute in the process of building visual mental models essential for deductive reasoning [Johnson-Laird et.al., 1991]. A nicely drawn picture can serve as a motivation, illustration, and explanation at the same time. Manipulations with physical models and figures of geometrical objects allow learners to get a better understanding through reorganization of the perceived information and construction of an appropriate structural skeleton for a corresponding mental model. Algebra tiles became a popular tool in teaching methods such as factoring trinomials, giving an area interpretation for formulas like $(x+1)(x+2) = x^2 + 3x + 2$. One of the main algebraic identities $(a+b)^2 = a^2 + 2ab + b^2$ is illustrated by the partition of a $(a+b)\times(a+b)$ square into four parts: two squares and two $a \times b$ rectangles. For a geometrical proof of the identity $a^2 - b^2 = (a+b)(a-b)$ one starts with the $a \times a$ square lacking a $b \times b$ square corner, then cuts it in two identical halves and flips one half to obtain a rectangle with dimensions $a+b$ and $a-b$ which establishes the identity. In this "geometrical-algebraic" approach we manipulate with geometrical images, notice properties and express them in symbols. Do these manipulations qualify as a proof of the algebraic identities? They definitely have a great power of convincing. Students can touch, experiment, learn, and conclude using a form of "manual thinking". Students with developed visual imagination do not need even to touch in order to internally see "why", in addition to perhaps other arguments, which they can also interpret visually.

Despite all the advantages of visual reasoning, its power is restrictive. First, algebraic identities also hold for negative and complex numbers. Is there a straightforward interpretation or modification of the area-based proof for them? Second, when we rearrange pieces, how do we know that we obtain exactly the shape we claim, e.g. a rectangle in our second example? Maybe it is only a visual illusion of a rectangle? In addition, simply presenting pictures will not transfer an idea to a learner [Arnheim, 1969]. The student needs to understand the structure of the object and its essential properties. The role of the picture is to help him to grasp the schematic idea of the phenomenon. Thus, artifacts and figures, so welcomed by our intuition, can not fully replace abstract geometrical objects for the purpose of rigorous proofs. A drawing is never precise, but it is often good enough to make a correct conclusion if the one who interprets it has an adequate mental model, i.e. is able to estimate measurements (angles, sides, etc), proportions, mutual position of its parts, possible and impossible configurations. A picture represents an idealistic situation, or even a whole family of them; it contains compressed information to be unfolded by the interpreter. From this perspective there are two important exercises. One of them is precise geometrical drawing using a ruler, protractor etc. Another one is the inspection of a figure and of marks such as right angle, equal segments, and parallel lines, and making as many conclusions as possible. This not only develops in children the sense of a particular figure, but also enhances their general ability to reason and build elements of proofs. A simple implication such as “I have an equilateral triangle, thus I know that the angles are equal”, could serve as a step in a longer derivation. But the point of the exercise is to make a large number of observations, to learn how to make a picture talk to you about its properties, to retrieve the information compressed in a drawing. Without this ability to make and justify simple conjectures students cannot progress in understanding and creating more elaborated proofs.

FROM SOPHISMS TO PROOFS

The visual “proof” that 64=65 attracted a lot of attention from my students.

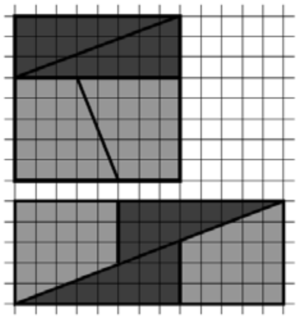

Figure 1: Visual “proof” that 64=65 by rearrangement of the parts of an 8X8 square into a 5X13 rectangle

I asked the students to explain the paradox without assistance or to seek my help after class if needed. Later, a group of confused students appeared in my office. They did not know where to start. Together we began to formulate and verify the ideas involved in the “proof”: (1) The area of a rectangle is equal to the product of the length by the width; (2) A flat figure can be cut in pieces, which can be rearranged to form a different shape of the same area. We agreed that these ideas were very natural, intuitively trustful, and that we used them without hesitation while solving other problems. Nevertheless, here we had come to a strange conclusion which contradicts common sense. More discussion led us to a general principle: (3) If the conclusion is false, there must be something wrong either with our assumptions or with the deductive process of making the conclusion. Although the articulation of this idea was a big step towards students' understanding of proofs in general, they could not progress in the given example. They cut an 8×8 square out of grid paper and rearranged the pieces according to my recipe to obtain a 13×5 rectangle. Everything in the process seemed correct, so where was the trick? It was evident that they had gotten stuck.

I decided to give them a break by focusing on something else. I happen to have several reproductions of the Escher's famous works, which students usually find very amusing. Among others there were the famous images of the impossible waterfall and triangle. Our discussion moved on; we talked about visual arts, realism or non-realism, and how one is able to draw images which contradict our perception of space. How then can we trust a picture? How can we verify which of the drawn compositions are possible in the space in which we live? What can be drawn? What is geometrically consistent and in what sense?

The students were giving simple examples such as “there is no triangle with sides having lengths of 1,3,5 units because 1+3<5.” Suddenly, someone expressed the idea that when a picture is drawn and some dimensions are known, then the others can not always assume arbitrary values: they may be estimated or calculated from the givens. I added this idea to our list of three items, and this apparently turned the group’s attention back to our paradox. Someone observed that the hypotenuse of the right triangle with sides 3 and 8 is equal to $\sqrt{73}$, while the trapezoid has sides 3,5,5, and $\sqrt{29}$. It took them few more minutes to realize that if the figure is correct then the diagonal of the rectangle must be equal to the sum $\sqrt{73}+\sqrt{29}$, whereas the hypotenuse of the triangle with sides 5 and 13 is a different number, $\sqrt{194}$. However, the equality is true if one approximates the radicals by rationals in the following way 13.9 =8.5+5.4, and this is the basis for the visual illusion. Another student looked at the angles of the right triangle with sides 3, 8 and the one with sides 5 and 13. He noticed that since $3/8 \neq 5/13$ (while $0.37 \approx 0.38$) it is just a visual illusion that the hypotenuse of the red triangle coincided with the diagonal of the rectangle. The slopes and length differences is invisible in this picture, but the area difference is obvious!

Students were very excited to discover the solution to the mystery. Some of them confessed that this was the first time they had paid so much attention to the proof itself, its logical structure, and the mathematical facts used. Most importantly, they were working with the picture and verified several ideas using their natural sense of geometry and logic. Some students decided to puzzle their friends with this paradox and asked whether the dimensions of the original square and the parts matter. They tried some arbitrary numbers and it did not work nicely. That was a fortunate turn of our discussion. I suggested that the dimensions involved in the construction, 3, 5, 8, 13, are not random, and they must be familiar to the students. Of course, this is a fragment of the Fibonacci sequence! But what does this sequence have to do with geometry? We learned previously that Fibonacci numbers have many interesting properties, and students started to list some of them from memory, but they did not seem relevant. Nevertheless, we recalled (in fact, re-derived) that the n-th Fibonacci number $F_n = (\phi^n - (1-\phi)^n)\big/\sqrt{5}$, where $\phi = (\sqrt{5}+1)\big/2$, is known as the golden ratio from its appearance in ancient art and architecture. Looking at the expressions for the slopes 3/8 and 5/13, we noticed that in the common denominator form the nominators of the fractions differ by 1, namely $3\times 13 - 5\times 8 = -1$. By analogy, the students checked 5, 8, 13, and 21 to find out that $5\times 21 - 8\times 13 = 1$. Since 13=5+8 and 21=13+8, they were able to rearrange a 13X13 square into a 8X21 rectangle and thus “proved” that 169=168. The slopes in this picture, 5/13 and 8/21, were even closer to each other that in my original example (their difference was 1/ (13X21)). This led the students to guess that there are an infinite number of area paradoxes with the same geometrical idea. To make sure, we checked a few other Fibonacci terms and conjectured that the product of two consequent Fibonacci numbers always differs from the product of their neighbors by one, $F_nF_{n+1} - F_{n-1}F_{n+2} = \pm 1$ (the sign depending on n being even or odd). If that is the case, we would have the slope difference decrease as n grows $|F_n / F_{n+2} - F_{n-1} / F_{n+1}| = (F_{n+1}F_{n+2})^{-1}$. The students were excited once again about their discovery of this connection of the paradox with Fibonacci numbers. They did not even notice how naturally they started to prove their conjecture! A proof was based on the above formula for the Fibonacci numbers in terms of the golden ratio. After a few mistaken trials and fixing the errors, they developed a pure algebraic proof, including two cases (n is even and n is odd). What is a picture related to this fact? One student said that the points with coordinates (F_{n+2}, F_n) approximately lie on a line, and they approach the line more closely as n increases. What is the slope of the line? They knew it was approximately 0.38, but what does it mean? Is there a relation with the Fibonacci sequence and the golden ratio? It took them a while to realize that the slope was ϕ^2; they then presented a calculus proof and used the explicit formula again. On that day they had many interesting discussions and arguments as they recalled, conjectured and proved their ideas to each other, -- a situation which does not often occur in a mathematical context. Their experience of success had built a certain maturity in the learners. My students demanded another paradox from me so they could demonstrate how clever they had become. Sure, I had another one; a proof that *all triangles are isosceles*. Consider an arbitrary triangle ABC. Let M denote a point of intersection of the bisector of the angle B and the perpendicular bisector of the side AC. For simplicity, assume that M lies inside of the triangle

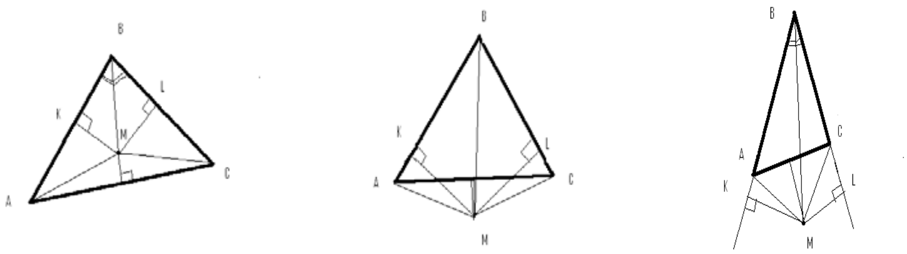

Figure 2: Proof that all triangles are isosceles. M lies inside or outside the triangle.

(Fig.2 left image). Let segment KM be perpendicular to the side AB, and LM to the side BC, where points K, L are on the corresponding sides of the triangle. Note that triangles MKB and MLB are identical right triangles since they both have the same hypotenuse BM, and MK=ML (property of a bisector). Therefore, BK=BL. Triangles AKM and CLM are another pair of identical right triangles, their hypotenuses AM=CM by the property of the perpendicular bisector. Therefore, AK=CL. Finally, AB=AK+KB=CL+LB=CB. The triangle is isosceles. QED.

It was a late afternoon, but the students did not want to leave. The proof laid out in front of them was shining in its simplicity. If not for the conclusion, no one would even suspect a lie; it was very much like one of those proofs they read in their books. The impossible conclusion drove the students to review (and prove) the bisector properties and the equality of the pairs of triangles. They got stuck again and remained helpless until someone finally decided to “act the problem out”, to draw a triangle and the bisectors. After few trials he proclaimed that the intersection point can not be inside the triangle. I accepted his argument, but to everyone’s surprise, there was a modification of the “proof” for the case when M lies outside the triangle (Fig.2) with the only possible difference that K, M lie at the extensions of the sides of the triangle and AB= KB-KA= LB-LC=CB. Again, it was not an immediate reaction, but the flaw was located: what I was able to sketch could not happen in a geometrical realm. When the picture was fixed we obtained AB= KB-KA , LB+LC=CB, thus AB and CB are not equal.

CONCLUSION

The duality of psychological and logical aspects of intellectual development implies a sharp distinction between informal and formal proofs. The first involves a great deal of experimentation and sense-making; the second tries to eliminate pure intuition and emphasizes rigor. The first reflects the process of constructing a mental model, the second is a manifestation of its completion. Traditionally, proofs appear in mathematical textbooks in a formal deductive way. Ironically, formal argumentation enables one to reason even without knowing the content. Consequently, pedagogical practice (and the opinions of such great mathematicians and teachers as F. Klein, H. Poincare, G. Polya, P. Halmos) suggests that before the formal stage, both immature learners and experienced researchers go through the contemplating and working levels of the informal stage in order to accumulate experience and develop an intuitive base for adequate interpretation and selection of various ideas. In this paper we gave an example of how visual paradoxes helped students to develop a sense of the purpose of proofs by examining the links between the given information and the conclusion -- the core of any deductive process. Their ability to understand and validate logical arguments was enhanced by the search for a flaw in the reasoning leading to a false conclusion. We have observed that the careful inspection of an impossible figure, which reveals its true structure, relations and properties, leads to an adjustment of the primary mental model and prepares cognition for making a deductive step and formation of a mathematical proof.

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WHAT CAN PRE-SERVICE TEACHERS LEARN FROM INTERVIEWING HIGH SCHOOL STUDENTS ON PROOF AND PROVING?

Boris Koichu

Technion – Israel Institute of Technology

This proposal outlines a study in the context of a graduate-level course on proof and proving in mathematics education. By the end of the course, the participants – pre-service teachers and graduate students – conducted task-based interviews with high-school students and reported on the interviewees' ways of proving and proof schemes. These reports along with the interview protocols were inductively analyzed in order to reveal the participants' understanding of proving and proof schemes. It was found that the participants displayed sensitivity to cognitive and social aspects of proving and to the role of teachers in developing the students' proving skills. It was also evident that some of them had difficulties with understanding the concept of proof scheme.

THEORETICAL BACKGROUND

The focus of the study

Mathematics education research suggests quite a number of useful theoretical constructs for understanding cognitive aspects of proof and proving (Mariotti, 2006; Harel & Sowder, 2007). Examples of such constructs include: *categorization of the roles of proof* in mathematics and mathematics education (Hanna, 2000); *proof schemes* (Harel & Sowder, 1998), distinction between *private and public aspects of proof* (e.g. Raman, 2003) and *the role of a classroom community* in shaping and validating proofs (Stylianides, 2007).

It is reasonable to assume that providing pre-service teachers with opportunities to look at students' proving through the lenses of these constructs can be beneficial for developing the teachers' own understanding of proof and proving in mathematics education. The outlined study partially tests this assumption, and therefore deals with the issue of teacher preparation within the "Cognitive aspects" theme of the ICMI Study19. Specifically, it aims at addressing the following questions:

1. Which cognitive and mathematical phenomena can pre-service teachers, who attended a course on proof and proving in mathematics education, recognize in task-based interviews with high school students?
2. How do the pre-service teachers analyze the interview protocols in terms of proof schemes?

Proof schemes

Harel & Sowder (1998) define a person's proof scheme (PS) to be what constitutes ascertaining and persuading for that person. This definition is deliberately student-centric. The taxonomy of proof schemes consists of seven major types of proof schemes, grouped into the classes of external conviction, empirical, and analytical proof schemes.

To recall, external conviction proof schemes depend (a) on an authority such as a teacher or a book (authoritative PS), (b) on mere appearance of the argument presented in a specific form, and not on its content (ritual PS), or (c) on symbol manipulations, when the symbols or manipulations have no coherent system of referents (non-referential or symbolic PS). Empirical proof schemes can be (a) inductive or (b) perceptual. Deductive proof schemes class consists of two sub-classes: (a) axiomatic and (b) transformational. Harel & Sowder (1998) point out that a person may possess and display different proof schemes, depending on a mathematical context or a situation in which proving occurs.

Task-based interviewing as a setting for exploring the interviewers

Since the subjects of the study play an unusual, for the subjects, role of interviewers and analysts, a brief argument concerning the potential of task-based interviews for revealing the *interviewers*' (and not only of the interviewees') reasoning is provided.

Task-based interviews in mathematics education research are usually used as an instrument for in-depth exploration of the subjects' reasoning and understanding (e.g., Schoenfeld, 2002). The interviews are rarely, if at all, used for investigating the interviewers, as it is done in the present study. However, the idea to ask pre-service teachers to act as interviewers and analysts is promising from both pedagogical and research perspectives.

First, it is well known that an interview situation bears a great learning potential for both interviewers and interviewees and opens a window in their reasoning and understanding (e.g., Koichu & Harel, 2007). Second, the role of an interviewer is in a line with the roles of a "close listener" (Confrey, 1994; Martino & Maher, 1999), an "informal assessor" (Watson, 2000) and a "teacher-researcher" (Cobb, 2000). It is widely accepted that all these roles are beneficial for professional development of mathematics teachers, so it can be expected that the role of an interviewer would also benefit them.

From the pedagogical point of view, pre-service teachers as interviewers have a natural opportunity to learn about conception of proof and ways of proving of their interviewees. From the research point of view, the teachers' ways of analyzing and reporting the evidence collected in the interviews reveal their own understanding of the previously studied theoretical constructs dealing with mathematical and cognitive aspects of proof and proving.

METHOD

Research settings

The study was conducted in the context of a 14-week course "Proof and justification in mathematics education" for undergraduate and graduate mathematics education students. The course was jointly taught by two lecturers[1]; 7 pre-service teachers and 5 graduate students attended the course. All the meetings were organized in accordance with the paradigm of active learning thought participation (cf. Sfard, 1998). An essential part of the course was devoted to reading and discussing contemporary research papers on proof and proving, including those mentioned in the previous sections. The students had also plenty of opportunities to be engaged in proving and then to reflect on their own experience in terms of studied theoretical constructs. During the course, the participants were given a series of homework assignments, either pedagogical-didactical or mathematical in nature. This paper is based on the final assignment of the course, which was given for work in pairs. In this assignment, the participants were requested:

A. To interview a "good" high school student in spirit of the interview described in Housman & Porter (2003). The interview design included engaging an interviewee in examining 4-5 mathematical conjectures and providing written proofs. Examples of the conjectures were: "The sum of three interior angles of any triangle is $180°$" and "If $(a+b)^2$ is even, then a and b are even"[2]. For each conjecture, the student was then asked: "How certain are you that the conjecture is true or false? How convincing is your proof to you? How convincing would your proof be to a peer? To a teacher? To a mathematician?" Additional recommended interview questions were: "What is a mathematical proof?" "To which extent proving is important in mathematics?", "Which types of proofs are you familiar with" etc.

B. To analyze the interview protocol in spirit of the papers by Housman & Porter (2003) and Stylianides (2007). The goal of the analysis was to formulate grounded conjectures about the interviewee's conception of proof, his or her ways of proving and proof schemes.

C. To write a report including: fragments of the interview protocol, the protocol analysis, discussion of the results in light of research papers studied in the course and personal reflection on the assignment.

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[1] The course had been initiated and designed by Prof. Orit Zaslavsky; I joined her in teaching the course in 2007. I am grateful to Prof. Zaslavsky for her support of various ideas that I brought with me, including the one presented in this paper.

[2] All the conjectures were discussed in the course before giving the assignment, so all the participants knew how to prove them and what difficulties high school students can have when trying to prove them.

The ways of data analysis

The data consisted of the notes made by the interviewees and interviewers during the interviews, fragments of the interview protocols and the final reports written by the participants of the course. First, the interview protocols were examined to find out which questions the interviewers asked during the interview in addition to the recommended questions and why. Second, the participants' analysis of the interview protocols was compared with the analysis of the same protocols by the author of this proposal; the apparent matches and discrepancies were indicated. Third, the written reports were inductively analyzed to find out which theoretical constructs from the papers studied in the course appeared there and how they were utilized to explain and interpret the observed phenomena. The latter analysis focused on signs of understanding and misunderstanding of the constructs by the participants; the central construct in question was that of proof scheme.

OVERVIEW OF THE FINDINGS

Generally speaking, the participants discussed in their reports the individual characteristics of the interviewees, their personal definitions of proof, ways of proving, including the ways of utilizing examples and counterexamples, proof thresholds, the interviewees' proof schemes and the deficiencies of their conceptions of proof due to (conjectured) insufficient attention to proof and proving in their mathematics classrooms. The reports and the analyses varied with respect to their deepness and relevance of the arguments, and pointed to different extents of understanding the explored phenomena. One participant did not overcome the stance of just an evaluator of mathematical correctness of the students' performance, whereas the rest tried to delve into the students' cognition. Consider an example[3].

From the interview protocol: One interviewee proved the conjecture about the sum of interior angles of any triangle by using the fact that an exterior angle of a triangle equals to the sum of two interior angles. She then asserted that she is fully convinced by her proof; that her classmates would also be convinced since she used in the proof mathematical symbols (e.g., an equation $180° - \alpha = \beta + \gamma$), that her mathematics teacher would also be convinced since she relied on "what is known in mathematics", but that a mathematician would probably be not convinced, since "my proof is incorrect…I used what I wanted to prove…I just reversed the proof". The latter assertion was made when the interviewers asked the girl to prove a statement about exterior angles of a triangle.

From the analysis by the pair of pre-service teachers: At the beginning, the interviewee was convinced in correctness of the statement since, in her opinion, it was a known mathematical fact, like an axiom. Therefore, it was an analytical/axiomatic proof scheme. On the other hand, she remembered that her

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[3] Due to space limitations, only one example is discussed, and only recaps of the interview protocol and of the analysis by the pre-service teachers are presented.

teacher taught the statement in school, so her certainty was also supported by belief in what the teacher had said, so it was also an authoritative proof scheme. When the interviewee proved the statement, she was convinced in its correctness by the power of mathematical symbols, which points to symbolic proof scheme. When she answered to the question about a mathematician, she understood that without a proof of a statement about exterior angles her proof would be incomplete. Finally, she was convinced in correctness of the statement about the sum of interior angle of a triangle, but not by the proof. It looks like the student believed in "classroom community" (cf. Stylianides, 2007), even though she was not familiar with this term ☺ [the smile sign appeared in the report], since she distinguished between the validity of her proof for members of her community (classmates and the teacher) and for a mathematician.

Discussion: The pre-service teachers demonstrated impressive sensitivity to cognitive and social aspects of the student's proving. The question about the proof of a statement on exterior angles was asked in time and pointed to the teachers' awareness of the circular nature of the student's proof. Some conjecture about the involved proof schemes look plausible, whereas others are poorly grounded in the data. For example, a conjecture about presence of analytical/axiomatic proof scheme, at best, looks as an instantiation of the Jourdain effect (cf. Brousseau, 1997). Interestingly, the analysis was emotionally loaded. It looked like the teachers enjoyed the process and were pleasantly surprised that the theoretical constructs studied in the course worked.

POTENTIAL CONTRIBUTION

The ICMI Study 19 asks is it possible/preferable to classify forms of proofs in terms of cognitive development. It also concerns the issue of how can teachers and mathematics educators use our knowledge about learners' cognitive development (see the ICMI Study 19 Discussion Document). The outlined study contributes to addressing these important questions by exploring the feasibility of teaching pre-service teachers the products of research on proofs and proving (e.g., proof schemes) and testing the opportunity to implement these constructs.

In summary, the study may have implications for both teacher professional development and testing the viability of theoretical constructs dealing with proof and proving. In addition, the methodology of exploring pre-service teachers acting as interviewers and analysts may be useful in other studies on developing mathematics teachers' knowledge base or in teacher education programs.

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WRITTEN PROOF IN DYNAMIC GEOMETRY ENVIRONMENT: INSPIRATION FROM A STUDENT'S WORK

Allen Leung

University of Hong Kong, Hong Kong SAR, China

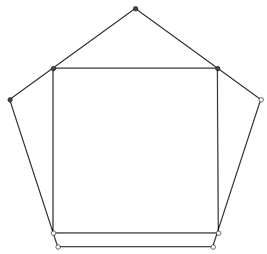
This paper continues the discussion in Leung and Or (2007) which reported and analysed how a Hong Kong Form 4 (Grade 10) student talked and wrote about his solution to a Sketchpad construction problem. In particular, the student produced a written "dynamic geometry proof" for a phenomenon that he observed in his exploration. I will recap the key discussions and continue with proposing possible syntaxes and semantics to "formalize" the student's "proof discourse" in the hope of opening up a direction to conceptualize proof in a DGE (Dynamic Geometry Environment).

BACKGROUND

Morris was an intelligent Form 4 (Grade 10) student in a Hong Kong secondary school. He was very keen in mathematics and a high achiever in the subject. He was an experienced Sketchpad user, often using Sketchpad to solve geometric problems on his own. Morris attended a Sketchpad workshop organized by the school's Math Club. During the workshop, the participants were taught the technique of relaxing a condition (Straesser, 2001) when solving construction problems in Sketchpad and using the TRACE function to visualize locus while dragging. Afterwards, the following problem was given to the participants as an exploration task:

Square inscribed in a regular pentagon

Investigate how to construct a square with 4 vertices lying on the sides of a regular pentagon as show in the figure on the right. Write down your method of construction and explain why your construction works.

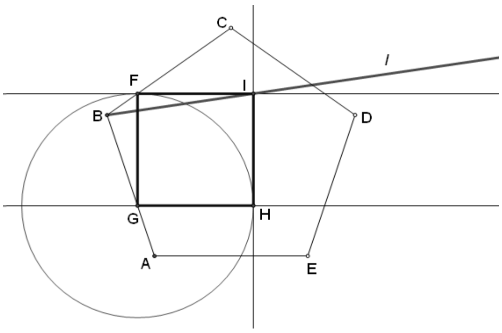


Morris solved the problem without much difficulty and he was asked afterwards to explain his construction.

THE INTERVIEW

The following is a transcribed excerpt from the interview conversation between Morris and the teacher in charge. During the interview, Morris had access to Sketchpad. (I: Interviewer, M: Morris)

1. M: Let a movable point (G) on this side (AB) of the pentagon. Construct a square (FGHI) like this. Then this point (I) would lie on this line (l) (see Figure 1a).



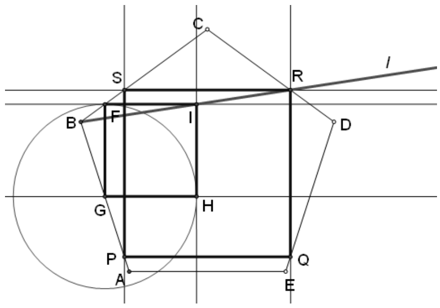


Figure 1a **Figure 1b**

2. I: Why? Do you mean when G varies, the locus of I is this straight line (*l*)?
3. M: Ym…. Yes. (Morris traced the locus of I to show that the locus of I is the straight line *l*).
4. I: So what are you going to do next?
5. M: So I project this line (*l*) and take the intersection (of *l* and CD) and draw a square … (Morris used the intersection R to draw a square PQRS as shown in Figure 1b).
6. (Morris then hid all the subsidiary lines and circles leaving only the pentagon and the square PQRS. Subsequently he dragged E arbitrarily to show that the constructed square is robust.)
7. I: Why? Why is the locus a straight line? Can you explain?
8. M: This line (FI) and this line (AE) are parallel. This side (BC) is fixed. Therefore this angle ($\angle$BFG) is fixed (when G moves). This triangle (ΔBGF) is similar. No matter how you move G this triangle is always similar. That is, the ratio of this side (BF) and this side (FG) is always a constant. Since this side (FG) and this side (FI) are equal, that means the ratio of BF and FI is always a constant. Since the included angle ($\angle$BFI) is the same, the triangle (ΔBFI) becomes similar, and hence the angle in the left ($\angle$FBI) is always the same. Therefore it comes out to be a straight line.
9. I: Why is this angle ($\angle$FBI) always the same?
10. M: Because of similar triangle!
11. I: Which triangle is similar to which?
12. M: Every triangle is similar! (Morris dragged G when he said so.)
13. I: You feel that they are similar?
14. M: They are always similar!
15. I: O.K.! How did you come up to this?
16. I don't know …. I just use an arbitrary method to prove this … the angle ($\angle$FBI) is a constant.
17. I: How do you know that you should prove this angle ($\angle$FBI) to be a constant?

18. M: Mm… it moves back and forth (Morris dragged G back and forth when he said so). This point (I) lies on the line (*l*) and we should look at this angle ($\angle$FBI).
19. I: Have you seen this problem before? Is this the first time you work on this problem?
20. M: I think so. (Morris had worked on a similar problem in a previous workshop.)
21. I: When you move (G) back and forth it reminds you that you should look at this angle ($\angle$FBI). Can I say so?
22. (Morris nodded his head slightly.)
23. I: Do you consider that you have solved the problem?
24. M: Ym … I think so.
25. I: Can you write down what you have already said? You can just write it down briefly.
26. M: Ym … I can try.
27. (After ten to fifteen minutes Morris presented the following written explanation.)

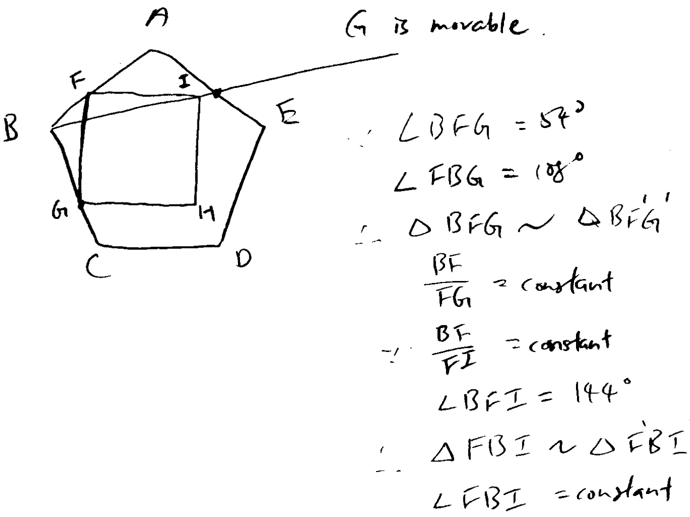


Figure 2: A scanned image of Morris' written proof

THE WRITTEN PROOF

The interviewer requested Morris at the end of the interview to write down his oral explanation. Morris produced an intriguing "formal" proof explaining that the locus of I under the movement of G is a straight line (Figure 2). The proof was written up in the format of a proof in Euclidean deductive geometry with a few DGE twists in it. There was a diagram depicting a static instant of the sequence of squares and the straight lines that passed through G and I. Beside the diagram was a statement "G is movable". Together the diagram and the statement formed a premise upon which subsequent arguments could be derived. However, any "logic" used hereafter must be one that could reflect the movability of G. Corresponding to the phrase "This triangle ($\triangle$BGF) is similar" (Line 8) that Morris used in his oral explanation, he wrote in the proof "$\triangle$BFG ~

$\Delta BF'G'$". Apparently, $\Delta BF'G'$ was not in the diagram. The primes that accentuated F and G seemed to symbolize the varying F and G under dragging. This was consistent with Morris' diachronic understanding of objects in DGE discussed above. He repeated this notation later in the proof with the statement "$\Delta BFG \sim \Delta F'BI'$". Another type of such diachronic expression that appeared in Morris's proof was "BF/FG = constant". The word "constant" had a deeper meaning than just being a numerical value; it meant invariant under variation via dragging. Thus the juxtaposition of a symbolic deductive proof formalism and a DGE-interpreted usage of symbols/signs seems to make Morris' written proof into a bridge that transverses the domains of experimental geometry (DGE) and deductive geometry (axiomatic Euclidean).

DGE SYNTAX AND SEMANTICS

From the above analysis there emerged a few ideas that might become significant when studying possible discourses in DGE.

1. Words like "movable", "become", "always", "constant" that connote (or is congruent to) motion, transition, invariance should be prominent in a DGE discourse. These words should be interpreted under the drag-mode or any other function in DGE that induces variations. Indeed, "Independent point is draggable" can be treated as a DGE axiom (for example, refer to the discussion in Lopez-Real & Leung, 2006).
2. Drag-sensitive objects in DGE are diachronic in nature. The concept of a whole could be a concept of continuous sequence of instances under dragging or variation (refer to discussion in Leung, 2008 on variation and DGE). Consequently, the denotation (or *congruent mode* of meaning, Halliday, 2004, p.14) of such objects may transcend the usual semantics of the spoken languages. For example, a singular "this" may actually mean many.
3. Writing up "formal" DGE proofs may involve using mathematical symbols or expressions that transcend the usual semantics of a traditional mathematical symbolic representation. For example, a DGE ΔABC may not point to a particular triangle; rather it represents all potential triangles ABC under dragging. In traditional axiomatic proof, one would say "for an arbitrary ΔABC". The diachronic nature of objects in DGE replaces the imaginary arbitrariness assumption in traditional mathematical proof.

I will use the following symbolic representations to capture these features:

$\dot{G}$	stands for an independently draggable point G
$\overset{\leftrightarrow}{d}(\text{Object})$	stands for the diachronic sequence of a DGE object under dragging
$\overset{\leftrightarrow}{d}_{\text{Object A}}(\text{Object B})$	stands for the diachronic sequence of a DGE Object B under dragging constrained on a DGE Object A
$\overset{\leftrightarrow}{\Rightarrow}$	stands for inductive implication under dragging

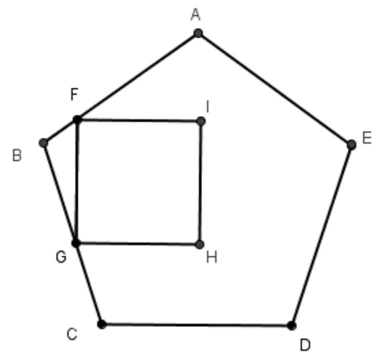
$\overset{\leftrightarrow}{m}$ (Object) stands for the diachronic sequence of the measured values for a DGE object under dragging

In this system of symbols, essentially crowned dot represents draggability, crowned double arrow represents dragging, subscript represents constraint or dependency, and brackets contain operand(s) under a dragging operation. With these I try to "re-write" Morris proof in a "DGE mixed" formal fashion:

<u>DGE Construction</u>:

a. ABCD is a robust regular pentagon

b. $\dot{G}$FIH is a robust square with $\dot{G}$ constrained on BC



$$\overset{\leftrightarrow}{d}_{\mathrm{BC}}(\dot{G}) \overset{\leftrightarrow}{\Rightarrow} \overset{\leftrightarrow}{d}\ (\mathrm{I})$$

<u>Prove</u> $\overset{\leftrightarrow}{d}$ (I) lies on a straight line

<u>DGE Proof</u>

1. $\overset{\leftrightarrow}{d}_{\mathrm{BC}}(\dot{G}) \quad \overset{\leftrightarrow}{\Rightarrow} \quad \overset{\leftrightarrow}{m}(\angle BFG) = 54^\circ$ and $\overset{\leftrightarrow}{m}(\angle FBG) = 108^\circ$

2. $\therefore \quad \overset{\leftrightarrow}{d}(\Delta BFG)$ are $\sim$ (AAA)

3. $\therefore \quad \overset{\leftrightarrow}{m}\left(\dfrac{BF}{FG}\right) =$ constant

4. GFIH is a square (DGE Construction b.)

5. $\therefore \quad \overset{\leftrightarrow}{m}\left(\dfrac{BF}{FI}\right) =$ constant

6. $\overset{\leftrightarrow}{d}_{\mathrm{BC}}(\dot{G}) \quad \overset{\leftrightarrow}{\Rightarrow} \quad \overset{\leftrightarrow}{m}(\angle BFI) = 144^\circ$

7. $\therefore \quad \overset{\leftrightarrow}{d}(\Delta BFI)$ are $\sim$ (SAS)

8. $\therefore \quad \overset{\leftrightarrow}{d}(\angle FBI) =$ constant

9. $\therefore \quad \overset{\leftrightarrow}{d}$ (I) lies on a straight line Q.E.D.

DISCUSSION

If one tries to write out a traditional Euclidean axiomatic proof for the above, a diagram like Figure 2 is needed and the main arguments involved in the proof are theorems about parallelism. In the DGE proof, the implicit critical feature behind the argument is that *the robust square GFIH is parallel translated along BC when G is being dragged along BC*. Thus, "Parallel translation under constrained dragging" may serve as a reasonable justification for certain phenomena in DGE (for examples, lines 1 and 6 in the DGE proof). The natural question to ask is whether this DGE proof can encapsulate the whole "DGE explanation"? May be more (modified) symbols are needed? Does it make pedagogical sense to formalize DGE proof? It is not my intention to propose that such a DGE proof should replace the traditional Euclidean proof; rather, my interest is in how traditional Euclidean proofs can be re-shaped in the DGE context if there is a semantic system in the DGE discourse. Dynamic visualization must be a critical feature in a DGE discourse, thus there is a need to lay down solid foundation upon which visual reasoning in DGE can be built upon. Developing a mathematically formal way to express ideas in DGE may form part of the foundation. Furthermore, having a semantic system to talk and write about DGE experiences may also pave a way to ease the tension that is often encountered between the experimental and theoretical aspects of DGE explorations. I hope this paper will serve as a stimulus to begin these undertakings.

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COGNITIVE AND LINGUISTIC CHALLENGES IN UNDERSTANDING PROVING

Kosze Lee, John P. Smith III

Michigan State University, United States

Drawing from a broader cognitive framework of human reasoning, this paper identifies cognitive and linguistic issues related to mathematical argument that are cognitive challenges for students learning to prove. These issues include students' pragmatic interpretations of the mathematical connectives and quantifiers, and their conceptions of the mathematical truth. These issues have implications to cognitively oriented research on and instruction in proof.

INTRODUCTION

Proof can be considered as a special form of argumentation in which deductive logic acts as a norm of warranting mathematical assertions (Selden & Selden, 2003). In attempting to prove mathematical statements, a student needs to attend to the deductive logical character of the task. This requires students to interpret and adopt logical constraints in the way as they are defined in deductive logic. If we view proving as a process of mathematical problem solving, then the arguments generated by students constitute their attempted solutions of deriving the conclusion required by the task. Students' strategic and content knowledge contribute to their successful production of a solution (Polya, 1954; Schoenfeld, 1985; Weber, 2001). However, students' interpretation of the underlying logical character plays a crucial role in structuring their arguments, during the process of proving. In other words, whether their arguments qualify as proofs also depend on their construal of the logical constraints imposed by the task.

The importance of proof schemes (Balacheff, 1988; Harel & Sowder, 1998), strategic knowledge (Schoenfeld, 1985; Weber, 2001), and relevant content knowledge (Weber & Alcock, 2004) have already been highlighted in many cognitively oriented studies of students' proving. On the other hand, the students' interpretation and adoption of the logical constraints in their proving process are given less attention. Our paper aims to explicate problems students experience in adopting deductive logic as the principal constraints of proving, especially when they are beginners at proof.

'REASONING FOR INTERPRETATION' AND 'REASONING FROM INTERPRETATION'

Recent cognitive science studies of human reasoning have shown that the ways that human subjects interpret deductive reasoning tasks matter to the kinds of inferences they make (Girotto, 2004; Stenning & Monaghan, 2004). In any such task, students have to interpret the task constraints according to their logical meaning. This involves interpreting the various logical terms such as conditionals ("if…then"), propositional connectives ("and/or") or quantifiers ("some/all"), and the notions of truth and falsity in ways that are congruent to the assignment of truth values as defined by logical truth tables. Two different cognitive processes in attempting deductive reasoning can be distinguished: (1) reasoning for an interpretation of these logical constraints and (2) reasoning to a conclusion based on these interpreted constraints, or in Stenning's terms, 'reasoning for interpretations' and 'reasoning from interpretations' (Stenning & Monaghan, 2004). The former process concerns students' adoption of deductive logic as their interpretation of the task, which can be observed in how they generate inferences. The latter process concerns students' knowledge of strategies for generating chains of inferences leading to conclusions, which is the goal of the task (Stenning & Lambalgen, 2004).

Cognitive issues related to 'reasoning for interpretations' are evident in the mathematics education research. Many students, who are learning to prove, are plagued by the interpretation of logico-mathematical terms and notions. Balacheff (1988) and Harel & Sowder (1998) documented the absence of the central understanding of deduction evident in the pre-college and college students' proof schemes. Secondary students experience difficulties in interpreting the logical relationships between the antecedent and the consequent of the logical implications (Hoyles & Küchemann, 2003). College students who are proficient in mathematics also hold alternative understandings of logical implication in asserting the truth value of conditionals (Durand-Guerrier, 2003).

The abovementioned cognitive issues can be categorized into: (1) the pragmatic interpretation of connectives (including the conditionals "if…then") and quantifiers, and (2) the conception of "true" and "false" (Stenning & Lambalgen, 2004). Though this categorization scheme originates from a non-mathematical task setting, we will use empirical studies in the mathematics education to illustrate the cognitive complexity underneath the adoption of deductive logic in proving.

In contrast, examples of 'reasoning from interpretation' processes are the students' applications of strategic knowledge and semantic-syntactic proof production (Weber, 2001; Weber & Alcock, 2004). In these studies, the students were mathematically enculturated to the logical terms and notions.

The pragmatic interpretations of connectives and quantifiers

How do students interpret the connectives and quantifiers in mathematical statements? Such a situation can be confusing because the logico-mathematical constraints are communicated through the connectives and quantifiers which also have their own meaning in everyday conversation. From a linguistic perspective, the uses of these terms often follow the maxims of pragmatics in everyday conversation – "be as informative and concise as possible" (Grice, 1989). For example, the quantifier "some" has the pragmatic meaning of conveying "some and not all" in everyday conversation concisely. It is also informative to the best of the speaker's knowledge, or "all' would be used instead. Likewise, the disjunctive "or" conveys the pragmatic meaning of "choose one option". However, their logico-mathematical meanings in the proving context convey otherwise: “some” indicates “some and perhaps all” and “or” indicate “choose at least one option”.

The pragmatics perspective is related to the sociolinguistic notion of “registers”, or the clusters of meanings conveyed through the same language through linguistic features of expression associated to different linguistic situations (Halliday, 1978). In mathematical communication, the “mathematical register” – the set of meanings and words appropriate in expressing mathematical thoughts – is conveyed through everyday language (Pimm, 1987). A typical example would be the different meaning of “expand” in algebra and in everyday life.

While the notion of “mathematics register” brings clarity to the interpretation issue, the pragmatics view pinpoints the cause of the issue: in conveying the logical constraints of the proving task to the student, the logico-mathematical meaning and the pragmatic meaning of the connectives and the quantifiers work against each other. Thus, a reasoning task like “All A are B. Does it follow that some A are B?” will yield “yes” as a logically valid conclusion but leads to “no” as a conclusion due to a pragmatic interpretation, “some and not all”.

The confusion of registers poses great challenges to students who are not acquainted to proving in distinguishing those meanings. The logico-mathematical meaning is hard to adopt. Often, the pragmatic interpretations take over and led them to respond differently to the task. In Hoyles & Kuchemann’s (2003) study, a number of students interpreted “if-then” conditionals as equivalent to their converses. Their responses suggested that the antecedent and the consequent were interchangeable and that the converse conveyed the same mathematical meaning. As the authors had noted, it might be due to the use of “if-then” sentences in everyday conversations which also conveyed the pragmatic meaning of “if-not, then-not”. Hence a statement “If it rains, I will bring an umbrella” uttered by the speaker often pragmatically implies its converse “If it doesn’t, I won’t bring one.”

Also, Durand-Gurrier (2003) has noted that college students who were good at mathematics had difficulties in interpreting the cases of false antecedent in the conditional “If n is an even number, then n+1 is a prime.” They mostly treated the cases of odd numbers as either irrelevant or undecidable. The odd numbers did not contribute any information to the inference of the consequent “n+1 is a prime” though this was logically true in mathematics. Their inconclusive inferences seemed to arise from their pragmatic interpretations of the false antecedent cases for their informative relevance, instead of their logical meaning.

Based on these studies in cognitive science and mathematics education, students’ learning to prove is likely complicated by the cognitive challenge of interpreting the proving task in accordance to its logical meaning instead of its pragmatic meaning. Pragmatic readings of the connectives and quantifiers in mathematical statements lead the students to non-mathematical conclusions.

The conceptions of truth and falsity

In cognitively-oriented studies of students' proving, researchers have posed reasoning (proving) tasks based on the accepted logico-mathematical notions of truth and falsity and implicitly assumed that students would interpret the tasks likewise. But in studies of deductive reasoning, the subjects had been found to mentally debate among different conceptions of logical truth which subsequently interfered with their inferences (Stenning & Lambalgen, 2004). Subjects were asked to examine the two rules – "if there is a U on one side, then there is an 8 on the other side" and "if there is an I on one side, then there is an 8 on the other side" – to determine which was true and which other was false. Some thought that absence of counterexamples did not qualify a rule to be true. Some contested the notion of whether not "true" means "false" and vice versa. Others held the notion that "true" rules could allow exceptions. However, in the studies of students' proof schemes and proving, the effects of the students' conception of these mathematical foundations are usually ignored (Balacheff, 1988; Harel & Sowder, 1998; Weber, 2001; Weber & Alcock, 2004).

In our work (Lee & Smith, 2007), we explored college students' proving processes and examined their conceptions of mathematical truth. Out of the six interviewees, two non-math majors taking college algebra held and applied conceptions of truth contrary to the standard mathematical notion.

One student held simultaneously the mathematical conception of truth and an alternative informal conception, and applied them at different points in the same task. In a proving task that involved quadratic functions and graphs, she first applied a "true means mostly true" conception to identify a particular quadratic graph as a member of a clearly defined class of quadratic objects. She was aware of how her inference was problematic to the mathematical conception of truth since the particular graph in consideration did not meet all the criteria defining the class. Subsequently, she used that graph as a counterexample to refute a general statement.

The other student was aware of the logical dichotomy between truth and falsity – "either right or wrong". However, her conception of mathematical truth contained an element of uncertainty. For her, any true mathematical statement always had an unknown exception. This notion of "unknown exception to the rule" introduced some psychological instability which undermined her conviction of mathematically true statements.

Each student's conception posed an epistemological obstacle to their learning of mathematical proving. If "true means mostly true", then the conceptual boundary between "true" and "false" becomes unclear and the strategy of proof-by-contradiction and finding counterexamples no longer qualify as means to warrant true statements. In addition, the notion of logical implication, which rests strictly upon the logical conjunction of truth values of its constituents, becomes fuzzy. If true means "there is always an exception to the rule" in mathematical assertions, then the cognitive purpose of removing one's doubts in proving about a mathematical assertion becomes impossible, let alone the social purpose of persuading others (Harel & Sowder, 1998). Indeed, one's conceptualization of the mathematical structure becomes fundamentally unstable.

CONCLUSION AND IMPLICATIONS

We have identified cognitive and linguistic issues in students' interpretation of tasks of mathematical proof. In particular, students' pragmatic interpretations of connectives and quantifiers, and their conceptions of mathematical truth result in difficulties in their construal of the logical character of the task. These difficulties are well illustrated by the studies in cognitive science and mathematics education.

Students' interpretation of proofs tasks as logical tasks may be a non-issue to students who are more mathematically enculturated to "the rules of the proving game." But for students who first entered "the proving game," the competition between the pragmatic interpretation and the logico-mathematical meaning of the proving task can pose deep cognitive challenges and hamper their competence in proving.

Truth and falsity are foundational issues in mathematics that have endured a long history of philosophical debate among intuitionism, logicism and formalism (Benacerraf & Putnam, 1964). Our work (2007) and the work of Stenning & Lambalgen (2004) have shown that the challenges of clarifying these mathematical foundations also exist for beginners with proof.

The distinction between the interpretation issues and proof construction issues in proving helps to explain students' proving process in the light of their interpretation of the task. More inquiries are needed to investigate the relation between students' interpretations and proof constructions, and also to see how instructional practices can attend to the cognitive challenge faced by students who are new to "the rules of the proving game."

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GOOD PROOFS DEPEND ON GOOD DEFINITIONS: EXAMPLES AND COUNTEREXAMPLES IN ARITHMETIC

Steven R. Lay

Lee University, Cleveland, TN, USA

All proofs depend ultimately on the underlying definitions. Unfortunately, the role of definitions in mathematics has been largely neglected in the United States grade school curriculum. Often careful definitions are not given and sometimes when they are presented, they are not valid. Concrete examples showing how definitions relate to proofs are presented in the context of ratios, exponents, and multiplication of fractions. The results of recent research with middle school mathematics teachers in southeastern Tennessee are included.

DEFINITIONS: THE FOUNDATION FOR PROOFS

The paper addresses question 9 of Theme 2, Argumentation and proof: What conditions and constraints affect the development of appropriate situations for the construction of argumentation and proof in the mathematics classroom?

There are many types of proofs. Some are straightforward and some are complex. But all proofs depend ultimately on the underlying definitions and the earlier results that have been derived from these definitions. Unfortunately, the role of definitions in mathematics has been largely neglected in the United States grade school curriculum. All too often what passes for a definition is more a description of how a concept is used rather than what a concept really is (Wu, 2002). For example, a common "definition" of a ratio is "a comparison of two quantities by division" (Glencoe, 2005, p. 264; Hake, 2007, p. 186; Holt, 2005, p. 342). This is certainly how a ratio is used, but if this is taken as the definition, then how can you explain to a student what it means to multiply two ratios? How do you multiply two comparisons?!! The only thing that can be multiplied are numbers. When ratios are presented as "comparisons" then students feel that they must learn a whole new set of rules for using them. What they have learned about numbers no longer applies. Furthermore, if one is trying to prove a property of ratios, then a precise definition of a ratio as a quotient of two numbers (and therefore itself a number) is necessary.

EXPONENTS

The situation with the definition of exponents is even more troubling. In this case a definition is usually given, but often it is not correct. There are two common definitions of exponents, illustrated with the expression 5^n.

1) 5^n is 5 multiplied by itself n times (e.g. Pintozzi, 2004, p. 48).
2) 5^n is a product of 5s where there are n factors (e.g. Holt, 2005, p. 84).

Let's use these definitions to try to prove that $5^1 = 5$ and $5^0 = 1$. Using the first definition, we encounter an immediate problem. If we are asked to multiply 5 by itself one time, we would certainly write 5×5. But this is 5^2, not 5^1. Likewise, 5 multiplied by itself two times would be $5 \times 5 \times 5$ or 5^3 not 5^2. So we see that the first definition is not valid, even for exponents greater than one. And it's anyone's guess why 5^0 should equal 1.

With the second definition we get $5^3 = 5 \times 5 \times 5$ and $5^2 = 5 \times 5$, as desired. But there is a problem with 5^1. What does it mean to say that 5^1 is a product of 5s where there is one factor? In order to have a product, there must be two factors. So a product with one factor makes no sense. And once again it's not at all apparent why a product of 5s where there are 0 factors should equal 1.

Using either common definition of exponents, it is not possible to prove that $5^1 = 5$ and $5^0 = 1$. The first definition is wrong for all exponents. The second definition doesn't hold when the exponent is 1 or 0. They must be treated as separate cases.

Fortunately, there is a better definition of exponents. It is not widely used, but it is mathematically correct and it has the beauty of including the cases when the exponent is 1 or 0.

> Definition of b^n: The exponent n counts the number of times that 1 is multiplied by the base b.

So we have, for example, when the base b is 5:

$5^3 = 1 \times 5 \times 5 \times 5$ (Start with 1 and multiply by 5 three times.)

$5^2 = 1 \times 5 \times 5$ (Start with 1 and multiply by 5 two times.)

When $n = 1$, we start with 1 and multiply by 5 one time: $5^1 = 1 \times 5 = 5$. When $n = 0$, we start with 1 and multiply by 5 no times: $5^0 = 1$. Proper definitions like this not only lead to better proofs, but also increase the students' ability to compute correctly and remember the "special case" when $n = 0$ (Lay, 2006).

MULTIPLICATION OF FRACTIONS

As an example of using a definition to prove a familiar algorithm, let us prove that $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. To do this, we will use the following definition and properties:

> Definition: Multiplying by $\frac{a}{b}$ is the same as multiplying by a and dividing by b.

> Properties: 1) Multiplying and dividing commute with each other. That is, multiplying a number by x and then dividing by y is the same as dividing by y and then multiplying by x. For example, 12 divided by 3 and then multiplied by 2 is the same as 12 multiplied by 2 and then divided by 3:

$$2\left(\frac{12}{3}\right) = 2(4) = 8 \quad \text{and} \quad \frac{2 \cdot 12}{3} = \frac{24}{3} = 8.$$

2) Dividing by x and then dividing by y is the same as dividing by their product xy. For example, 12 divided by 2 and then divided by 3 is the same as 12 divided by their product 6:

$$\frac{\frac{12}{2}}{3} = \frac{6}{3} = 2 \quad \text{and} \quad \frac{12}{2 \cdot 3} = \frac{12}{6} = 2.$$

Now for the proof that $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a\left(\frac{c}{d}\right)}{b} \qquad \text{Definition of multiplication by } \frac{a}{b}.$$

$$= \frac{\frac{ac}{b}}{d} \qquad \text{Property 1.}$$

$$= \frac{ac}{bd} \qquad \text{Property 2.}$$

Using valid definitions as a foundation to proving common computational algorithms and other properties of numbers is the essence of mathematics and should be emphasized in the school curriculum. But we cannot expect teachers and students to construct valid proofs when we give them inadequate definitions.

THE RESULTS

For the last two years, Lee University has partnered with Edvantia, a regional educational research organization, and 11 school districts in southeastern Tennessee to teach their middle school math teachers how to use better definitions in their explanations and proofs. To date, 52 teachers have attended our two-week summer institute called Improving Numeracy and Algebraic Thinking (INAT). Prior to each institute the teachers were given a challenging pre-test of their ability to explain and prove pre-algebra concepts. The range of scores on the pre-test has been from 2% to 38%, with a median score of 18%. Following the intervention and the introduction of better definitions, the teachers' scores on a similar assessment ranged from 20% to 100%, with a median score of 88%.

This dramatic improvement in their ability to explain why math properties and algorithms work the way they do was also reflected in their post-institute evaluation surveys. 98% of the teachers strongly agreed that their new knowledge could be incorporated into their classroom teaching, with the other 2% agreeing. Many of the teachers also indicated by their written comments that they were returning to their classrooms with greater confidence and enthusiasm.

We are now in the middle of a 3-year project to determine the impact these teachers will have on the success of their middle school students, as measured by the students' scores on state mandated assessments. This research is funded in part by a Mathematics and Science Partnership (MSP) grant from the Tennessee Department of Education and an Improving Teacher Quality (ITQ) grant from the Tennessee Higher Education Commission.

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MULTIPLE PROOF TASKS: TEACHER PRACTICE AND TEACHER EDUCATION

Roza Leikin,

University of Haifa, Israel

One of the questions raised by the discussion document of ICME study-19 involves types of proofs. In this paper I consider multiple –proof –tasks (MPT) that require finding more than one proof for a particular mathematical statement. Mathematical distinctions between different proofs serve a basis for the definition of MPT. Proof spaces are suggested herein as a useful tool for exploring individual or group performance on MPTs. In the paper I outline MPTs as an effective research and didactical tool as well as a tool for teachers' professional development.

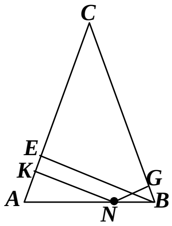
MULTIPLE PROOF TASKS

Mathematics educators agree that linking mathematical ideas by using more than one approach to solving the same problem (e.g., proving the same statement) is an essential element of the developing of mathematical reasoning (NCTM, 2000; Polya, 1973, Schoenfeld, 1985). Problem solving in different ways both requires and develops mathematical knowledge (Polya, 1973) and flexibility and creativity of the individual's mathematical thinking (Krutetskii, 1976; Silver, 1997; Tall, 2007).

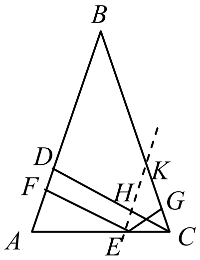
I define multiple solution tasks as tasks that explicitly require from a solver solving a mathematical problem in different ways (Leikin, 2007; Leikin & Levav-Waynberg, 2007). Some problems require proofs, and thus analogously to the previous works, I call them Multiple Proof Tasks (MPTs). Bellow I describe mathematical, psychological and didactical perspectives on MPTs. The examples used to illustrate the ideas are taken from multiple studies performed during the last six years focusing on employment of MPTs in teacher practice and teacher education.

DEFINITION

Multiple-proof tasks are tasks that contain an explicit requirement for proving a statement in multiple ways. The differences between the proofs are based on using: (a) different representations of a mathematical concept (e.g. proving formula of the roots of a quadratic function using graphical representation, using symbolic representation in canonic form, using symbolic representation in a polynomial form); (b) different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic (Task 1, Figure 1); (c) different mathematics tools and theorems from different branches of mathematics (Task 2; Figure 1); or (d) different tools and theorems from different subjects (not necessarily mathematical)

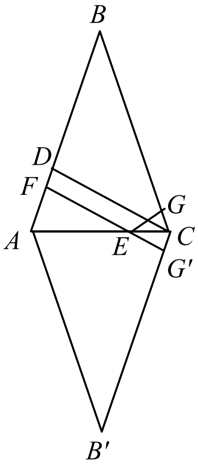


Task 1 ABC is an isosceles triangle: AC=CB
Point N is on AB
NG perpendicular to CB; KN perpendicular to AC
BE altitude to AC
Prove in as many ways as possible:
NK + NG = EB



Proof 1.1: Congruence
Construction: *EH*||*AB*
DHEF – parallelogram
⇒*DH*=*FE*
Let's prove *HC*=*EG*
4.1a: *ΔEKC*-isosceles triangle,
DH=*FE* as altitudes to lateral sides
4.1b: *ΔEHC*≅*ΔCGE*

Proof 1.3 Symmetry
Construction:
ΔAB′C symmetrical to *ΔABC* about *AC*
DC=*FG* (parallel segments between parallel lines)
EG′=*EG* symmetrical
EG′+*FE*=*FG*
FE+*EG*=*DC*

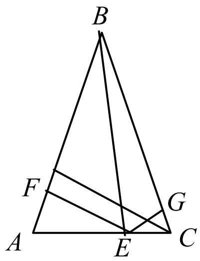


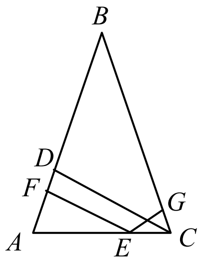
Proof 1.2: Area
Construction: *BE*

$$S_{ABE} + S_{EBC} = S_{ABC}$$

$$\frac{AB \cdot EF}{2} + \frac{BC \cdot EG}{2} = \frac{CD \cdot AB}{2}$$

$$\Rightarrow EF + EG = CD$$





Proof 1.4: Similarity
ΔAEF≈ΔACD≈ΔCEG
4.4a: Proportion
4.4b: Side sum

Task 2: ***Prove in as many ways as possible:*** Of all the rectangles with a given perimeter *P*, the square has the maximal area.

Proof 2.1: Calculus

$P = 4p;\quad f(x) = x \cdot (2p - x);\quad f'(x) = -2x + 2p;\quad f'(x) = 0;\ f_{\max}(x)\quad x = p$

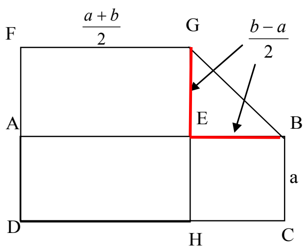
$x = p$ is the side of the square with perimeter *P*=4*p*

Proof 22: Algebra

$f(x) = x \cdot (2p - x)$ is a parabola with vertex (max) at $x = p$

2.a) according to the vertex formula;
2.b) according to symmetry of the parabola on the segment]0; 2*p*[

Proof 2.3: Geometry and algebraic manipulations



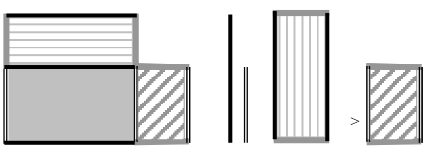
ABCD is a rectangle with perimeter *P* and sides *a* and *b* (without loss of generality $a < b$); DFGH is the square with perimeter *P*, its side is $\frac{a+b}{2}$.

$S_{ABCD} = S_{AEHD} + S_{HEBC};\quad S_{DFGH} = S_{AEHD} + S_{AFGE}$

$S_{HEBC} = a \cdot \frac{b-a}{2};\quad S_{AFGE} = \frac{a+b}{2} \cdot \frac{b-a}{2}$

$a < b \Rightarrow a < \frac{b-a}{2} \Rightarrow S_{HEBC} < S_{AFGE} \Rightarrow S_{ABCD} < S_{DFGH}$

Proof 2.4: Geometry



Proof 2.5: ***Symmetry considerations***
Of all the figures with a given perimeter **P,** the most symmetrical has the maximal area

Figure 1: Examples of a MPT

The mathematical distinctions between the proofs serve as a basis for developing mathematical connections between different representation of mathematical concept, between their properties and between different fields of mathematics. In other words, MPTs allow developing connected mathematical knowledge in students and their teachers.

MPTS AS A RESEARCH TOOL

Proof Spaces

In order to explain the potential of MPTs as a research tool I suggest the notion of ***proof spaces***, which are the collections of proofs of a statement that individuals or groups can produce (see elaborated definitions in Leikin, 2007, cf. example spaces defined by Watson & Mason, 2005).

An *expert proof space* of a MPT is the fullest set of proofs of a statement known at a given time. This space may expand as new proofs of the statement are produced (see Figure 1 for an expert proof space of Tasks 1 and 2). *Individual proof spaces* are subsets of an expert proof space. According to a person's capability to produce proofs to a multiple proof task with or without prompts we differentiate between an *available personal proof spaces* consisting of proofs that a solver produces without any help of others and *a potential personal proof spaces* consisting of proofs that one can produce with the help of others (cf. the concept of ZPD defined by Vygotsky, 1978). Solution spaces may be differentiated according to their conventionality: *Conventional proof spaces* include proofs displayed in curriculum-based instructional materials (e.g., Proofs 1.1, 2,1, Figure 1 for Israeli curriculum). *Unconventional proof spaces* include either proofs that are not included in curriculum-based instructional materials (e.g., Proofs 1.3; 2.3, 2.4, 2.5, Figure 1) or curriculum-based proofs applied in an unusual situation (e.g., Proofs 1.2, 1.4, 2.2 Figure 1).

An expert proof space of a task can be held by a teacher when he or she plans a lesson with a particular Task. It may be held by a research mathematician in his/her professional work aimed at expanding the space, it can be held by an educational researcher when planning a research. We compare individual proof spaces with the expert proof spaces in order to characterize one's problem-solving expertise, connectedness of his/her mathematical knowledge and his/her creativity.

Example: Exploring mathematical creativity with MPS

Fluency, flexibility and novelty are main components of creativity (Torrance, 1974). MPTs were shown (Leikin & Lev, 1997) as an effective research tool for examination of one's creativity: *flexibility* refers to the number and the availability of different proofs generated in the individual solution spaces, *novelty* refers to the conventionality of proofs and their availability in the individual solution spaces, and *fluency* refers to the pace of solving procedure as well as the ability to produce multiple proofs. The study compared creativity of gifted students (GS) with creativity of excelling (but not identified as gifted) students (ES) and with creativity of regular students (RS) focusing on MPTs. We showed that differences between creativity of GS and creativity of RS are task dependent. There were no differences between the number of proofs produces by GS and RS on curricular-related MPT. However available unconventional proofs were found in proof spaces of GS only. On un-conventional MPT there were meaningful differences in the proofs and their availability produced by GS and those produced by ES: thinking of GS were shown as more flexible and original. Proof spaces of RS differed from those held by ES and GS. They included only a small number of proofs, did not include unconventional proofs, and on the unconventional tasks were empty for most of the RS.

DEVELOPMENT OF TEACHERS' KNOWLEDGE AND BELIEFS

Our studies showed that MPTs are powerful tool for the development of teachers' knowledge and beliefs (Leikin & Levav-Waynberg, in press).

Example: Geometry course for pre-service mathematics

During the 56-hours course, pre-service mathematics teachers (PMTs) were asked systematically to proof geometry statements in at least 3 different ways and find statements in regular textbooks that may be proved in at least 3 different ways. As a result the PMTs developed their expertise in proving mathematical statements in general as well as in providing multiple proofs to mathematical statements in particular. These changes occur along with the changes in teachers' beliefs.

At the beginning of the course only 5 of 12 PMTs succeeded to prove that the *median to hypotenuse in a right triangle equals half of the hypotenuse*. Four of them produced one proof only and one of the teachers suggested two proofs. The collective solution space for this problem consisted of 3 proofs of 11 proofs in an expert solution space.

At the end of the course all 12 PMTs proved successfully the statement in Task 1 (Figure 1): Individual solution spaces included 1 proof for 4 PMTs (all of them did not succeed proving the theorem in the pre-test), 2 proofs for 5 PMTs and 3 proofs for 3 PMTs. The collective solution spaces included all the proofs presented in Figure 1.

We observed also changes in PMTs' beliefs about the effectiveness of MPTs in teaching mathematics as shown in the excerpts bellow.

Changing "WHY?"

> **Revital – before the course**: Why do I need this? This is too much. I had a very good teacher and I think I know geometry, however, he never required from us to solve problems in different ways. It may happen that two students have different solutions, but there is no time in the lesson to address all of them.

> **Revital – after the course**: First of all this is fun. At some point you feel you enjoy it -- enjoy solving and enjoy knowing. You tell "Wow, I can do it!". At the beginning [of the course] I did not believe this would happen. Why we did not learn in this way in school?

Changing "HOW?”

Mona – before the course: I can see that different people can have different solutions. But how can I do it alone?

Mona – after the course: When we solved the tasks in the group one is always surprised by how differently people think. We always had at least 3 or 4 solutions of a problem. So you learn from other people solutions and start believing that this is possible.

MPTs AS A DIDACTICAL TOOL

The transitions from a systematic setting, in which teachers learn through solving MPTs, to a craft setting where teachers are expected to implement MPTs are not trivial. We found two main types of implementation of MPTs in the classroom: teacher-initiated and students-initiated implementation.

In a *teacher-initiated implementation*, the teachers create a didactical situation in which students are required to produce multiple proofs. Searching various proofs becomes a part of a didactical contract between the teacher and the students. Teacher-initiated MPTs may differ with respect to their openness. A teacher can plan either *guided proofs*, when he/she outlines to students several directions in which proofs can be performed and the students have to perform the proofs; or *un-guided proofs*, when the students have to find tools appropriate for the statement and also are expected to produce proofs using them.

In a *student-initiated situations* a teacher does not plan MPTs as a part of planned learning trajectory and the lesson development depends on students' ideas and teacher's flexibility. The requirement of proving a statement in more than one way can be raised by students. It happens either when they don't understand the first proof offered or when they find an alternative proof and want to share it with the teacher and the other students.

As a concluding remark, note that students' collective proof spaces are the main source for the development of their individual solution spaces – i.e., development of more connected mathematical knowledge (see excerpt "Mona - after the course" above). When approaching MPTs, students see the variety of ideas and tools, they learn to appreciate other students' thinking. Moreover teachers, when implementing MPTs, usually broaden their "expert" space based on the proofs suggested by the students and thus learn mathematics through teaching. Teachers' familiarity with MPTs and their awareness of the importance of PMTs for students' learning allow them to be more flexible and sensitive in mathematics classroom especially in various student-initiated situations (Leikin & Dinur, 2007).

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HOW CAN THE GAME OF HEX BE USED TO INSPIRE STUDENTS IN LEARNING AMTHEMATICAL REASONING?

Chun-Chi Lin

Department of Mathematics

National Taiwan Normal University, Taiwan

In this article, we report on our experiment in putting together two types of mathematical arguments, proofs by contradiction and constructive proofs, in the mathematical subject, called Nash's strategy stealing proof in the game of Hex. Both of these two types of proofs are very common and important but have opposite characters in mathematics and mathematical thinking. Usually students are not used to indirect way of mathematical reasoning, like proofs by contradiction. It is the main goal of this experiment to inspire students by showing the power of proofs by contradiction in playing the game Hex.

INTRODUCTION

One interesting question we raised is: whether we can inspire students in learning abstract mathematical reasoning by playing mathematical games in classes. To answer this question, we choose the game of Hex in our experiment. In applying this mathematical game to teaching, we found that there exist at least three general purposes: (i) Connections; (ii) Reasoning and Proof; (iii) Problem Solving. In this article we only discuss the aspect of part (ii).

In our experiment, we try to put together two types of mathematical arguments, *proofs by contradiction* and constructive proofs, in a mathematical subject. In mathematics and mathematical reasoning, both of these two types of proofs are very common and important, but have opposite characters. It is an interesting task to introduce the more abstract way of thinking, *argument by contradiction*, so that students would naturally accept this indirect but powerful approach in analyzing problems.

In our classes, we first introduce the game of Hex, its invention, rules. Then we let students play each other on several different sizes of Hex boards, usually from 5 x 5-boards to 11 x 11-boards. At this stage, some students had found that the first player could always win when the board size is 5 or 6. After more evidence in their plays, we let students discuss to find a winning strategy on Hex boards of small size. This activity will give students a lot of fun and challenging questions to solve. Students usually at this stage would have been warmed up to see the beauty and power of *Nash's strategy stealing proof*. In fact, many students were amazed by Nash’s abstract way of reasoning. It is exactly the main goal of this experiment to impress students by showing the powerful approach of *proofs by contradiction* in playing the game of Hex.

Hopefully, such kinds of experience would motivate students to apply and learn more this way of mathematical thinking. Moreover, this example provides students a good story for comparison and interaction between constructive proofs and proofs by contradiction.

BACKGROUND AND SETTING

The game of Hex is a very intriguing mathematical game. It is always great fun to play for two persons, often Black vs. White like Go. It was invented independently by Peit Hein (a Danish mathematician, designer, and Poet) in 1942, and by John Nash (an American mathematician who was awarded the Nobel prize in economics in 1994) in 1948. Later, it was introduced to general public by Martin Gardner in *Scientific American* in 1959, see Gardner (1957). Each game of play often takes 10 minutes on an 11 x 11-board size. Players put their pieces on empty hexagons of the board in turn. Just like Go, the first turn is usually Black's. Black wins as she/he connects the opposite black sides of the board with a chain of black pieces. Similarly, White wins when she/he connects the opposite white sides of the board with a chain of white pieces.

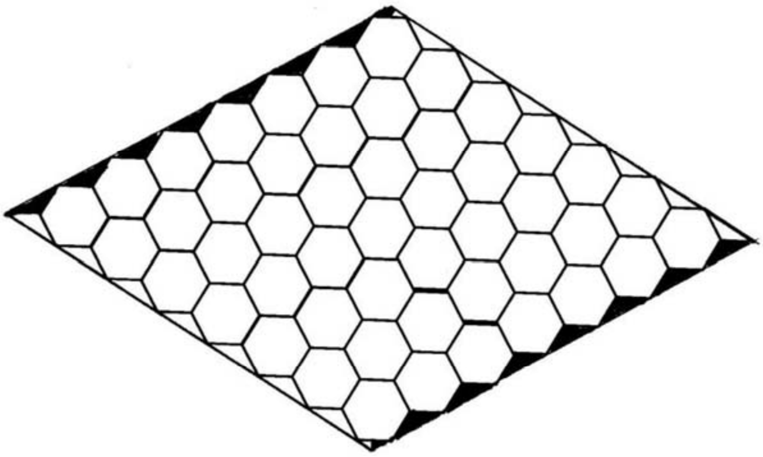

Figure 1: The Hex Board of Size 7×7

One of the interesting characters of this game is that it would never end in a draw. In other words, one of two players must win even if none of them want to win. It is obvious that the winner is unique. This property makes Hex to be a very satisfying game. In particular, most popular games, including (Chinese) Chess, Go, do not have this property. In fact, there is a mathematical proof for this property: the existence of unique winner (or in other words *the game of Hex cannot end in a draw*). Piet Hein stated this property in his first presentation of the game. A rigorous proof resembling Hein's intuitive ``proof'' was given by David Gale in Gale (1979), which is based on graph theory and shows that this property is essentially Brouwer Fixed Point Theorem.

The other interesting property is the existence of a winning strategy for the first player of Hex (or in other shorter words *the first player can always win*).

According to the author's knowledge, the proof of this property was first given by John Nash, which was also called a *strategy stealing proof*, see Milnor (1995) or Kuhn et al (2002). Namely, assume that there is a winning strategy for the second players in Hex, then the first player pretends to be second player so that the first one just applies the winning strategy to his own game. Since this argument implies occupying a cell which already occupied, following the winning strategy he should just make any arbitrary move. Because no cell can be a disadvantage, this strategy will give the first player a win, which contradicts the initial assumption. Therefore, there must exist a winning strategy for the first player.

Nash’s proof naturally motivates the following question: *does there exist a sufficient fast algorithm to find a winning strategy for the first player of Hex on any board size of n x n?* In fact this question has been a challenging problem in computer science, see Anshelevich (2002). Since Nash's proof is not a constructive one, a proof by contradiction, it does not provide useful information in finding a winning strategy. Moreover, due to the great branching factor of Hex, searching for an algorithm of winning strategy might not be done in polynomial time. In particular, Hex may not even belong to a NP-problem, but a class of much harder problems --- the so-called PSPACE (for a quick introduction, see Maarup’s website: http://maarup.net/thomas/hex/). For example, to find a winning strategy on a Hex board of 7 x 7 size is already a very complicated case to analyze and a very time-consuming task for computers like PC, see Anshelevich (2002).

Our experiment

Although Nash's strategy stealing proof is simple and beautiful, it is not easy for most people to come up with this idea of proof by themselves. Moreover, without believing and understanding Nash's proof, it is not clear why a winning strategy must exist.

To let students believe in the existence of winning strategy, we start from the cases of n=2, 3, 4, which are quite easy to be explained and analyzed. I also designed some puzzles in these cases for students to learn some basic techniques in playing Hex. Besides, I used the software, Hexy, developed by the Russian mathematician Anshelevich to demonstrate some techniques and to play with students on a projector. Students were often attracted by this computer game. After learning the basic techniques, I let students to play each other so that they will be more familiar with the game on several different sizes of Hex boards. The size of Hex boards ranges from 5 x 5 to 14 x 14.

Students would gradually find that on a smaller board size, like 5 x 5, or 6 x 6, the first player is indeed easier to win. However, playing on a board of bigger size, like 14 x 14, the winning advantage for the first player is not clear. I took this common feeling of students to question students if they believe that the winning advantage for the first player does really exist or not.

It is then a good time for students to find out a winning strategy on the board size n=5, which is a good exercise in classes. I let students to find out the winning strategy in groups, as the same ones when they played. Besides, I'd like to mention that after playing the game it is easy for students to feel that a proof by induction seems not possible to work. Moreover, I didn't let students discuss the case of n=6 in classes, but left it as a project for motivated students to explore after classes.

Finally, I demonstrated *Nash's strategy stealing proof*. I showed how this proof motivates more interesting problems in computer science as mentioned above.

DISCUSSION AND CONCLUSION

For the purpose of introducing Nash's proof, it is efficient to give students hints step by step so that they can re-discover Nash's strategy stealing proof by themselves. However, in this approach students would lose the opportunity to compare the two types of arguments, constructive proof and non-constructive proof. Both of these two types of arguments are very common in mathematics. A comparison between these two, showing their advantage and disadvantage, would inspire students in learning mathematical reasoning, especially *proofs by contradiction*.

In our experiment, we first made students (i) believe that there does exist a winning strategy for the first player, then (ii) find out a winning strategy when the board size is between 2 and 6, and finally (iii) enjoy and appreciate the beauty and power of Nash's strategy stealing proof.

We would like to mention that there also exists other aspect of using the game of Hex in mathematics classes, for example, in introducing various types of fixed point theorems (like the so-called *Brouwer Fixed Point Theorem*) or in lecturing *game theory*. We would like to discuss these experiments in the future.

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PROOF AND PROVING IN A MATHEMATICS COURSE FOR PROSPECTIVE ELEMENTARY TEACHERS

Jane-Jane Lo, Raven McCrory

Western Michigan University & Michigan State University, U.S.A.

In this study, we propose a fourth element of proof to supplement Stylianides' (2007) definition and we define three levels of proving activities in mathematics courses for prospective teachers. Class episodes from one instructor are used to illustrate how these three levels are instantiated and how the four elements of proof emerge in his attempts to teach mathematical argument and proving to future elementary teachers.

INTRODUCTION

The mathematics education community worldwide is facing the challenge of improving students' ability to prove and reason mathematically at all grade levels. Prior research has shown that young children can make legitimate mathematical arguments and even formal arguments that count as proof (Maher & Martino, 1996; Stylianides, 2007). Yet, studies have also shown that many prospective and practicing elementary teachers hold a procedure-based view of mathematics and to verify mathematical statements, they rely on external authority (textbook or instructor) or accept a few examples as evidence of truth (Simon & Blume, 1996). One way to address this weakness is through mathematics courses specially designed for prospective teachers.

In his study of elementary classrooms, Stylianides (2007) defines proof as follows:

> Proof is a *mathematical argument,* a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
>
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community;
>
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291).

Stylianides uses this definition to analyze instruction involving proof and illuminate possible actions teachers may take to support proving activities in their classrooms. We propose that in order to understand the proving activities in mathematics courses, we need to include a fourth element in the definition:

> 4. It is relative to objectives within the context (*context dependence*) which determine what needs to be proved.

This fourth element of proof is related, but not identical, to using a *set of accepted statements*. In one case, it may be adequate to prove that the result of adding 2 and 3 is 5 by counting objects; in another, the proof may require showing that the addition operation is in fact the correct operation to use. An example below will illustrate this point more clearly.

In this study, we build upon Stylianides work and seek to understand proving activities in mathematics courses for prospective elementary teachers. We propose a three-level hierarchy of proving activities in mathematics courses for prospective teachers. Class episodes from one instructor will be used to illustrate how these three levels are instantiated and how the four elements of proof emerge in his attempts to teach mathematical argument and proving to future elementary teachers.

THEORETICAL FRAMEWORK

Researchers have identified three major roles for proof: to prove, to explain, and to convince (Hanna, 1990; Hersh, 1993). In the context of K-12 mathematics, there is a growing consensus that proof should not be taught as a meaningless exercise that is used only to establish formally the truth of a statement, but should also be taught as a tool that can be used to explain why a statement is true. The definition of proof proposed by Stylianides (2007) quoted earlier reflects this viewpoint.

Research on mathematical knowledge for teaching (cf., Hill, Ball & Rowan, 2008) as well as the conceptualization of proof outlined briefly outlined above suggest that prospective teachers need to learn more than just constructing a valid mathematical proof. They also need to be able to connect such understanding to issues related to their students' conceptions, curricular materials, and instructional techniques that might arise from proving activities in elementary classrooms. A better articulation of the goals for learning about proof in mathematics courses for future teachers is required.

GOALS FOR PROVING ACTIVITY IN MATHEMATICS COURSE FOR FUTURE TEACHERS

We posit that future elementary teachers need to learn proof at three different levels:

a. Proof as a mathematical tool or technique for showing or verifying that something is true (or false). This is what we might teach in a high school geometry class, or what we might expect a student in a number theory class to do.

b. Proof as a mathematical object with particular characteristics and standards. This includes making explicit the steps of a proof, the representations used, the assumptions on which the proof is based, and all three of Stylianides elements of proof. It implies explicit discussion of proof itself, rather than only using proof to show that something is true.

c. Proof as developmental with the level of assumptions, arguments and representations depending on the students' age and grade level. A future teacher needs to think about what children at a particular grade level could be expected to know that could be used in a proof; what kind of arguments they are capable of making; and what kinds of representations they can use. This knowledge needs to be developed explicitly, and it draws on knowledge of content, students, teaching and curriculum, cutting across nearly all aspects of knowledge for teaching. The developmental nature of proof is related to the 4^{th} part of the definition of proof given above, but the context dependence of proof is not only about developmental level; it is also about goals and objectives of any given proving activity.

These three levels correspond roughly to doing proofs, understanding the nature of proof, and adapting the concept of proof to different developmental levels, all important for mathematics teachers at any level of education.

In the remaining part of this paper, we will use episodes from a lesson on divisions of fractions taught by one of our case study instructor, Pat, to illustrate the usefulness of the above conceptualization.

Episodes from A Lesson on Divisions of fractions

The teacher, Pat (a pseudonym) presented the following word problem to the class:

> A batch of waffles requires ¾ of a cup of milk. You have two cups of milk. Exactly how many batches of waffles could you make?

In a classroom where the instructional focus is on procedure, an explanation such as the following may be accepted as "proof" by the classroom community.

> (1) This is a division problem. I divide 2 by 3/4. To do so, I multiply 2 by 4/3. 2 times 4 is 8. 8/3 is the same as 2 2/3. So the answer is 2 2/3 batches.

In his classroom, Pat gave explicit instructions for his students to draw pictures to model their solution process, to explain every quantity, diagram, and step in their reasoning with reference to the original problem, and to come up with a number sentence that matched the given problem. These steps were routine in his classroom, part of the *modes of argumentation* and *modes of argument representation* that were common to this class as illustrated by the following accepted proof for the correct answer 2 ¾ batches that was the result of a collective classroom effort.

> (2) 2 cups of milk is equivalent to 8/4 cups. Two ¾ cups make 6/4 cups. 8/4 – 6/4= 2/4. So I know I can make at least 2 batches. The remaining ½ cup is equivalent to 2/4 cup and can make 2/3 of a batch. Thus the answer is 2 ¾ batches.
>
> The following picture was drawn on the board to support this proof.

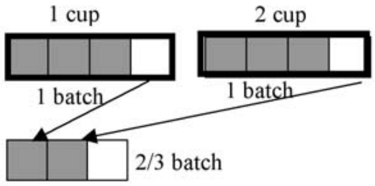

Figure 1: Student explanation

The explanation in Figure 1 was agreed upon as a correct explanation of the right answer, but the discussion of the problem did not end here. Pat asked the class to think about what was wrong with writing 2 1/2, the answer given by some students and a common error made in this type of division problem. Pat's question illustrates the *contextual dependence* of proving. In this instance showing that a wrong answer was wrong became as important as showing that the right answer was right. Several issues emerged during the subsequent discussion. For example, one student pointed out that the unit for 2 was batches (of cookies) but the unit for ½ was cup (of milk), so they could not be put together. Mathematically, only quantities of the same unit could be added together or subtracted from one another. Furthermore, Pat drew attention to the fact that in the picture, "two shaded squares" (at the bottom) were used to represent both two "1/4 cups" and two "1/3 batches". In other words, each square could be conceptualized as ¼ cup or 1/3 batch depending on what was being counted as the whole.

Pat pushed students further to write "a number sentence that matches the problem you were asked to solve". He aimed at getting students to justify their choice of operation. He drew upon prospective teachers' experience working with elementary students on one-step word problem involving whole numbers:

> Pat: When you worked with your kids, you said something like … you have 3 fish bowls, each bowl has four fishes in it. How many fishes are there altogether? The kids would take 4 blocks, 4 more blocks and 4 more blocks, and go 1, 2, 3, 4, 5… Okay 12, there are 12. When you said, write a number sentence, what do you want the kids to write?
>
> Students: 3x4=12.
>
> Pat: 3x4=12 so that's what you want them to write."

Students offered several number sentences for the cookie problem, including 2 x 4/3. Pat pushed for a sentence that included only the numbers in the problem, resulted in the right answer, and could be justified within the context of the problem. These were standard *modes of argument representation* in Pat's class.

After the sentence 2 ÷ 3/4 was offered, he asked "Why is it a division? If it is division, what is it about this problem to make it a division?" This was not a trivial task for the students. The first attempt was essentially a description of steps to come up with the answer 2 2/3, that did not address the "why division" question. Another student set up an algebraic equation: ¾ × x = 2, as number of cups per batch times the number of batches equaled to the totally number for the cups. To solve for x, one had to divide ¾ from both sides that led to 2 ÷ 3/4. This was not accepted by Pat because it was not a mode of reasoning accessible to elementary students. This is another example of the *contextual dependence* of proof:

although this explanation undoubtedly worked for the students in Pat's class, it was not acceptable given the objectives of the lesson. Finally, the idea of measurement division was offered: giving the size of the group (e.g. 3/4 cup), how many groups of that size can be formed from the given quantities (e.g. 2 cups)?

The discussion could have ended right there if the sole purpose of the discussion had been to justify that division was the correct operation to model the given word problem. But Pat continued, drawing his students' attention to a related word problem: "A batch of waffles requires 3 cups of milk. You have 8 cups of milk. Exactly how many batches of waffles could you make?" He pointed to similarity in terms of the physical action one might take to solve these problems (e.g. repeated subtraction) as well as the pictorial representations (e.g. 8 separated boxes to represent "8" or 2 sets of four connected boxes to represent "8/4"). The lesson ended with Pat reminding the class that the challenge they themselves faced when coming up with a number sentence to model a fraction division word problem was of the same nature of the challenges elementary students faced when they tried to abstract the operational sense of division from their physical modeling activity in the whole number context.

DISCUSSION

In this paper, we built on the definition of proof proposed by Stylianides (2007) in an attempt to characterize the proofing activities in mathematics courses for prospective elementary teachers. We found that it was necessary to consider the goal of the activity in order to determine what will be accepted as a mathematically valid proof by the classroom community.

In Pat's class we see examples of the construction of proof using all four of the elements from Stylianides (as modified by our 4^{th} element). We also see the three levels of proof: At level 1, students constructed a proof that 2 2/3 was the correct solution to the problem. At level 2, Pat drew explicit attention to the nature of proof when he set up the requirements for an acceptable argument. In essence he told his students in this episode and at many other times during the semester that a proof must account for every number in the problem and solution, must make sense within the context of the problem, and must result in a correct answer. At level 3, Pat made explicit the developmental nature of proof, rejecting an algebraic proof because it would not work for the future students of these future teachers. Level 3 was again apparent when he used a simpler example to emphasize what young students might experience in their proving activities.

Analysis of this episode has illustrated one instructor's effort to achieve multiple goals through proving activities. By drawing students' attention to various issues such as modes of representations, modes of reasoning, common student errors, and the abstraction involved from whole number to fraction operations, Pat provided his students with ample opportunities to develop mathematical knowledge that would be needed for engaging elementary students in reasoning-based proving activities. Future studies are needed to examine the effect of such explicit attention on students' ability to do proof, to understand the nature of proof and to connect their content knowledge of proof to the knowledge of students, teaching, and curriculum as they continue their professional path.

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TEACHERS MATHEMATICAL REASONING DOES MATTER

Nancy Lacourly, María Leonor Varas

Centre for Advanced Research on Education and
Centre for Mathematical Modelling, Universidad de Chile.

This paper presents the results of a study that assessed the quality of mathematical instruction and other precedents in teachers, students and schools, as well as its impact on student learning. It proves that, both the teacher's knowledge and the quality of classroom activities are significant predictors of the students' understanding and capacity for applying the Pythagorean Theorem. This impact is superior to other variables, including the performance of the students in a previous test.

INTRODUCTION

In Chilean schools, proofs are very seldom incorporated in math lessons. On the contrary, the inquiry methodology for introducing a new idea is the most popular, as it is supposed to allow the students to 'discover' important results by themselves. Such is the case with the Pythagorean Theorem that is taught at the 7th grade. As this grade is part of elementary school, the teachers in charge of teaching math at this level are frequently all-purpose teachers instead of specialists in mathematics.

According to the codification used by the TIMSS 1999 Video Study (TIMSS 1999) 'inquiring' forms part of 'mathematical reasoning' and its use should promote a deeper reflection and understanding. This incorporation of inductive thinking – being beneficial – can also confuse the students or even teachers about the value of the deductive method and its inevitable role in mathematics. With this misunderstanding, it is very frequent to 'discover' theorems without any warning about the limitations of an unproven conjecture. Such lack of precision can have important consequences at the level of generality that this theorem is supposed to possess. Furthermore, not having any preparation with the scientific method, it is rare for the teacher to apply the rules of 'experiment design' to correctly plan the trial activity. For this reason, the conclusions can turn up even more dubious. An example: if the goal is to discover the Pythagorean Theorem through paper cuttings to compare the areas of the square built on the sides of the triangle, it is convenient to try it with various types of triangles (obtuse triangles and acute triangles, as well as right triangles), in order to connect the property with the presence of the right angle.

Even when no formal proof is performed in math classes, the teaching of a theorem can vary among teachers with different degrees of knowledge about a particular proof, and about the role of proofs in general. The understanding that the students can achieve from this result will be necessarily limited if the teacher can't explain why the Pythagorean theorem is named like that, when it was used long before Pythagoras was born.

How much of the teacher's knowledge is brought to learning?

The Swiss-German 'Pythagoras' study (Klieme 2004) (paper in preparation), proved that the quality of the proof that the teacher conducts in classes is a powerful predictor of learning achievements. This study assessed the quality of various aspects of mathematical instruction, by video observation of introductory classes of the Pythagorean Theorem in 40 different courses distributed between Switzerland and Germany, in which the inclusion of a proof was requested.

The Chilean study of the same name used the instruments of the European study and added two elements: a test for directly assessing the knowledge of teachers and, in assessing the teaching quality, added a scale on the quality of the inquiry activity. It proved that both factors effectively predict the learning gain for the students and that its impact is greater than the academic training of teachers.

METHODOLOGICAL ASPECTS

The results presented here are part of the larger study mentioned above (Varas 2008), which assessed the quality of mathematics instruction and contrasted this with the learning outcomes of children from the 7th grade. The analysis was conducted according to the methodology used by the 'Pythagoras' international comparative study, developed by the Institute for International Pedagogical Research of Germany (DIPF) and the Institute of Education of the University of Zürich (Switzerland), adapting and validating their instruments as well as developing new ones.

Our study included video-recording of the first three lessons on the Pythagorean Theorem and taking several tests and questionnaires along the 2007 school year. 802 students and 21 teachers were part of those classes.

As in the European case, the teachers were asked to include a proof. Using multilevel analysis, we try to link the learning gain with individual factors of the student and with characteristics of the teacher, instruction or school. At the first level –individual– test results of knowledge and skill in geometric visualization, beliefs and valuations are considered. At the group level we account for the type of school, the teacher's academic preparation, quality of the classes, knowledge, beliefs and valuations of mathematics and its teaching.

In the European study a central assumption is that the quality of the taught proof increases a pupil's understanding of the Pythagorean Theorem, as well as the ability to apply it (Lipowsky 2005).

In the Chilean case, teachers failed to incorporate a proof of this theorem and it was necessary to apply them a test to know what their level of knowledge was in this regard. The majority of the teachers believed to have made a proof through inquiry activity. Furthermore, instructional practices observed in the videos of classes were not conducive to mathematical reasoning in any of its expressions. The most popular activities of inquiry, designed to 'discover' the Pythagorean Theorem, fail to make their contribution to the development of reasoning. This is due to the avoidance of all aspects of distinction between conjecture and mathematical truth, thesis and assumptions, anecdote and generality. These aspects were recorded and coded with a quality indicator of the inquiry activity. The codification of the videos was made by three trained experts, with high inter-rater reliability.

In this paper we analyze the models that explain the performance of students in a post-test, conducted after the three classes were filmed, with variables of the student and variables of the teacher, school and course. The student variables we consider are the test of the beginning of the school year and the ability for geometric visualization. The considered group level variables are:

1. The teacher's knowledge about proofs of the Pythagorean Theorem and the role of proofs in math.

2. Quality of the inquiry activity with which they present or 'make-to-discover' the Pythagorean Theorem registered in the videos.

3. Level of training or preparation of the teacher: elementary school teacher or high school math teacher.

4. Type of school: public or private.

5. Average of the class in a test, to consider the 'peer effect' produced in a pupil by the academic quality of its companions.

The information collected was submitted to analysis of reliability and consistency of the dimensions assessed, through factor analysis, Cronbachs-alpha and item-test correlation. The hierarchical-linear models allow the study of the impact that the characteristics of teachers could produce in the learning outcomes, controlling the previous knowledge and skills at the student level.

The results are also conclusive in this case. The first two variables are always contributing significantly to the learning profit, while in the contrary; the type of school rarely makes it significant.

RESULTS PRESENTATION AND ANALYSIS

At the end, all the relevant statistical information from seven models is summarized in a table, which shows various combinations of the above factors. In all of them, the dependent variable is the results of the post-test, which assesses the understanding of the Pythagorean Theorem and its direct applications.

The simplest multilevel model is of the following type:

$$Y_{ij} = \beta_{oj} + \beta_{1j} X_{ij} + r_{ij}$$

$$\beta_{oj} = \gamma_{oo} + \gamma_{o1} Z_j + u_{oj}$$

$$\beta_{1j} = \gamma_{1j} + u_{1j}$$

Where Y_{ij} is the result of the student i, of teacher j; X_i is an explanatory variable of the same student and Z_j is an explanatory variable of teacher j. The Greek letters correspond to the parameters that the model fits in and the remaining terms are terms of random errors, some of which can be removed when its variance is not significant.

The criteria used to select these models consider the reliability, the significance of the parameters and the percentage of variance explained by the model (multiple correlations).

The selected models have a percentage of explained variance higher than 50% and a good significance of the parameters. Only in models 1 and 3 the reliability is poor, which is succeeded in models 2 and 4, by deleting the random term from the constant. It is noteworthy that in this type of study, one cannot expect a high percentage of explained variance due to the small size of the sample of teachers – result of the cost of such experiments – and the absence of some explanatory variables that were not within our scope, such as socio-economic variables.

We choose to present several models instead of one model with more predictive variables, because the model that could contain all the predictive variables does not increase the percentage of explained variance regarding each of these seven models, and the parameters lose significance.

The first test is not as significant as the variables of the teacher. In fact, its correlation with the post-test is only 20%.

The quality of the teacher – both in preparation, and in knowledge and quality of the taught mathematical reasoning – is what best predicts the results of the students, even more so than the level at which they are found at the beginning of the school year. The teacher makes the biggest difference.

We point out the consequence of these findings for the teacher's preparations program. It is recommended to include the teaching of proofs to teacher students, even in countries where proofs are not supposed to be taught at school.

Summary of models output

			Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7
Level 2	Teacher knowledge	T-value	5,660	4,492	4,863	4,019	2,284	2,645	2,094
		p-value	*0,000*	*0,000*	*0,000*	*0,000*	*0,037*	*0,017*	*0,053*
	Quality Inquiry activity	T-value	2,334	2,324	--	--	--	--	--
		p-value	*0,034*	*0,002*	--	--	--	--	--
	Teacher preparation	T-value	--	--	3,072	3,194	3,091	5,232	--
	Elementary/ Math	*p-value*	--	--	*0,009*	*0,002*	*0,008*	*0,000*	--
	School type	T-value	--	--	--	--	--	--	-2,010
	Private/Public	*p-value*	--	--	--	--	--	--	*0,062*
	Average Postest	T-value	13,247	10,703	13,451	11,250	--	--	14,460
		p-value	*0,000*	*0,000*	*0,000*	*0,000*	--	--	*0,000*
	Average first test	T-value	--	--	--	--	-0,032	--	--
		p-value	--	--	--	--	*0,975*	--	--
	Intercept	T-value	-13,486	-10,748	-13,764	-11,537	--	--	-8,536
		p-value	*0,000*	*0,000*	*0,000*	*0,000*	--	--	*0,000*
Level 1	First test	T-value	1,924	1,981	1,993	1,833	1,092	1,894	--
		p-value	*0,070*	*0,064*	*0,062*	*0,070*	*0,074*	*0,074*	--
	Geometrical ability	T-value	--	--	--	--	--	--	3,098
		p-value	--	--	--	--	--	--	*0,007*
	Intercept	T-value							
		p-value							
Variance component	Intercept variance	Value	0,006	--	0,007	--	0,213	0,185	0,283
		p-value	*0,382*	--	*0,291*	--	*0,000*	*0,000*	*0,018*
	Slope variance	Value	0,018	0,016	0,018	0,018	0,034	0,033	0,0002
		p-value	*0,001*	*0,014*	*0,001*	*0,012*	*0,001*	*0,001*	*0,055*
	Residual variance	Value	1,162	1,167	1,180	1,181	1,180	1,164	1,071

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GENERIC PROVING:
REFLECTIONS ON SCOPE AND METHOD

Uri Leron and Orit Zaslavsky

Technion – Israel Institute of Technology, Israel

We analyze the role of generic proofs in helping students access difficult proofs more easily and naturally. We present two examples of generic proving – a simple one on numbers and a more advanced one on permutations – and consider the strengths and weaknesses of the method by reflecting on these examples. A classroom scenario a-la Lakatos is used to bring out the role of the teacher in generic proving. Finally, we speculate on the question: Which proofs are more or less amenable to generic proving?

A generic proof is, roughly, a proof carried out on a generic example. We introduce the term *generic proving* to denote any educational activity surrounding a generic proof. Our paper is organized as reflection on two examples of generic proving, one simple and elementary and the other more advanced.

TWO EXAMPLES OF GENERIC PROVING

Theorem: A natural number which is a perfect square (i.e., the square of another natural number) has an odd number of factors.

For example, the number 16 has 5 factors (namely: 1, 2, 4, 8, 16), and 25 has 3 factors (namely: 1, 5, 25).

Generic Proof: Let us look for example at 36 (a perfect square). We want to show that it has an odd number of factors. There are several ways in which 36 can be written as a product of two factors. We systematically list all such factorizations: 1×36, 2×18, 3×12, 4×9, 6×6.

All the factors of 36 appear in this list. Counting the factors, we see that the factors appearing in all the products, except the last, come in pairs and are all different, thus totaling to an even number. Since the last product 6×6 contributes only one factor, we get a total of odd number of factors. Specifically, for the case of 36, we have $2\times4 + 1 = 9$ factors.

For our second example we have chosen a theorem from group theory: *Every permutation has a unique decomposition as a product of disjoint cycles*. (These terms will be explained as the proof folds out.) In order to highlight both the mathematical and educational aspects of generic proving, we will present the theorem and its proof via a classroom scenario in the style of Lakatos (1976). As in Lakatos (and to accommodate severe space limitations), the scenario will be highly idealized and abridged. In realistic classrooms we are not likely to meet such bright and motivated students who ask all the right questions. Still, we believe that a similar scenario can happen in realistic classes, except that it will be more meandering and will require more time and effort.

In their previous lesson, the students in this scenario had already learned and practiced the definition of a permutation (a one-to-one function of the set {1, 2, … , n} onto itself), and the 2-row notation for permutations. They have also learned when two permutations are equal (i.e., when they are equal as functions), and how to multiply permutations (i.e., compose them as functions).

[1] Teacher: Let us look at an example of a permutation[1] and see if we can find anything interesting about its structure – how it can be constructed from simpler permutations (somewhat like how numbers are constructed from their prime factors).

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 1 & 2 & 4 & 7 & 3 & 5 \end{pmatrix}$$

For example, let's start at 1, and follow its path as we apply the permutation over and over again, thus: $\sigma: 1 \to 6 \to ...$

[The students work in teams, continuing what the teacher has started: $\sigma: 1 \to 6 \to 3 \to 2 \to 1.$]

[2] Alpha: It came back to 1! There is no point going on, since it will just repeat the same numbers.

[3] Teacher: Right. This part of the permutation is called *a cycle*, and is written (1 6 3 2). It is a special kind of permutation, in which each number goes to the next one on the right, except the last one, which goes back to the first number in the cycle. (The numbers that don't appear in this notation are understood to be mapped to themselves.) Note that the same cycle can also be written as (6 3 2 1), (3 2 1 6), or (2 1 6 3) since they are all equal as functions.

Let's see if we can find more cycles in our permutation. The letters 1, 2, 3 are already used up, but 4 is not, so let's repeat the same game starting with 4.

[The students work in their teams to find the path of σ starting at 4.]

[4] Beta: 4 goes to itself; we cannot construct a cycle.

[5] Teacher: Since we see that $4 \to 4$ (σ leaves 4 unchanged), we write this as (4) and call this a *trivial cycle* (or cycle of length 1). It is equal to the identity function I. What do we do next?

[6] Students: Construct the cycle starting at 5 (the next unused letter).

[The students construct the path $5 \to 7 \to 5$, and the corresponding cycle (5 7).]

---

[1] Reminder: In this graphical representation of the permutation, the numbers in the bottom row are the images of the corresponding numbers in the top row. Thus $\sigma(1) = 6,\ \sigma(2) = 1,\ \sigma(3) = 2$, etc.

[7] Students: Now all the numbers 1,2,3,4,5,6,7 are used up, we can't construct any more cycles.

[8] Teacher: Right. We can't and we needn't; we have now found all the cycles of our permutation. In fact, if we recall the definition of permutation product (as function composition), we can see that our original permutation is actually equal to the product of its cycles!

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 1 & 2 & 4 & 7 & 3 & 5 \end{pmatrix} = (1\ \ 6\ \ 3\ \ 2)(4)(5\ \ 7)$$

How do we know this? Take 1 for example. You can see that on both sides 1 goes to 6, and similarly for all other letters. (This is not surprising: it's how we constructed the cycles.) Hence the permutations on the two sides are equal as functions. Notice that no number appears in two (or more) cycles on the right-hand side. The cycles are therefore said to be *disjoint*.

We summarize our work so far by saying that the permutation *has been decomposed as a product of disjoint cycles*.

[9] Gamma: Can we always do this? Can we decompose *any* permutation as a product of disjoint cycles?

[10] Teacher: (to the class) Well, what do you think?

[11] Gamma: Why shouldn't we just repeat the same process for any permutation?

[12] Delta: Wait a minute! What if this procedure didn't work? We were lucky that 2 went back to 1 in the first cycle, but what if it didn't? What if it went back to 6 for example? Then we wouldn't have a *cycle*.

[13] Teacher: If we had 2 going to 6, and earlier we also had 1 going to 6, then we would have both $\sigma(1)=6$ and $\sigma(2)=6$. Is this possible?

[14] Epsilon: No, this is impossible, because a permutation is a one-to-one function so we can't have $\sigma(1)=\sigma(2)$.

[15] Teacher: That's correct, therefore our procedure will always yield *cycles*. For similar reasons, we can't have the same letter appear in two different cycles, because this too would violate the one-to-one property of the permutation [we skip some technical details here]. This guarantees that our procedure will generate *disjoint* cycles.

[16] Teacher: (summarizes) We have just constructed together a generic proof of the theorem: *Every permutation can be decomposed as a product of disjoint cycles*.

From here, the classroom activity could continue in several directions, including a discussion leading to a generic proof of the *uniqueness* of this decomposition. An excellent homework assignment for advanced students could be to formalize and generalize the foregoing generic proof to show that the theorem holds for *any* permutation.

REFLECTIONS ON SCOPE AND METHOD

Reflecting and generalizing from these examples some important and interesting issues emerge. We list these as questions with brief hints at possible answers. We hope to discuss these questions in more length and depth at the conference.

Q1. *What is a good generic example in the context of a generic proof?*

GPBA (the Gist of a Possible Beginning of an Answer): An example simple enough to be easy and familiar for the students, but complex enough to be free of distracting special features, thus having the potential of representing the "general case" for the students. In Mason and Pimm's terms (1984), a generic example should allow us "to see the general through the particular". Movshovitz-Hadar (1988) asserts that a generic example should be "large enough to be considered a non-specific representative of the general case, yet small enough to serve as a concrete example"; We suggest that "size" should be replaced by a measure of the *complexity* of the example. For example, we consider 36 a good generic example for a generic proof of the "perfect square" theorem above, while 4, 16, 25 or even 169 ($= 13^2$) would have been too special (e.g., have too few factorizations). In a similar vein, for the cycle decomposition theorem, we have chosen as generic example a permutation on 7 digits, having cycles of lengths 1, 2 and 4. A shorter permutation on 6 digits would have been possible, with cycles lengths of 1, 2 and 3, but we deemed that orderly sequence to be too special and possibly misleading.

Q2. *What are the strengths of generic proofs?*

GPBA: They enable students to engage with the main ideas of the complete proof in an intuitive and familiar context, temporarily suspending the formidable issues of full generality, formalism and symbolism. While a complete formal proof may be beyond the reach of almost all school children (e.g., Healy & Hoyles, 2000; Stylianides, 2007), we could imagine a classroom activity whereby even elementary school children learn the generic proof of the perfect square theorem, and produce their own versions for other examples. Indeed, they would most likely feel that they were carrying out *the same* proof. In complicated proofs, it is possible to build up the complexity gradually, via a chain of successively more elaborate *partial* generic proofs, each highlighting finer points of the proof that were not salient in previous steps.

Q3. *What are the weaknesses of generic proofs?*

GPBA: The main weakness of a generic proof is, obviously, that it doesn't really prove the theorem. The 'fussiness' of the full, formal, deductive proof is necessary to ensure that the theorem's conclusion infallibly follows from its premises. In fact some of the more subtle points of a proof may not easily manifest themselves in the context of the generic proof: some steps which "just happen" in the example may require a special argument in the complete proof to ensure that they will always happen. In the generic proof of the cycle decomposition theorem, for example, cycles just "close back" to their first element, and the cycles just "turn out" to be disjoint. In fact, if we had not been careful, we could have completed the generic proof without even mentioning the essential one-to-one property of permutations. Since these crucial issues do not naturally come up during the generic proving, the teacher's initiative here is crucial.

Q4. *Not all proofs are equally amenable to a generic version. Can we characterize the proofs (or parts thereof) that are so amenable?*

GPBA: This fascinating and difficult question is a blend of mathematical and educational aspects. An answer would likely involve the form and structure of the proof, but the effectiveness of a generic version is expressed in terms of its ability to render the main ideas of the general proof accessible to students.

What we can induce from the examples is that if a proof involves an *act of construction* (of a mathematical object or procedure, a decomposition or factorization as in our two examples), then this construction can be effectively presented via a generic example. Often an act of construction in a proof is hidden by the mathematical formalism, but may be revealed by "structuring" the proof (Leron, 1983, 1985), whence a generic version of the proof may become feasible.

But constructions in proofs always have to satisfy several predetermined conditions, and the proof that they indeed satisfy these conditions may not naturally come up in the work on the generic example. In our example above, the decomposition of permutations (the construction act) must satisfy the conditions that the factors are disjoint cycles. In our idealized scenario, the teacher was lucky to have these issues brought up by the students themselves, but in more realistic situations more educational initiative and creativity on the part of the teacher may be required.

Some proofs may not seem on the surface to be amenable to a generic version because of their structure or logical form, or the nature of the mathematical objects involved; for example, proof by contradiction or proofs involving infinite objects. But even in such cases, we can often isolate some constructive element that can be presented via a generic example. We mention three such examples. One, Euclid's proof of the infinitude of prime numbers. The basic construction here (*given any finite set of primes, construct a new prime not in the set*) can be presented via a generic example. Two, Cantor's proof that the real numbers are uncountable, where a new element is constructed by the diagonal method. the diagonal method itself (*given a rectangular table of numbers, construct a row different from all the rows in the table*) can be first introduced via a small finite generic example and then gradually extended to the infinite case (Leron & Moran, 1983). Three, Lagrange's theorem on finite groups: *The order of a subgroup divides the order of the group*. Since this theorem concerns a relation on the collection of all finite groups, it is hard to see at first glance how it can be helped by a generic example. But since the *proof* involves a construction (a partition of the group into cosets) it is possible to devise classroom activities that demonstrate the main ideas of the proof on a generic example (Leron &, Dubinsky, 1995).

In contrast to these examples, we may consider the Heine-Borel theorem from analysis: *A subset of R^n is compact if and only if it is closed and bounded*. Because the theorem and its proof deal with complex logical relations between infinite collections of infinite objects, it is hard to see how they can be effectively demonstrated on an example.

CONCLUSION: WHAT WAS THIS AN EXAMPLE OF?

This paper was in itself an example of a very general method for dealing with complexity in the face of the limited resources of human working memory: the method of *successive refinement*s. In this method – which is prevalent in computer science but could be just as effective in mathematics and mathematics education – one gradually approaches a complicated target system (such as a proof or a software system) via a chain of simpler versions of that system. In the case of generic proving, the chain of successive refinements moves from the specific (an example) to the general, but it is also possible to move along other dimensions, such as from the intuitive to the formal (by gradually formalizing an argument), or from the global to the local (by gradually adding technical details). This method can be very effective in helping teachers and students bridge the difficult gap from common sense to formal mathematics.

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GEOMETRICAL PROOF IN THE INSTITUTIONAL CLASSROOM ENVIRONMENT

Víctor Larios-Osorio

Autonomous University of Querétaro (UAQ), México

Claudia Acuña-Soto

Center for Research and Advanced Studies (CINVESTAV) of IPN, México

Classroom proof in middle school is an important part of mathematics that must be taught, but its approach will depend as much on the reference institution (mathematics) as on the individual school practices. This article, therefore, reflects on this topic to illustrate the current need to define a suitable vision of geometric proof for the classroom that considers (locally applied) rigor as an axis of functioning, clearly distinguishing the need to obtain the mathematical meaning applied to the process in question as well as identifying the role of the teacher, particularly in the use of dynamic geometry software.

INTRODUCTION

Proof in mathematics as the mean of validating knowledge should be incorporated in teaching according to each level, in an equivalent to the paradigm called Geometry II, which Kusniak[1] et al. (2007) define as "Natural and axiomatic Geometry based on hypothetical deductive laws related to a set of axioms close as possible on the sensory reality."

But what should be the required conditions for introducing students to axiomatic proof modulated by sensory reality? Establishing these conditions gives us a motive for reflection which we would like to explore in this work, in particular relating to the role of the institution, in the sense of Godino and Batanero (1994), as common practice-oriented communities and the role of explicit rigor and the construction of inferences in activities to make plausible conjectures in computer environments.

The reference to institutions of *mathematics professionals* and *"teachers" of mathematical knowledge* should be considered because the meaning given in mathematics education to an object of this environment that is determined in principle by whoever gives that meaning in mathematics itself.

The epistemology that has been key to the way proof has been viewed in mathematics enables us to observe the thought structures related to rigor and

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[1] The three proposed paradigms are "Geometry I (Natural Geometry with source of validation closely related to intuition and reality (…). Geometry II (Natural and axiomatic Geometry based on hypothetical deductive laws related to a set of axioms close as possible on the sensory reality). (…) Geometry III (formal and axiomatic geometry) (…) the set of axioms is independent of reality and should be complete in the formal sense". (Kuzniak, Gagatsis, Ludwig, & Marchini, 2007, pág. 955)

validation. Learning the reasons behind proof in these terms enables us to learn mathematics that make sense, supported by mathematics itself.

PROOF AT MIDDLE SCHOOL

Balacheff (1987) makes a distinction between *demonstration* (*démonstration*) and *proof* (*prevue*) in terms of validation so that it can be said that mathematics develops the first while mathematics teachers only handle the second.

Going beyond this distinction, it is important to emphasize the institutional meanings and practices relevant to the mathematical community and the school. This difference is not only an adjustment in the type of validation and logical structure used to obtain it, but also considers the diverse functions that it fulfills in teaching as well as the structure, context of origin and objectives of teaching.

The consideration of these differences is advantageous to teaching because the emphasis of classroom mathematical proof at school **lies not in its structure but in its making sense and having meaning.**

Let's consider that a proof for middle school level is:

1. An implicitly rigorous proof of a mathematical fact.
2. That is based on arguments that have as their primary function to convince (the speaker and those around him or her).
3. To give an explicitly rigorous explanation for such fact.
4. And whose structure is organized based on inference and deductive argument.

By *deductive argument* we mean that which is based on deductive reasoning that has a ternary structure in which the premises, general property and conclusion intervene:

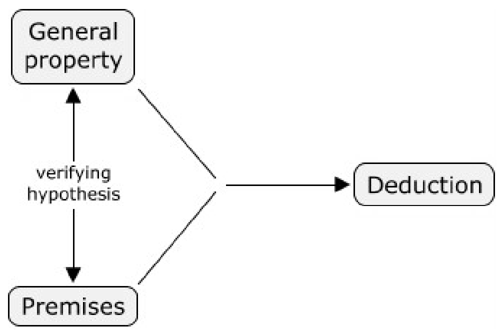

Figure 1: Deductive reasoning schema.

If we take the meaning of proof as a central issue, then we may be supported in the *Cognitive Unity of Theorems* notion (Boero, Garuti & Mariotti, 1996), which suggests the existence of a continuity between the production of a conjecture and the construction of its proof.

In principle, explorations into this continual process would enable the production of a statement to be validated in addition to the arguments that could be used in the process. In middle level teaching, or its equivalent, in the aforementioned Geometry II, a formal proof is not necessarily produced, "but their deductive reasoning shares many aspects with the construction of a mathematical proof" (op. cit. page 126).

ON DYNAMIC GEOMETRY SOFTWARE

The characteristics of dynamic geometry support the exploration of environments with opportunities for new forms of interaction with mathematical objects opening the door to making generalizations from specific cases through use of the drag function and animation.

Nevertheless, the dynamic images in software easily take on evidentiary value for those who handle them leading to the development of the well-known process of transforming evidence into proof and proof into evidence: the result is the inhibition of the need to produce justifications with local deductions or, indeed, proofs.

This problem is fundamentally related to the view being formed of proof and the functions that can be developed in the classroom; if we accept that a proof only explains and convinces regardless of the arguments, then the simple observation of the screen and the use of the tools provided by the software could be enough.

Mariotti and Maracci (1999) appear to go in this direction, partially at least, when they suggest the combined use of software with open questions. They point out that when confronted by this kind of problematic situation an explanation and persuasion is sought, rather than validation, so that, "the resulting argument may be very successful in explaining an answer, but completely inadequate as proof of a conjecture" (page 266).

Occasionally, and for reasons related to the didactic contract, it is necessary to explicitly ask for something to be "proven" (even though it has not yet been observed) so that students feel obliged to proving it and therefore reflect on the mathematical conditions supporting it, although, strictly speaking, the final product is not solely the production of a proof.

Michael de Villiers (1995), when he confronted this situation, found that it is precisely the task of explaining the observation and the intervention of the teacher which helps to build deductive reasoning from the observations made with dynamic geometry software.

FINAL REMARKS

The paradigms proposed by Kuzniak and his collaborators coincide with the cognitive development of individuals and allow us to revisit the idea of institutional practices (Godino & Batanero, 1994) since then **there is a need to define a suitable vision of geometric proof for the classroom that considers (locally applied) rigor as an axis of functioning, clearly distinguishing the need to obtain the mathematical meaning applied to the process in question.**

The teacher is a mediator in the process of cognitive unity, encouraging conjecture in the students, thus bringing them closer to the inferential process typical of mathematical thought, even when it is posed close to the tangible world and from there moves towards the diverse paradigms.

We believe that conjectures are essential because when they become statements to be demonstrated they are accompanied by validation. *Conjectures* and *theorems* are statements with epistemic value and a theoretical statute. In both cases the statements are validated but their processes of validation have distinct characteristics: conjectures are related to semantic aspects and theorems to syntactic aspects. In other words, the differences in the semantic and syntactic aspects are not apparent in the statements themselves, but these aspects determine the corresponding validation methods.

The truth of the conjecture, based on its pertinence and "strength", in reality, contains a high degree of intuitive conviction on the part of the individual since he is convinced that the conjecture is "true" for structural convenience. In actual fact, however, there is only one acceptance that "it can be true".

We propose, therefore, the following schema of the process to be followed:

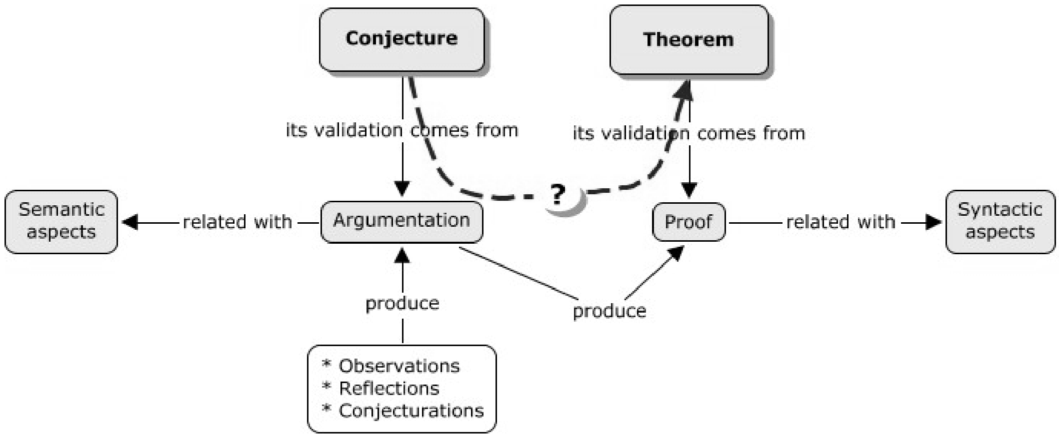

Figure 2: Conjecture-theorem relationship schema.

We maintain that the thesis of cognitive unity of theorems is coherent with this schema since conjecture, the product of exploration, is accepted or rejected through argumentation and observation that enable the construction of the proof. When this last has been performed then the statement which began as a conjecture becomes a theorem.

While there is cognitive continuity in the processes of building conjectures and proving them, it exists in semantic terms (Pedemonte, 2001); not so in the structural (Duval, 1999) and epistemological aspects (Arzarello, Olivero, Robutti & Paola, 1999), since their forms (structure) and the intentions of each of those processes, and their products, are different.

It is undeniable, then, that software can provide students with tools (linguistic, graphic and even inferential) to express proof or answers, since it is a semiotic mediator that influences knowledge building, including language and argumentation (Larios, 2005). This fact makes some observations or justifications refer directly, either explicitly or implicitly, to the software (its commands, characteristics or architecture). This suggests a possible need to emphasize to students that in their observations and proof there should be a distinction between the characteristics of the software and the geometric properties. This, in turn, implies a "purging" of the statements they can propose.

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PROMOTING STUDENTS' JUSTIFICATION SKILLS USING STRUCTURED DERIVATIONS

Linda Mannila and Solveig Wallin

Åbo Akademi University, Finland

Being able to explain the process of solving a mathematical problem is essential to learning mathematics. Unfortunately, students are not used to justifying their solutions as emphasis in the classroom is usually put on the final answer. In this paper, we describe how students can become used to explicate their thinking while solving a problem or writing a proof in a structured and standard format using structured derivations. We also present the results from an analysis of upper secondary school students' argumentation skills from using this approach in a course on logic and number theory. Our findings suggest that the structured derivations format is appreciated by the students and can help promote their justification skills.

BACKGROUND

"Mathematics is not just about identifying the truth but also about proving that this is the case" (Almeida, 1995, p. 171). Learning to argue about mathematical ideas and justifying solutions is fundamental to truly understanding mathematics and learning to think mathematically.

The National Council of Teaching Mathematics (NCTM) issues recommendations for school mathematics at different levels. In the current documents (NCTM, 2008), communication, argumentation and justification skills are recognized as central to the learning of mathematics at all levels.

According to Sfard (Sfard, 2001), thinking can be seen as a special case of intrapersonal communication: " [o]ur thinking is clearly a dialogical endeavor where we inform ourselves, we argue, we ask questions, and we wait for our own response […] becoming a participant in mathematical discourse is tantamount to learning to *think* in a mathematical way" (p.5). Although it is important to be able to communicate mathematical ideas orally, documenting the thinking in writing can be even more efficient for developing understanding (Albert, 2000).

Justifications are not only important to the student, but also to the teacher, as the explanations (not the final answer) make it possible for the teacher to study the growth of mathematical understanding (Pirie & Kieren, 1992). Using arguments such as "Because my teacher said so" or "I can see it" is insufficient to reveal their reasoning (Dreyfus, 1999). A brief answer such as "26/65=2/5" does not tell the reader anything about the student's understanding. What if he or she has "seen" that this is the result after simply removing the number six (6)?

Nevertheless, quick and correct answers are often valued more in the classroom than the thinking that resulted in those answers. It is common for students to be required to justify their solution and explain their thinking only when they have made an error – the need to justify correctly solved problems is usually de-emphasized (Glass & Maher, 2004). As a result, students rarely provide explanations in mathematics class and are not used to justify their answers (Cai et al., 1996). Consequently, the reasoning that drives the solution forward remains implicit (Dreyfus, 1999; Leron, 1983).

In this paper, we will present an approach for doing mathematics carefully, which aids students in documenting their solutions and their thinking process. We will also present the results from the analysis of students' justifications from a course using this approach. The aim is to investigate the following questions:

- How does the use of structured derivations affect students' justifications?
- What advantages and drawbacks do students experience when using structured derivations?

STRUCTURED DERIVATIONS

Structured derivations (Back et al., 1998; Back & von Wright, 1999; Back et al., 2008) is a further development of Dijkstra's calculational proof style, where Back and von Wright have added a mechanism for doing subderivations and for handling assumptions in proofs. With this extension, structured derivations can be seen as an alternative notation for Gentzen like proofs.

In the following, we illustrate the format by briefly discussing an example where we want to prove that $x^2 > x$ when $x > 1$.

•	Prove that $x^2 > x$, when
-	$x > 1$
‖-	$x^2 > x$
≡	{ Add $-x$ to both sides }
	$x^2 - x > 0$
≡	{ Factorize }
	$x(x - 1) > 0$
≡	{ Both x and x-1 are positive according to assumption. Therefore their product is also positive. }
	T

The derivation starts with a description of the problem ("Prove that $x^2 > x$"), followed by a list of assumptions (here we have only one: $x > 1$). The turnstile (‖-) indicates the beginning of the derivation and is followed by the start term ($x^2 > x$). In this example, the solution is reached by reducing the original term step by step. Each step in the derivation consists of two terms, a relation and an explicit justification for why the first term is transformed to the second one. Justifications are written inside curly brackets.

Another key feature of this format is the possibility to present derivations at different levels of detail using subderivations, but as these are not the focus of this paper, we have chosen not to present them here. For information on subderivations and a more detailed introduction to the format, please see the book and articles by Back et al.

Why Use in Education?

As each step in the solution is justified, the final product contains a documentation of the thinking that the student was engaged in while completing the derivation, as opposed to the implicit reasoning mentioned by Dreyfus (1999) and Leron (1983). The explicated thinking facilitates reading and debugging both for students and teachers.

Moreover, the defined format gives students a standardized model for how solutions and proofs are to be written. This can aid in removing the confusion that has commonly been the result of teachers and books presenting different formats for the same thing (Dreyfus, 1999). A clear and familiar format has the potential to function as mental support, giving students belief in their own skills to solve the problem. As solutions and proofs look the same way using structured derivations, the traditional "fear" of proof might be eased. Furthermore, the use of subderivations renders the format suitable for new types of assignments and self-study material, as examples can be made self-explanatory at different detail levels.

STUDY SETTINGS

The data were collected during an elective advanced mathematics course on logic and number theory (about 30 hours) at two upper secondary schools in Turku, Finland during fall 2007. Twenty two (22) students participated in the course (32 % girls, 68 % boys). The students were on their final study year.

For this study, we have used a pre course survey including a pretest, three course exams and a mid and post course survey. The pretest included five exercises, which students were to solve. They were also asked explicitly to justify their results. The surveys included both multiple choice questions and open-ended questions for students to express their opinions in their own words.

For each course exam, we have manually gone through and analyzed three assignment solutions per student, giving us a total of 198 analyzed solutions (22 students * 3 exams * 3 solutions). In the analysis, we focused on two things: the types of justification related errors (JRE) and the frequency of these.

RESULTS AND DISCUSSION

Justification related errors in the exams

The analysis revealed the following three JRE types:

- *Missing justification*. A justification between two terms in the derivation is missing.

- *Insufficient or incorrect justification*. E.g. using the wrong name of a rule or not being precise enough, for instance, writing “logic” as the justification, when a more detailed explanation would have been needed.
- *Errors related to the use of mathematical language*. Characterized by the student not being familiar with the mathematical terminology. For instance, one student wrote “solve the equation” when actually multiplying two binomials or simplifying an inequality.

The pre course survey indicated that the students had quite varied justification skills. Over half of the students disagreed with the statement “I usually justify my solutions carefully” and an analysis of the pretests showed that many students did do quite poorly on the justification part, especially for the two most difficult exercises (over 50 % of the students gave an incorrect or no explanation). Also, the nature of the justifications was rather mixed: whereas some gave detailed explanations, some only wrote a couple of words giving an indication of what they had done.

The exam assignments included surprisingly few JREs taking into account the skills exhibited by students in the pretest. The overall frequency of JREs stayed rather constant throughout the course: a JRE was found in 15-20 % of the 66 assignments analyzed for each exam. Most students who made a JRE of a specific type, made only one such error in the nine assignments. Note that this is one erroneous justification comment throughout all three exams. Only six students made more than one JRE of a specific type.

Missing justifications were the most common JRE in the second exam (11 % of students), whereas students did mainly insufficient/ incorrect justifications in the first and third exam (9-12 %). Errors related to mathematical language stayed fairly constant in all exams (3-6 %).

The low number of missing justifications in the first exam is understandable given the character of the assignments (short, familiar topics). In the second exam, new topics had been introduced, resulting in a larger number of missing justifications. This however decreased in the third exam, suggesting that students had got used to always justifying each step. The slightly increased number of insufficient/ incorrect justifications in the third exam can be explained by the third exam being the most difficult one. The main point here is to note that the overall frequency of JREs was low.

Survey results

The mid and post course surveys revealed students’ perceived benefits and drawbacks of using structured derivations. Our analysis showed that 77 % of the students stated that the solutions were much clearer than before. Further another 77 % suggested an increased understanding for doing mathematics.[1]

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[1] The quotations have been freely translated from Swedish by one of the authors.

> "At first I found it completely unnecessary to write this way, but now I think it is a very good way, because now I understand exactly how all assignments are done."

> "I actually liked this course (rare when it comes to mathematics), structured derivations made everything much clearer. Earlier, I basically just wrote something except real justifications. Sometimes I haven't known what I've been doing."

The main drawbacks, according to the students, were that the format made solutions longer (32 % of students) and more time consuming (55 % of students). This is understandable, as the explicit justifications do increase the length of the solutions and also take some time to write down. The justifications, however, were considered a source of increasing understanding, thus the time consumption might be regarded something positive after all. In fact, we believe it is a large benefit, as it helps promote quality instead of quantity.

The students also noted that structured derivations required more thinking. Moreover, they recognized that the format helped them make fewer errors partly because they had to let it take time to write down the solutions.

> "In this course the calculations become more careful since you take the time to think every step through."

> "[Using the traditional format, you] can more easily make mistakes when you calculate so fast."

Another interesting finding was that students seemed to believe that justifications were not part of the solutions when doing mathematics in the traditional format. Describing the traditional way they do mathematics, they e.g. noted:

> "You don't have to explain what you do!… It's enough to get a reasonable answer."

> "You lack explanations for why you do things the way you do."

A final remarkable observation was the lack of completely negative comments. Comments starting out in a negative tone ("It takes much time", "I don't like all the writing"), all ended up positive ("… but I understand what I do better", "…but I make fewer errors"). In our opinion, this is a promising finding.

CONCLUDING REMARKS

The format and results presented in this paper, suggest that it is possible to get students to start justifying their solutions better. If you want to do something carefully, it will take some time and effort. "Quality before quantity" is something that, in our opinion, should be emphasized also in mathematics education.

The focus on also explaining solutions raises a new challenge - how do we get students to choose an appropriate level of detail for their justifications. While talented students may feel comfortable using "simplify" as a justification, this might not be sufficient for weaker students. A certain level of detail thus needs to be enforced at least at the beginning of a new topic, in order to ensure that students truly are learning the topic at hand.

Another question raised that merits further investigations is what type of justification should be preferred (name of a mathematical rule, natural language description of the process, i.e. what is done in the step)? The impact of the type of justification ("simplify" compared to a longer description) on the quality/ correctness of a solution also deserves attention.

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THE ART OF CONSTRUCTING A TRANSPARENT P-PROOF

Aliza Malek and Nitsa Movshovitz-Hadar

Technion – Israel Institute of Technology

This paper is devoted to lessons about constructing Transparent Pseudo-Proofs (abbr. TPPs) drawn from an empirical study of the impact of undergraduate students' exposition to TPPs of theorems in Linear Algebra. The following issues are discussed: (i) an appropriate formal proof that the TPP will reflect; (ii) an adequate value for the particular case that will become the pivot for developing the TPP; (iii) a satisfactory style that avoids anything specific to the particular case from entering the proof; (iv) the 'level of transparency' to suit the mathematical background of the target audience (exemplified by The Three Dots phenomenon); (v) the 'level of generality' to suit the instructor's goals.

INTRODUCTION

A transparent proof was defined as a proof of a particular case which is small enough to serve as a concrete example, yet large enough to be considered a non-specific representative of the flaw of the arguments in the proof of the general case, so that one can see the general proof through it as nothing specific to the particular case enters the proof (Movshovitz-Hadar, 1988). As a transparent proof is not a proof (of the general case), the term was modified to "Transparent Pseudo Proof" (Movshovitz-Hadar and Malek, 1998), abbreviated TPP. In the past two decades the term "generic proof" has also been used in the literature, and many researchers addressed the issue of using them to "sweeten the bitter pill" of proving in various levels of schooling.

Several studies in the past two decades examined the place of conjecturing, proving and reasoning through particular cases, or examples, in various school levels. However only seldom did researchers touch the issue of employing examples in tertiary level mathematics, and the investment it takes to construct a good example for proof and proving. What is it that makes such an example a "good one"? This is the focus of discussion in this paper.

Several issues related to employing examples as a pedagogical tool in the teaching of various levels of mathematics were published in recent years. Some of these studies investigated the way students can learn from worked-out examples (e.g., Atkinson et. al., 2000). These studies show that learning from worked-out examples is very effective as it is faster and it reduces mental load from the learner (Mwangi & Sweller, 1998). Atkinson et. al (ibid) conclude their work by suggesting further studies to deal with the question: “How can examples of authentic problem solving be designed to reduce cognitive load and promote acquisition of transferable cognitive structures?” (p. 211).

Intrigued by Atkinson (ibid), we conducted an empirical study of the use of a particular type of worked-out examples, which are concerned with proof rather than with problem solving.

Before we proceed, let us note that a TPP is, quite obviously, *not* a full proof to a mathematical statement, but surely it is a full proof to a particular case of it. Hence, although it cannot replace the formal proof, it might be considered temporarily as a step towards the proof. In addition, being a full proof of a particular case of the statement, a TPP can be considered a worked-out example to proving in general.

Within our empirical study we developed, experimented and evaluated the impact of Transparent Pseudo-Proofs (TPPs) on the acquisition of transferable cognitive structures related to proof and proving. As the results of the study show, students benefited from their exposure to TPPs in all these areas. (For a detailed account of the study, its theoretical framework, its results, and implications, see Malek and Movshovitz-Hadar 2008).

A number of TPPs were carefully constructed for that study. This paper describes lessons derived from the researches' experience in composing them. These lessons may prove useful for the practitioner who struggle with the construction of TPPs, which is an art in itself.

WHAT DOES IT TAKE TO CONSTRUCT A TPP?

From the very definition of TPP three basic actions to be carries out in the process of construction of a TPP follow immediately.

1. Choosing an appropriate formal proof that the TPP will reflect. Since a TPP is a didactic derivative of some formal proof, its preparation must start from choosing an appropriate formal proof that the TPP will reflect. In other words, it is an act of matching the TPP to the proof we wish to introduce, which is rather different of course from merely putting down an independent proof of a particular case.

2. Choosing an appropriate value for the particular case that will become the pivot for developing the TPP itself. According to the TPP definition the value should be carefully chosen so that it is neither too small nor too large. For example a proof of a theorem dealing with some properties of a polynomial cannot be reflected by a proof about a binomial (as it is very likely to become too specific), nor by a proof about a 17 addends polynomial although it appears as a "random" choice (as it may become too cumbersome).

3. Making sure that nothing specific to the particular case enters the proof. All along the construction of the TPP one must be very careful in following the general proof so that nothing specific to the particular case enters the proof. If something like that happens, the proof may become non-transparent which may make it impossible for the student to come up with a proof of the general case or even worse, lead her or him to a false generalization. (See an example in Movshovitz-Hadar 1997.)

These three actions need not necessarily take place in that order. Further major considerations are due as discussed in the rest of this paper.

GENERALITY LEVEL IS TO BE DETERMINED

Mathematical statements have many general elements that are replaced by specific ones in a TPP. The question is, is it necessary to replace them all?

To focus on the replacement of general features of a mathematical statement for the sake of constructing a TPP, let us look for example at the statement: "The sum of the *n* roots of order *n*, $x_1,...,x_n$, of a complex number *z*, is 0". The general elements in it are: (i) "…the roots $x_1,...,x_n$". (ii) "…of order *n*"; (iii) "… a complex number *z*"; To construct a TPP one may choose to replace one or more of the three general elements by an appropriate specific one, obtaining various particular cases of different levels of generality. For instance:

- Only *n* is replaced (*n*=5): "The sum of the 5 roots of order 5, x_1, x_2, x_3, x_4, x_5, of a complex number z, is 0". (See the sample TPP above).
- Only *z* is replaced ($z = 1 + i$): " The sum of the *n* roots of order *n*, $x_1,...,x_n$, of the complex number $z = 1 + i$, is 0".
- Both *n* and *z* are replaced ($n = 5,\ z = 1 + i$): "The sum of the 5 roots of order 5, x_1, x_2, x_3, x_4, x_5, of the complex number $z = 1 + i$ is 0".
- All three general elements are replaced: "The sum of the 6 roots of order 6 of the complex number $z = -64$, namely: $x_1 = \sqrt{3} + i,\ x_2 = 2i,\ x_3 = -\sqrt{3} + i,\ x_4 = -\sqrt{3} - i,\ x_5 = -2i,\ x_6 = \sqrt{3} - i$, is 0".

It is noteworthy that the last instance, the least general one, is inviting a proof by checking the sum which is not a transparent proof. However it might be valuable for increasing the intuitive trust in the truth of the general statement.

The 4 different specific statements listed above, demonstrate what we suggest calling a *Generality-level Pyramid* for the given general statement. In its base rests the most specific case, namely one of those in which *all* general elements are replaced by specific ones. There are lots of such cases and choosing the one to treat (if at all !) is a matter of consideration. Going up the pyramid, the generality level increases, and the "volume" of alternatives gets smaller. Since one may choose to replace a few general elements, it is generally true that the higher the number of general elements replaced, the less general the particular statement becomes (e.g. the 2nd, 3rd and 4th instances above). However, within a certain number of replacements (e.g. the 1st and 2nd instances above where only one element is replaced) various options are open for choosing the elements to be replaced, hence the hierarchical order of levels in the pyramid can be argued.

Clearly the theorem we are trying to prove is usually on top of the pyramid, however, for the sake of open-endedness, the Generality-level Pyramid of each mathematical statement should be seen as a truncated one, thus leaving an option for further generalization of the theorem in question. (For example the statement discussed above is a particular case of the statement: the sum of all *n* roots of a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ of degree *n* ($a_n \neq 0$) is $-\frac{a_{n-1}}{a_n}$) .

TRANSPARENCY IS IN THE EYES OF THE BEHOLDER

Although a TPP is constructed carefully to reflect all the major steps in the formal proof, what might appear as transparent to one student, is not necessarily transparent to another one. This leads to the notion of *relative transparency* of a TPP. Since a TPP is aimed at making the difficult parts of the corresponding formal proof more accessible to the learner, it is important to firstly analyze the points in the formal proof which may become obstacles to the specific target audience of learners in view of their background and mathematical abilities. Consequently, in constructing a TPP the instructor, aware of potential obstacles typical to the specific target audience, needs to carefully construct the crucial steps so that as many students as possible can see the formal proof through it.

For example let us consider the theorem: For any natural number n, the sum of the roots of order n of a complex number z is 0 and their product equals: $(-1)^{n+1} z$. Here is a TPP for this theorem (on the left) that reflects a formal proof (on the right).

<table>
<tr><th>A Transparent Pseudo Proof (n=5)</th><th>A Formal Proof</th></tr>
<tr><td colspan="2">Recall that the roots of order n of a complex number z are the complex roots of
$x^n - z = 0$
Recall also that: $x^n - z = 0$ has n complex roots $x_1, x_2, \ldots, x_n$, which satisfy:
$x^n - z = (x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_n)$</td></tr>
<tr><td>As you know the sum of the roots and their product, appears in the binomial expansion of $(x-x_1)(x-x_2)$ as $x^2-(x_1+x_2)x+(-1)^2 x_1x_2$.
Let's take a look at a larger particular case
The roots of order 5 of a complex number z are the roots of the equation
$x^5-z=(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)=0$
$(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)=$
$x^5-(x_1+x_2+x_3+x_4+x_5)x^4+\ldots+(-1)^5 x_1x_2x_3x_4x_5 \Rightarrow$
$x^5-z = x^5-(x_1+x_2+\ldots+x_5)x^{5-1}+\ldots+(-1)^5 x_1\cdot x_2\cdot\ldots\cdot x_5$
On the left hand side the only positive power of x is 5, in other words the coefficients of any other positive power, such as x^{5-1} on the left hand side, is 0. Hence the coefficient of x^{5-1} on the right hand side must also be 0.
We get $(x_1+x_2+\ldots+x_5)=0$
Comparing the coefficients for x^0 we get:
$?-(1)^5 x_1\cdot x_2\cdot\ldots\cdot x_5 = -z?$
i.e., $(-1)^{5+1} x_1\cdot x_2\cdot\ldots\cdot x_5 = z$
QED (n=5)
Can you now prove the general statement on your own?</td><td>$x^n - z = (x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_n) =$
$x^n-(x_1+x_2+x_3+\ldots+x_n)x^{n-1}+\ldots+(-1)^n x_1\cdot x_2\cdot x_3\cdot\ldots\cdot x_n = 0$
Comparing the coefficients of
x^{n-1} and x^0 we get:
$(x_1+x_2+\ldots+x_n)=0$
and
$?-(1)^n x_1\cdot x_2\cdot\ldots\cdot x_n = -z?$
$\Rightarrow$
$(-1)^{n+1} x_1\cdot x_2\cdot\ldots\cdot x_5 = z$
QED</td></tr>
</table>

In the TPP, the three dots which represent "etc., etc.," appear three times in the expression: $x^5 - (x_1 + x_2 + \ldots + x_5)x^{5-1} + \ldots + (-1)^5 x_1 \cdot x_2 \cdot \ldots \cdot x_5$. Not all occurrences may be clear to some students, as they are not all equally obvious. The middle occurrence is particularly cumbersome. For these students a more detailed proof (of the particular case) is needed. E.g. replace

$$(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5) =$$
$$x^5 - (x_1+x_2+x_3+x_4+x_5)x^4 + \ldots + (-1)^5 x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \ldots$$

by

$$[(x-x_1)(x-x_2)(x-x_3)(x-x_4)](x-x_5) =$$
$$[\{[x^2-(x_1+x_2)x+(-1)^2 x_1 \cdot x_2](x-x_3)\}(x-x_4)](x-x_5) =$$
$$[\{x^3-(x_1+x_2+x_3)x^2+\ldots+(-1)^3 x_1 \cdot x_2 \cdot x_3\}(x-x_4)](x-x_5) =$$
$$[x^4-(x_1+x_2+x_3+x_4)x^3+\ldots+(-1)^4 x_1 \cdot x_2 \cdot x_3 \cdot x_4](x-x_5) =$$
$$x^5 - (x_1+x_2+x_3+x_4+x_5)x^4 + \ldots + (-1)^5 x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \ldots$$

In this insert one can see that for n=2 the sum and the product of the roots "pop in". For n=3 and n=4 they appear again in the appropriate place. In addition in these cases, there is one occurrence of three dots, which is the less obvious one, and the reader may realize that their meaning is not merely "etc., etc.," but "the details are not really important for the proof (as the sum and the product are the focus of attention)".

Anecdote: "Etc., etc.," Vs. "Whatever". To illustrate the point mentioned above here is an anecdote from the empirical study.

Sara had trouble with the polynomial expansion of $(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_n)$. As she got the handout she read it out loud, but soon slowed down and seemed puzzled. So she received the more detailed handout. At the single occurrence of "three dots" in the n= 3, 4 expansion, she said: "dot, dot, dot", and stopped only to make sure she was able to complete the missing details. However, in reading the expansion of n=5, $x^5 - (x_1 + x_2 + \ldots + x_5)x^4 + \,..+ (-1)^5 x_1 \cdot x_2 \cdot \ldots \cdot x_5$, she said "etc., etc.," in the first and third occurrence, but as she moved towards the second one, she glanced at the interviewer, waived her hand and said: "whatever" and she did not bother to complete the details.

She proceeded by writing a proof for n=6 saying: "It does not really matter, 6 or 600, the argument is just the same." Then she pointed at the polynomial expansion and said: "This goes on and on. In each step we rely on the previous one, hmm…it is actually a proof by induction. Let me see - - assuming for k that…" and she wrote down freely the transition step:

$$(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_{k+1}) = [(x-x_1)(x-\mathrm{x}_2)\cdot\ldots\cdot(x-x_k)](x-x_{k+1}).$$

Stopping for a second she said: "Yes, I see, it can work for k+1", and she went on writing while saying: "$= x^{k+1} - (x_1 + x_2 + \ldots + x_{k+1})x^k + \ldots + (-1^{k+1})x_1 x_2 \ldots x_{k+1}$." Again, she distinguished between the two kinds of "three dots", addressing them by different words.

To wrap it up, when composing a TPP, it is useful to construct a Generality-level Pyramid for that theorem, and chose the generality level suitable for the pedagogical goals and the mathematical background of the target audience. A

TPP need not avoid *all* the general elements. Some may and possibly should remain. After all, Transparency is in the eyes of the beholder.

END NOTE: SOME RESERVATIONS

Although many proofs can be introduced through a transparent version, it is not clear that for *every* proof a TPP can be composed. Other methods of introducing proof and proving such as guided discovery approach might be useful in such cases (For an example see Movshovitz-Hadar, 2008 in press).

Additionally, there is some risk in using TPPs, as they may obscure the necessity for a general proof if an overdose of TPPs is assigned. If students are not challenged to attempt a general proof to start with, or at least following the introduction of a TPP, they may dismiss of the need to prove a theorem altogether, thus jeopardizing the bearing of mathematics knowledge embedded in proof (Rav in Hanna and Barbeau, 2008).

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TEACHING AND LEARNING A PROOF AS AN OBJECT IN LOWER SECONDARY SCHOOL MATHEMATICS OF JAPAN

Mikio Miyazaki — *Shinshu University, Japan*

Takeshi Yumoto — *Kaisei Junior HS Nagano, Japan*

This study concluded that the teaching and learning a proof as an Object in lower secondary school mathematics involves at least 4 aspects. In deriving this conclusion we firstly pointed out that the teaching and learning of proofs as an Object can be the basis both for the teaching and learning a proof as an activity and for the teaching and learning the functions of proof in practical lessons of proofs in school mathematic. Secondly, we identified four aspects with the teaching and learning a proof as an Object through practical lessons (11 lesson hours) on the "parallel lines and angles" unit found in the second grade of lower secondary school.

PRACTICAL PROBLEMS WITH THE TEACHING AND LEARNING OF PROOFS

While obtaining all the knowledge we gain over our lifetimes we tend to organize and develop it after having clarified with proofs that it is based on evidence and logical thinking. Hence, since proofs can become a "driving force" in a person's productive activities throughout their lifetimes, it is necessary that they can appreciate the actual meaning and significance of proofs through education.

In recent years the importance of proofs has been re-established internationally and they now occupy a major position in curriculums (Mariotti, 2006). In Japan the teaching and learning of proofs have been practiced at lower secondary school for about half a century. According to the results of the National Assessment of Scholastic Attainments in Japan (practiced in 2007), however, it is clear that the present situation with teaching and learning of proofs in lower secondary school is rather problematical. In order to acquire new directions for use in improving that situation it is necessary that the fundamental "chunk" of the long-term teaching and learning of proofs be revealed, and then their "substance" explored.

Accordingly, this study focuses on the teaching and learning a proof as an Object and aims at specifying the different aspects that exist within.

FUNDAMENTAL NATURE OF TEACHING AND LEARNING A PROOF AS AN OBJECT

In mathematics education it has been considered that at least three kinds of aspects exist within a proof and when proving something.

The first aspect is realized when a proof can be seen as a structural object, which consists of the following components: particular propositions, universal propositions and deductive reasoning. Seeing a proof as an Object enables clarifying what the components of a proof and the relationships between them are, how a proof is composed using those components and their relationships, and in addition why a proof needs the structure that it is made up of.

The second aspect is realized when a proof can be seen as an intelligent activity. (Balacheff, 1987; Harel & Sowder, 1998 etc.) Seeing a proof as an activity enables clarification of what it is that supports proof activity, what is involved in the activity.

The third aspect is realized when a proof can be seen with the focus on its roles and functions in mathematics, empirical science and the real world. (de Villiers, 1990; Hanna & Jahnke, 1993 etc.) Seeing a proof as a function enables clarification of how a proof contributes to human activities, why it is necessary, and how the functions of a proof get embodied.

Long-term practical lessons on proofs are developed with the teaching and learning supported by the reciprocal relationships between three perspectives of a proof. Especially, at the start of lessons, teaching and learning proofs are practiced informally. Furthermore, in order to make an informal proof more formal, teaching and learning the structure of a proof is set, which then enhances the quality of proof activities. In addition, the quality of them can be reflected in the teaching and learning of the functions of a proof.

Hence, in long-term practical lessons on proofs, teaching and learning a proof as an Object is imperative in enhancing the quality of teaching and learning a proof as an activity, and the quality then enables the teaching and learning of the more advanced functions of a proof. In fact, teaching and learning a proof as an Object can play a fundamental role in long-term practical lessons.

ASPECTS OF TEACHING AND LEARNING A PROOF AS AN OBJECT

From Reducing a Proof into its Components to Integrating them as a Proof

Proofs can be reduced into propositions and reasoning. Conversely, a proof can be composed through an appropriate combination of propositions and reasoning. Hence, in the teaching and learning a proof as an Object, not only the reduction of a proof into propositions and reasoning, but also the integration of the reduced propositions and reasoning are necessary.

Therefore, in the planning and practice of lessons, we need to understand the following issue: teaching and learning a proof as an Object shifts gradually from reducing a proof into its components to integrating them as a proof.

Teaching and Learning a Proof as an Object in the Unit "Parallel Lines and Angles"

In the national curriculum of Japan one of the aims at the early elementary school level is to develop a foundation of a proof and proving through stating the reason and writing it down, with the teaching and learning of formal proofs commencing in the second grade of lower secondary school. In the geometry curriculum in particular, the intention with the unit "Parallel Lines and Angles" is to primarily clarify the reasons for the properties and relationships of angles. The unit "Structure of a Proof" then introduces the mechanism and method of a proof.

Finally, the intention with the unit “Properties of Triangles” and the unit “Properties of Quadrangles” is to formally prove the various properties of figures.

However, even after the unit “Structure of a Proof”, only a few students can understand a formal proof. And therefore in subsequent units not only a lot of students but also teachers face serious difficulties with their learning and teaching of proofs.

Hence in the unit “Parallel Lines and Angles”, which comes before the Unit “Structure of a Proof”, we cooperatively developed and practiced eleven lessons so that students could gradually establish a base for the structure of proof. The teacher who put into practice those lessons has taught mathematics at lower secondary school for over 20 years. A series of lessons was put into practice with 35 second graders from an attached lower secondary school of a national university from September the 2nd to October the 4th 2004.

Four Aspects within Teaching and Learning a Proof as an Object

Aspect I: recognizing/constructing a proof roughly

In the first lesson of the unit “Parallel Lines and Angles”, the teacher intended to organize the properties of figures, which the students had learned at elementary school, through solving the problem of finding the size of angle $\angle x$ in a concave quadrangle: 35 degrees, 80 degrees and 20 degrees (Fig. 1). In this problem a lot of students merely used calculations to find the angle, but did not write the properties of the figure that was the base of their calculations. On the right (Fig. 1) is what Mariko wrote. She described the reason for her answer of “135 degrees” as being a division into two triangles. Based on the wrong assumption that the 80 degree angle could be bisected she used the universal proposition that “The sum of internal angles of a triangle is 180 degrees”, and thus determined “180-(40+35)” and “180-(40+20)”. The teacher questioned her on why she had calculated that, to which Mariko replied, “Why did I?” and was lost in thought for a while.

Therefore, while Mariko could describe a particular proposition as a numerical expression she could not distinguish the universal proposition from the particular proposition that was necessary in this problem. A similar situation was found with many of the other students in the class.

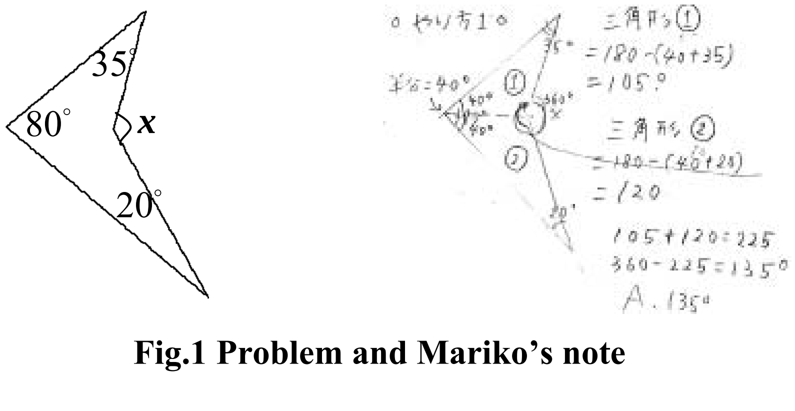

Fig.1 Problem and Mariko’s note

Aspect II: making the distinction between universal propositions and particular propositions

In the second lesson, in solving the above-mentioned problem, Ryota found the size of the vertically opposite angle of $\angle x$ (135 degrees) using the universal propositions of "The sum of internal angles of a triangle is 180 degrees" and "The sum of internal angles of a quadrangle is 360 degrees", which had already been accepted by the class. Ryota then only stated "Because it is so" and wrote his answer of 135 degrees on the board while saying "$\angle x$ is 135 degrees".

His answer solicited a lot of questions similar to "Why did you immediately find it to be 135 degrees?" from the other students. Since this type of question was in accordance with the aim of the lesson, the teacher raised the question of "Why are these two angles the same?" Another student then showed that when the same angle (45 degrees) was combined with each vertically opposite angle it resulted in a straight angle (180 degrees). In addition, another student showed that when two lines which meet at a right angle are moved in symmetry around their point of intersection the size of the vertically opposite angles is always equal. Following this the teacher then wrote, "Vertically opposite angles are equal", as the fifth "theorem" (universal proposition). In the second lesson through to the seventh lesson, while learning the properties of parallel lines and angles and the sum of polygonal internal/external angles, theorems (universal propositions), which the students had implicitly used in proofs, were verbalized, and the geometrical properties and relationships organized as "a list of theorems".

Aspects III: recognizing/making the deductive relationship between a universal proposition and a particular proposition

In the eighth lesson through to the tenth lesson, students posed problems for use in explaining how to find angles using the eleven "theorems", and solved them with each other. In the ninth lesson, students wrote four kinds of answers on the board for a problem with the unnecessary information (Fig. 2), and they considered which of the 11 "theorems" accepted by the class to use. At this time, the discussion became heated on which "theorem 3: A straight angle is 180 degrees", or "theorem 10: The sum of internal and external angles is 180 degrees", was used to get the expression "180-46=134" in answer (b) (Fig. 2). While the teacher and the students discussed what had been used from the list of "theorems" "accepted by the class" (universal proposition) in order to find the angles and why it was appropriate to consider that a universal proposition should be used, the students learnt that the deductive relationship between a particular proposition and a universal proposition in that the former was derived from the latter using universal specialization reasoning. In fact, after this lesson, Mariko wrote her following impression of the lesson on her worksheet: "I think it is wrong to equate theorem 3 with theorem 10 because theorem 10 is effective only in the case where internal and external angles actually exist, although in any case theorem 3 indicates that a straight line is 180 degrees regardless of internal and external angles."

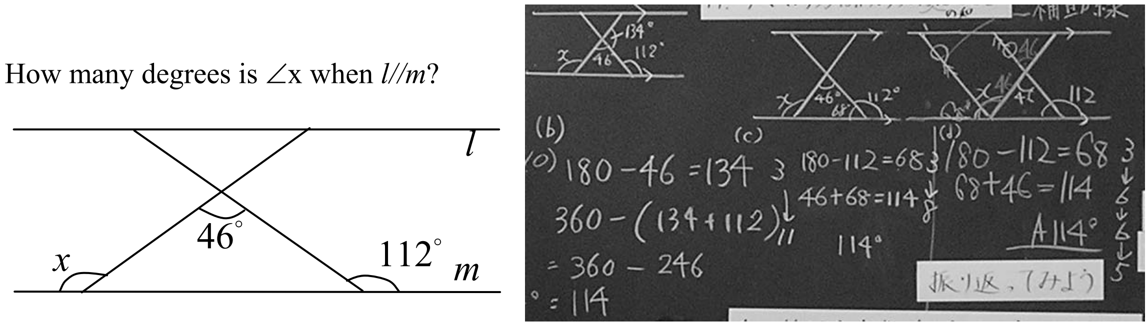

Fig.2 One of the posed problems and students' answers on the board

Aspects IV: organizing all the deductive relationships between universal propositions and particular propositions

In the ninth class, in order to clarify the differences between the four kinds of answers that resulted from the problem (Fig. 2), the teacher asked for each answer, "Which of our theorems are used and in what order?" The students added the number of "theorems" and arrows such as "$3 \rightarrow 8$" beside their answer, thus revealing how they used them in their answer. (Fig. 2) Next, in the tenth class, concerning Yu's answer to another problem, when adding the numbers and arrows the same as before, the students developed five types of usage. Then, in the eleventh class, the teacher showed these types (Fig. 3), and the students discussed which could be appropriately used in accordance to their interpretation of Yu's answer.

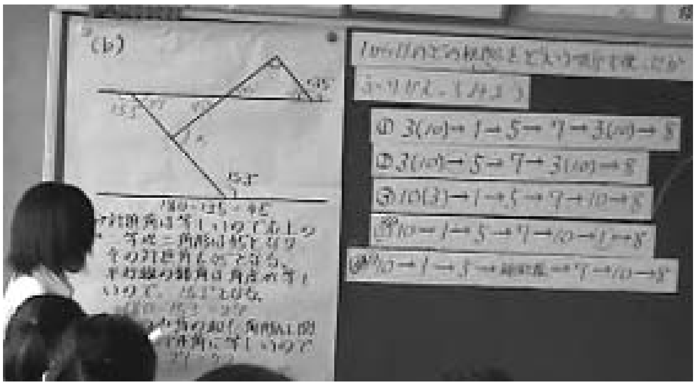

Fig.3 Five ways of interpretation for Yu's answer

CONCLUDING REMARKS

In developing and practicing the eleven lessons on the unit "Parallel Lines ad Angles" we came to a mutual recognition that the teaching and learning a proof as an Object moved gradually from reducing a proof into its components to integrating those components as a proof. In the practice of those lessons the teaching and learning a proof as an Object commenced with a rough recognition/construction of a proof. (Aspect I) Subsequently, since a lot of students were not conscious of the existence of universal propositions, which are implicitly used in their proofs, our intention was that the students would verbalize the universal propositions and then distinguish them from particular propositions. (Aspect II) Next, based on the distinction of students recognizing that a particular proposition could be derived from a universal proposition, they could then actually derive a particular proposition from a universal proposition. Namely, our intention was for the students to be able to recognize and construct the deductive relationship between a universal proposition and a particular proposition. (Aspect III) Finally, in order to integrate the relationships between universal propositions and particular propositions the students explored how to use universal propositions in a proof. Here, the intention was for students to be able to organize the relationships between universal propositions and particular propositions. (Aspect IV)

A long-term framework for the socio-cognitive development of proofs (Balacheff, 1987 etc) and a comprehensive framework for teaching and learning of proofs in lower secondary school mathematics (Kunimune, 1987) has already been proposed. These frameworks provide a long-term vision and view, but proved difficult to use in short-term practical lessons. Opposing this, the four aspects specified in this study may allow the teaching and learning a proof as an Object in lower secondary school mathematics to be more focused upon, and become frameworks that enable the development of a series of lessons for a certain unit.

The following are issues for further research:

- What kinds of aspects are there in the teaching and learning a proof as an activity and the teaching and learning of the functions of a proof?
- What kind of aspects can be specified in the long-term teaching and learning of proofs with the focus on the interaction between the structure, activities and functions of proof?

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BREAKDOWN AND RECONSTRUCTION OF FIGURAL CONCEPTS IN PROOFS BY CONTRADICTION IN GEOMETRY

Maria Alessandra Mariotti*, Samuele Antonini**

**Department of Mathematics, University of Siena, Italy*

*** Department of Mathematics, University of Pavia, Italy*

In this paper we aim to contribute to the discussion on the role of images in respect to proof, elaborating on two examples of indirect arguments produced by students. The analysis is framed by the construct of figural concepts (Fischbein, 1993) and by the model for indirect proof introduced in (Antonini & Mariotti, 2008). According to the general assumption that the harmony between the figural and the conceptual aspect is required for productive reasoning, we will present two cases where the production of an indirect argument can be interpreted as emerging from the need of restoring such harmony. In particular, the quest for an image after the contradiction seems to be crucial to support the passage from the 'absurd' to the validation of the original statement.

INTRODUCTION

From the early work of Bishop (1983), a number of research studies have investigated the role of visualization in mathematical thinking, as thoughtfully discussed by Presmeg (2006). Interest has been devoted by mathematicians and philosophers to the potential of images in relation to mathematical proof, and specifically in the case of proof in Euclidean Geometry (Hanna & Sidoli, 2006). Although Geometry has to be considered a theory completely independent of any reference to reality, the importance of pictures and diagrams has been widely discussed, in relation not only to demonstration but also to discovery (Giaquinto, 1992; Mancosu et al., 2005). If deductions make sense within a theoretical context, their meaning and their justification value often refers to representations of geometrical figures, either external or internal (see Norman, 2006 for a recent discussion on the epistemic value of diagrams). Different contributions can be found in Mathematics Education literature referring to the role of visualization in the solution of geometry problems (for instance, Duval, 1998; Fischbein, 1993), in particular within a dynamic Geometry environment (for instance, Laborde, 1998; Mariotti, 1995). The use of diagrams, and the defence of their legitimacy in relation to the issue of proof, as well the difficulties that they may originate are the core of the study of Richard (2004).

The relationship between theoretical aspects and figural aspects becomes crucial in the case of proof by contradiction, where reasoning may need specific theoretical control for the lack of an adequate support of figural representation. Quite often in the case of proof by contradiction the properties derived at the theoretical level may conflict with the available images – either external or internal – related to the geometrical figure in play. This issue becomes of particular interest when the range of external representations is enlarged to include images generated within a Dynamic Geometry environment, as it is the case of *pseudo objects* described by Leung & Lopez-Real (2002).

In this paper we aim to contribute to the discussion on the role of images in respect to proof, elaborating on two examples of indirect arguments produced by students. The analysis will be framed by the construct of figural concepts (Fischbein, 1993) and by the model for indirect proof introduced in (Antonini & Mariotti, 2008).

THE NOTION OF FIGURAL CONCEPT

According to Fischbein (1993), activities in elementary (Euclidean) geometry involve mental entities that cannot be considered either pure concepts or mere image. Geometrical figures, as involved in geometrical reasoning, are mental entities that simultaneously possess both conceptual properties (as general propositions deduced in the Euclidean theory) and figural properties (as shape, position, magnitude). Fischbein called them *figural concepts* (Fischbein, 1993). The theory of figural concepts provides us with an efficient theoretical tool suitable to analyse cognitive processes by considering images and concepts intimately interacting in geometrical problem solving. The complete symbiosis between conceptual and figural aspects is only an ideal situation, difficult to reach because of different constraints. Mistakes and difficulties can be efficiently explained in terms of the missed or incomplete fusion (harmony) between figural and conceptual aspects, whilst productive reasoning can be explained by the fact that the two components blend in genuine figural concepts (see, for example Mariotti, 1993; Mariotti & Fischbein, 1997). Though with age and instruction this symbiosis tends to improve, difficulties may persist, mainly when conflict emerges, as may be the case with indirect arguments involved in proof by contradiction.

INDIRECT PROOF

In recent papers (Mariotti & Antonini, 2006; Antonini & Mariotti, 2008) we presented a model to describe and explain the processes involved in an indirect proof (with this term we mean both proof by contradiction and proof by contraposition).

Given a statement - called the *principal statement* - the model describes the structure of an indirect argumentation leading to its proof. We distinguish two levels. In the *theoretical* level a direct argument is developed to prove what we called the *secondary statement* - obtained assuming as hypothesis the negation of the principal statement or of its thesis, and as thesis respectively a contradiction or the negation of the hypothesis of the principal statement. In the *meta-theoretical* level it is performed the proof of a *meta-theorem* relating the proof of the secondary statement with the validation of the principal statement. Although this meta-theorem is usually left implicit, giving for grant that it constitutes a ‘natural’ passage, difficulties are described.

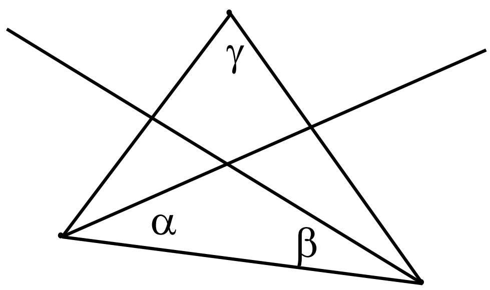
The notion of figural concept described above may offer an effective tool to describe the cognitive processes involved in an indirect proof in Geometry. For instance, the dialectics between the figural and the conceptual may explain the difficulties met in the proof of the secondary statement as a lack of an adequate control to overcome the potential discrepancy between the conceptual and figural aspect. In the following, according to the general assumption that the harmony between the figural and the conceptual aspect is required for productive reasoning, we will present two cases where the production processes of an indirect argument can be interpreted as emerging from the need of restoring such harmony. In particular, the quest for an image after the contradiction seems to be crucial to support the passage from the 'absurd' to the validation of the original statement.

RESTORING THE FIGURE

The following examples are drawn from a wide-ranging research study concerning indirect proof. The study involved university students and students attending the last year of the high school, and had the objective of investigating processes related to indirect proof (see Antonini & Mariotti, 2008). The excerpts are drawn form the transcripts of two interviews carried out with pairs of students, all of them familiar with elementary Euclidean geometry. During the interviews a problem was proposed and the students were asked to cooperate in solving it. In the following example the following geometric open problem is considered: *What can you say about the angle formed by two angle-bisectors in a triangle?* The solutions provided showed interesting examples of indirect arguments leading to a contradictory conclusion.

"There is no triangle any more": the case of Elenia and Francesca

Elenia and Francesca are two students enrolled in the first year of the Biology Faculty. After a short exploration of the possible configurations, the case of orthogonality is under scrutiny. In the following excerpt Elenia is speaking and Francesca does not intervene. After having deduced that "if the angle between the angle bisectors is right then $2\alpha+2\beta=180$" the conversation goes on:



46 E: … there is something wrong.
47 I: Where?
48 E: In 180.
49 I: Why?
50 E: Because, it [180] is the sum of all the three interior angles, isn't it? […]
53 I: Yes.
54 E: Right.
55 I: And then?
56 E: And then there is something wrong! They should be $2\alpha+2\beta+\gamma=180$ […]
60 E: …and then it would become $\gamma=0$…
61 I: And then?

62 E: But equal to 0 means that it isn't a triangle! If not, it would be so [she joins her hands]. Can I arrange the lines in this way? No... […]

85 E: And then essentially there is no triangle any more.

86 I: And now?

87 E: …that it cannot be 90 [degrees].

88 I: Are you sure?

91 E: Yes. […] because, in fact, if γ=0 it means that… it is as if the triangle essentially closed on itself and then it is not even a triangle any more, it is exactly a line, that is absurd.

The first statement, e.g. the fact that the sum of the two angles (2α and 2β) is 180°, is correctly deduced. This conclusion contrasts with a theorem of the theory, thus producing a contradiction. This would be sufficient to validate the fact that the assumption about the angle between the angle bisectors is false. Nevertheless Elenia seems to be surprised (46-56) and puzzled by this conclusion. She does not conclude the deduction about the angle bisectors, rather she declares that "*there is something wrong*", probably because she can not imagine the corresponding configuration: there is a break between the conceptual and the figural aspect. Further investigation leads her to conclude that the measure of the third angle has to be 0. This new conclusion generates a new figure where two sides of the triangle collapse in a segment making the third angle 'disappear' and the whole triangle disappear with it (85. *There is no triangle any more* …). The segment emerges from seeking harmony between figural and conceptual aspects, from the need of generating a geometrical figure respecting the deduced properties "γ=0". The feeling of surprise and the claim that something is wrong came from the impossibility of conceiving the figural component of the deduced properties. The new figure is not 'impossible' or 'fictitious', rather it makes clear for the students that the assumption about the angle bisectors is false. In fact, the last argument (91) makes explicit how the recovered image of a triangle closed into a segment accounts for the impossibility for the angle to be right: the harmony between the figural and the conceptual is recovered and that supports the move from the contradiction to rejecting the original hypothesis.

"It would become a quadrilateral": the case of Paolo and Riccardo

Paolo and Riccardo are high school students of a Scientific school (grade 12). After considering and excluding the case that the angle between the angle bisectors is acute, Paolo and Riccardo approach the case of orthogonality. (In the interview they named K and H the angles that Elenia and Francesca named 2α and 2β.)

61 P: As far as 90, it would be necessary that […] K/2=45, H/2=45 […].

62 I: In fact, it is sufficient that […] K/2 + H/2 is 90.

63 R: Yes, but it cannot be.

64 P: Yes, but it would mean that K+H is ... a square […]

65 R: It surely should be a square, or a parallelogram

66 P: (K-H)/2 would mean that […] K+H is 180 degrees...

67 R: It would be impossible. Exactly, I would have with these two angles already 180, that surely it is not a triangle. […]

71 R: We can exclude that [the angle] is $\pi/2$ [right] because it would become a quadrilateral.

Similarly to the previous case, passing through a theoretical argumentation (deduction), Paolo and Riccardo arrive to the condition that the sum of K and H is a straight angle, as a necessary condition for the angle bisectors being orthogonal. But Riccardo acknowledges the impossibility of this condition (63) immediately followed by Paolo who produces its geometrical interpretation. Seeking a figural interpretation of the 'absurd' conclusion generates an adaptation of the figure: "*it surely should be a square, …*" (65). The falsity of the original assumption is now acceptable, "*because it would become a quadrilateral*", e.g. not a triangle.

DISCUSSION AND CONCLUSIONS

As already discussed (Antonini & Mariotti, 2008), it may happen that the absurd is easily and correctly reached (in this case, "the sum of two of the three angles is 180°), but this is not sufficient to complete the proof, rather it produces the feeling that something is wrong. In fact, in this case the conclusion leads to a rupture between the figural and the conceptual aspect, a rupture that in both cases generates an impasse. In order to overcome this impasse and complete the argument the students try to restore the unity between the two aspects. The conclusion, absurd from the theoretical point of view, is reinterpreted at the figural level. Its consequences are figurally elaborated as far as a new figure is produced in order to restore the harmony: a segment for Elenia and a quadrilateral for Paolo and Riccardo. Once the harmony is restored the argument can be developed: in both cases the new figures negate the existence of a triangle and consequently provide the missing step to validate the falsity of the assumption.

A new element emerges from the previous discussion related to the dialectics between the figural and the conceptual aspect. After experiencing the break, and because of it, the students seek to recover an image consistent with the condition theoretically achieved. The new image offers the guide for an explaining argument bridging the gap between the achievement of an absurd conclusion and the validation of the principal statement.

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ARGUMENTATIVE AND PROVING ACTIVITIES IN MATHEMATICS EDUCATION RESEARCH

Juan Pablo Mejía-Ramos **Matthew Inglis**

University of Warwick, U.K. *Loughborough University, U.K.*

Our goal in this paper is to identify the different argumentative activities associated with the notion of mathematical proof and present the results of a bibliographic study designed to explore the extent to which each of these activities has been researched in the field of mathematics education. We argue that the comprehension and presentation of given arguments are important, but under-researched, mathematical activities related to proof.

INTRODUCTION

Proof is widely agreed to be central to the activity of mathematicians, however it is also a notoriously difficult concept for students to learn. These two factors have led to recurring discussions of proof and proving in the mathematics education literature. One of the most influential frameworks used to situate such discussions is the theory of proof schemes (Harel & Sowder, 1998). Harel and Sowder defined a person's proof scheme to be the processes they use to become certain of the truth of a mathematical statement, and to convince others of this certain truth. Their study provided a detailed classification of the different proof schemes used by college students, noting that many used non-deductive ones.

Of particular interest for our purposes is that Harel and Sowder considered a wide variety of situations when studying their students proof schemes; including problem exploration activities, 'proof that…' tasks, 'true or false' tasks and 'explain why…' tasks. It is unclear whether, in each of these situations, students focus solely on the truth of statements (or, if they do, whether they focus solely on gaining certainty). Indeed, Healy and Hoyles (2000) noticed that students would often prefer different arguments for presentation to a teacher than they would for convincing themselves, suggesting that the task activity is an important factor when analysing proving behaviour.

Our primary goal in this paper is to classify different activities associated with proof, with reference to task contexts. Our underlying assumption is that each of these different activities could, in principle, cause different behaviour (whether this is the case or not, of course, is an empirical matter).

ACTIVITIES CONCERNING PROOF

When laying out a preliminary map of the different activities which mathematicians engage in, Giaquinto (2005) suggested that for any piece of mathematics there are three associated general activities: making it, presenting it, and taking it in. Within the context of proof and argumentation these three general activities correspond to: constructing a novel argument, presenting an available argument, and reading a given argument. However, the behaviour associated with these three distinct activities is likely to vary between contexts. A mathematician presenting an argument as part of a journal article, for example, may well behave differently from if she were presenting an argument in an undergraduate lecture course. One of the reasons that different contexts produce different behaviour is that the goal the individual has in mind at the time is likely to vary. De Villiers (1990), following Bell (1976), proposed that proof has five main functions or goals: verification (establishing the truth of a statement), explanation (providing insight into why a statement is true), systematisation (organising results into a deductive system), discovery (the discovery or invention of new results), and communication (the transmission of mathematical knowledge). De Villiers's categorisation suggests that each of the three general activities related to mathematical argumentation can be performed with these different functions in mind. For example, someone may present an argument in order to persuade a given audience of the conclusion's truth, to provide them with insight into why it is true, or to demonstrate the argument's validity in a given system. Similarly, one might read an argument with the intention of understanding it, or in order to evaluate how persuasive, explanatory, or valid it is.

Our framework of sub-activities concerning proof emerges from considering how different goals (including the ones discussed by De Villiers) may guide each of Giaquito's general types of activities. This leads us to differentiate between three distinct types of construction activities, which we call exploration of a problem (related to discovery), estimation of truth of a conjecture (including verification), and the justification of a statement estimated to be true (related to both explanation and systemisation). Similarly, we identify two main reading activities: the comprehension of a given argument and the evaluation of an argument with respect to a given set of criteria. Finally, we differentiate between presentations in which a given argument is used to: convince a given audience of the argument's claim, explain to a given audience why the claim is true (one way of communicating knowledge), demonstrate the argument's validity to a given audience and demonstrate to an expert one's understanding of the given argument. Table 1 summarises this framework in terms of what is the initial given situation in each sub-activity, its particular goal and its expected product.

ACTIVITIES RESEARCHED IN MATHEMATICS EDUCATION

In order to investigate the extent to which the mathematics education literature has focussed on each of these activities, we conducted a bibliographical study aimed to analyse the type of tasks used in empirical research on the notion of mathematical proof and argumentation.

Method

The bibliographical study involved conducting a search of education articles discussing the notion of mathematical proof or argumentation, filtering those that discussed tasks related to this notion (either to illustrate a theoretical viewpoint or as part of an instrument in an empirical study), and classifying these tasks according to their given conditions, their implicit goal and expected product.

Construction	Problem exploration	Estimation of truth	Justification
Given	A problem situation	A conjecture	A statement estimated to be true
Goal	Answer an open-ended question	Estimate the truth of the conjecture	Justify the statement
Product	An argument with a new statement as claim	An argument with the conjecture as claim and a non-neutral qualifier	An argument with the given statement as claim

Table 1a: Activities associated with the construction of a novel argument

Reading	Comprehension	Evaluation
Given	An argument	An argument and a set of criteria
Goal	Understand the argument	Assess the argument against the given criteria
Product	Possibly sub-arguments with the given argument's statements as claims	An assessment (yes/no or continuous) and possibly a justification of the assessment

Table 1b: Activities associated with the reading of a given argument.

Presentation	Conviction of an audience	Explanation to an audience	Demonstration: validity	Demonstration: understanding
Given	An argument and an audience			An argument and an expert
Goal	Convince the audience that the claim is true using the argument	Explain to the audience why the statement is true using the argument	Demonstrate to the audience the validity of the argument	Demonstrate to the expert that one understands the argument
Product	A variation of the given argument			

Table 1c: Activities associated with the presentation of a given argument.

In order to avoid sampling biases, we conducted the search using only one database, the Education Resources Information Center (ERIC), a large online digital library of education research. We searched the ERIC database for all journal articles with the keywords 'argumentation' and 'mathematics', or 'proof' and ‘mathematics’ (i.e. Publication Date: pre-1966 to 2008; Keywords (all fields): argumentation AND mathematics, proof AND mathematics; Publication Type: Journal Articles; Education Level: Any Education Level).

This search (conducted in August 2008) produced a list of 641 articles, which included a large number of articles that were irrelevant to our study (e.g. *American Mathematical Monthly* articles presenting actual mathematical proofs, and non-empirical articles published in professional journals as *Mathematical Teacher*). Consequently, from these 641 articles, we selected those that appeared in one of seven journals chosen for their tradition of publishing empirical mathematics education research: *Cognition & Instruction, Educational Studies in Mathematics, For the Learning of Mathematics, Journal of Mathematical Behaviour, Journal for Research in Mathematics Education, Mathematical Thinking and Learning* and *ZDM*. Our final sample contained the 131 articles from the original list that had been published in one of these journals.

We then searched each of these 131 articles for any task related to the notion of proof. Each of these tasks was classified into one or more of the nine sub-activities in our framework, depending on the initial conditions of the task, its primary goal and product. For instance, in one of the articles in our sample, Recio and Godino (2001) discussed undergraduate students’ responses to the following task: “Prove that the difference between the squares of every two consecutive natural numbers is always an odd number, and that it is equal to the sum of these numbers.” This particular task gave students a specific statement and asked them to prove it. Therefore, this task was classified as involving the justification of a given statement. Selden and Selden (2003) asked mathematics undergraduates to read purported proofs and decide whether or not they were proofs. Clearly, Selden and Selden’s task involved reading a given argument, with the goal of evaluating it against the criteria of validity. Therefore, these tasks were classified as involving the validation of a given argument.

Results

Table 2 presents the number of articles that discussed at least one task related to a sub-activity in our framework. From those articles in our sample that discussed specific tasks, the majority (82 papers) addressed students’ construction of novel arguments, some (24 papers) involved students’ reading of given arguments and none focused on the presentation of a given argument. In particular, only 3 articles addressed tasks related to the comprehension of a given argument and none of the articles discussed tasks directly focussed on the presentation of an argument to demonstrate students’ understanding of it.

Main activity	Sub-activity	N
Construction	Exploration of a problem	44
	Estimation of the truth of a conjecture	16
	Justification of a statement	22
Reading	Comprehension	3
	Evaluation–Miscellaneous (e.g. ‘convincing, explanatory?’)	10
	Evaluation–Validation (e.g. ‘is it a proof?’)	11
Presentation	Conviction of an audience	0
	Explanation to an audience	0
	Demonstration of validity	0
	Demonstration of understanding	0

Table 2: Number of articles discussing tasks in each sub-activity

DISCUSSION

What argumentative activities related to the notion of proof do students normally engage in when learning mathematics? This is an interesting question that could be studied empirically. Nevertheless, from our own experiences as mathematics students and teachers at the secondary school and undergraduate level we suspect that there are three main proving activities in which students regularly engage when learning mathematics:

- Construction of novel arguments: in the exploration of a given problem situation, while estimating the truth of a given conjecture (e.g. when addressing ‘true or false’ questions), or when asked to justify/prove a statement they had not seen before (mainly in classroom activities or assignments);
- Reading arguments given by teachers/lecturers or presented in books with the objective of understanding them;
- Presenting arguments that they had previously read in order to explain these arguments to their peers (during class), or to demonstrate to their teachers that they understand them (normally in exams).

It is hard to determine the relative importance of each of these activities in the learning of mathematics without data from a detailed empirical study on the types of argumentative activities that students engage in during classroom activities, when working on homework assignments, and taking exams. However, we hypothesise that (i) the comprehension of given mathematical arguments and (ii) the presentation of these arguments to demonstrate one’s understanding of them, are two of the key activities involved in the assessment of undergraduate students' proving skills: students spend long periods of time trying to understand proofs in mathematics textbooks and lecture notes, and then present these arguments (or parts of them) to their teachers in exams with the aim of demonstrating their understanding. If this is indeed the case, our findings suggest that we, mathematics educators, know very little about students' behaviour in some of the main types of activities involved in the assessment of their proving skills, which in turn may become the type of activities many students focus on, precisely because of their involvement in assessment.

To summarise, in this paper we have presented a framework of activities associated to the notion of proof, which builds on the specific given conditions and the goals guiding the construction of a novel argument, the reading of an argument and the presentation of a given argument to a given audience. We have also discussed the findings of a bibliographical study on the type of tasks discussed (and employed) in empirical research in mathematics education. These findings suggest that researchers in the field have tended to concentrate on understanding a relatively small subset of the activities associated with mathematical argumentation and proof. In particular, we have suggested that two key argumentative activities involved in the assessment of students' proving skills have yet to receive substantial research attention: the comprehension of given mathematical arguments and the presentation of an external argument to demonstrate one's understanding of it.

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PROOF STATUS FROM A PERSPECTIVE OF ARTICULATION

J. Mamona-Downs* and Martin Downs

**University of Patras, Greece*

In this paper, we discuss and illustrate a position that arguments made in Mathematics may succeed in achieving correct answers but fail to qualify as proofs due to a lack of full articulation, causing the reader to have to decipher the content of the presentation.

INTRODUCTION

It has been long recognized by educational literature that proof does not aim for completely rigorous exposition (see e.g., Hanna & Janke, Thurston, 1996). Naturally, this raises a question; if an argument is made that succeeds to obtain the correct/required result, on what grounds is the argument accepted as a proof or not?

Much research has treated this question in terms of conviction; for example the phrase 'convince yourself, convince a friend, convince an enemy' is quoted and is endorsed (in part at least) in many papers in the field. Some other researchers find that such a 'test' is misleading (e.g., Austin, 1995). It is quite common for students' written scripts to impart the germ of a solution when read by an attentive reader of suitable experience, even though what the students wrote was inchoate and highly fragmented. What lacks is clear exposition.

In tasks involving abstract mathematics, accessing channels of expression that allow lucidity is not necessarily easy for students, Downs & Mamona-Downs (2005), Thurston, (1995). Students can hold very intricate mental argumentation without being able to convey this into ratified mathematical frameworks. This can cause students (especially those studying mathematics at university) to show frustration in not being able to express themselves as they wish.

Another situation is that students do hold the mathematical tools that they need, but have difficulties in handling the associated symbolism correctly. Even though the student may have captured the gist of a correct argument, abuse of usage of symbolism can give evidence that mental organizational processing sometimes over-rides the proper management of the exposition.

In the two previous paragraphs, two circumstances have been described where an argument sufficient to obtain a correct answer fails to constitute a proof. In this paper, we will illustrate these two circumstances and then discuss their educational significance.

In closing this introduction, perhaps a note of clarification is needed. As far as this paper is concerned, we do not regard the term 'proof' simply to refer to the demonstration of propositions that were given to the students. Rather, we see proof in terms of students' work that yields a result, and an argument that backs it;

we are interested whether the argument 'qualifies' as a proof or in any action a student might take to make the argument into a proof. (Note that the context of this study will be on project work, so that there is time for students to revise the final presentation.) The judgement about status between argument/proof is left to the researcher. This differs with some extant research that examines students' evaluation of the work of other students, such as Selden & Selden, (2003).

THE ILLUSTRATING TASK

Background Details

The example used was raised in project work taken by mathematics undergraduates attending a problem-solving course, taught by one of the authors. The course largely followed the ideas of Polya concerning strategy making and heuristics, Polya (1973) and of Schoenfeld concerning metacognition, especially executive control and accessing knowledge, Shoenfeld (1992). Further, the conversion of mental argumentation into accepted mathematical frameworks was given some stress. The course was intended for the attending students to improve their own problem solving skills and mathematical literacy. The course is elective and it is open to mathematics students at any year of the undergraduate program except the first (due to curriculum constraints). Half of the weight of the grading of the course was assigned to the students' contribution in project work. Each student was involved in one project. The students worked individually or in small groups of two or three. The time that the students had to complete their work was 3 weeks. The design of the projects was not particularly concerned about 'openness'. Instead, a sequence of tasks was given where the resolving of the later tasks likely requires either an indirect or direct reference to the solution of previous tasks. Usually, each project is assigned to two groups. After the project was handed back and appraised, we asked each group (as a body) for a semi-structured interview of 1-3/2 hours.

The illustrating task below was answered correctly by the two groups assigned to the relevant project, however their results appear in differing algebraic expressions. The interview mostly was targeted to elucidate how the students achieved and chose the form of their presentation. Group 1 had two students (one of high ability, the other medium), Group 2 had two students (both of high ability).

The task

The subject of the project at hand concerned the calculation or application of the greatest power of a natural number dividing another given natural. In fact the tasks always concerned the case of greatest powers of 2, though some were open to generalization to other integers. Printed on the assignment sheet was the following definition:

"Given a natural number n, the symbol $2^r \| n$ means that the number r is the greatest whole number for which 2^r divides n".

This symbolism is commonly used in textbooks of Number Theory and the students had past experience using it. The particular task used is as follows:

$Part\,(a): Let\ n \in N. Suppose\ that\ r_n\ satisfies\ 2^{r_n} \,\|\, 2^n!\ \ Find\ \ r_n.$

$Part\,(b): Let\ m \in N. Suppose\ that\ s_m\ satisfies\ 2^{s_m} \,\|\, m!\ \ Find\ \ s_m.$

The two different forms of r_n raised for part (a).

The two groups of students assigned to this particular project will be denoted Group 1 and Group 2. In the project work, Group 1 attained the result $r_n = 2^n - 1$, whilst Group 2 gave their final answer as:

$$r_n = \sum_{i=1}^{n-1}\left(\frac{i.2^{n-i}}{2}\right) + n.$$

The reasoning used for part (a)

For Group 1, the reasoning is presented below (translated into English from Greek):

"We know that from the numbers 1, 2, 3, …, 2^n, there are 2^{n-1} numbers which are divided by 2. We note that from the numbers 1, 2, 3, …, 2^{n-1}, there are 2^{n-2} numbers that are divided by 2. We note that from the numbers 1, 2, 3, …, 2^{n-2}, there are 2^{n-3} numbers that are divided by 2. Continuing to the end we have that $2^n!$ = 1.2.3…2^n is divided by 2 raised to the power

$2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2 + 1$. "

For Group 2, the approach used basically followed the lines of the argument below:

Let $S := \{1, 2, 3, \ldots, 2^m\}$.

For i=0, 1, 2, …,m define $S_i = \{s \in S : 2^i \| s\}$, so

$$r_m = \sum_{i=1}^{m} |S_i|.i$$

Then $|S_i|$ is calculated.

However the above is somehow a 'cleaned up' version of the one given in the script of Group 2. The main discrepancy is that Group 2 made their exposition in terms of equivalent classes, determined by the equivalent relation '~' on N defined by $\alpha \sim \beta \Rightarrow \exists\, r \in N$ such that $(2^r \| \alpha) \wedge (2^r \| \beta)$.

The reasoning used for part (b)

The process developed by Group 1 in part (a) when generalized also covers part (b); this is not considered here.

We transcribe the presentation of part (b) for Group 2 below with lines numbered for later reference.

1. *We name n as the biggest power of* 2 *which is not bigger than m.*
2. *That is we have that* $2^n \leq m < 2^{n+1}$
3. *Let* $\theta < m$, *so* θ *is a term in* $m!$
4. *We put* $n = n_1$ *and put* $\theta = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_k} + a$
   where $n_1 > n_2 > \ldots > n_k$ *and* $a < 2^{n_k}$.
5. *We have* $2^b \| a \Rightarrow a = 2^b \pi \Rightarrow \theta = 2^b(2^{n_1-b} + 2^{n_2-b} + \ldots + 2^{n_k-b} + \pi) \Rightarrow \theta \sim a, \quad \pi : odd$
6. *Suppose* $\theta_j = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_k} + a_j$, *with* $j \in \{1, \ldots, 2^{k+1}\}$
7. *According to the above then* $\theta_j \sim a_j \Rightarrow \prod_{j=1}^{2^{k+1}} \theta_j \sim \prod_{j=1}^{2^{k+1}} a_j = (2^{k+1})!$
8. *Then* $m! = (2^{n_1})! \prod_{j=1}^{2^{n2}} \theta_j \prod_{j=1}^{2^{n3}} \theta_j \ldots \prod_{j=1}^{2^{nk}} \theta_j \sim (2^{n_1})!(2^{n_2})! \ldots 2^{n_k}!$

   $= \prod_{i=0}^{n} ((2^i)!)^{a_i} \quad , a_i = 1 \text{ or } 2.$
9. *Next* $2^s \| m! \Rightarrow 2^s \sim m! \Rightarrow 2^s = \prod_{i=0}^{n} ((2^{r_i}))^{a_i} \Rightarrow s = \sum_{i=0}^{n} a_i \times r_i$
10. *From part* (a) *we have that* r_i *is equal to* $\sum_{j=1}^{i-1} (\frac{j \times 2^{i-j}}{2}) + i$

COMMENTING OF THE TWO GROUPS' PRESENTATIONS

For group 1, it is fairly easy to decipher what the students are 'doing' and why. But it is still very much up to the reader to link what they write with the mathematical environment of the given task. Because of this, their presentation failed to refer to a system to which all the objects and actions that appear are explicitly related. Without 'exterior' interpretation, a literal reading has little meaning. For these reasons, the students display a mental understanding as is evident in the isolated phrases that do appear, and they 'had' a secure argument in their minds, but this could not be properly articulated and so could not be accepted as a proof. Two further points should be made. First concerns the brevity of their expression. Given the intuitive base in answering, they could have made an attempt to fill up their reasoning in an informal way. In the interview, the students gave the impression that trying to add some extra expression would only serve to remove further precision from their argument. They expressed disaffection about their exposition, did not regard it as a proof, but at the same time they could not see a way to make it as one. The second point is that the students did not make recourse to induction. Generally the students in the course were adept in carrying out and applying the method of induction. Once the students obtained the expression $r_n = 2^n - 1$ informally, a fairly straightforward induction proof is available. Why did they not think of doing this? In the interview, the students were not so clear on this issue. We contend that they had invested so much on their original line of thinking that induction may not be so naturally considered, or it would in some sense 'degrade' the aesthetics held by the students in their first approach.

Group 2 introduced the equivalence relation $\alpha\sim\beta \Rightarrow \exists\ r\in N$ such that $(2^r \| \alpha) \wedge (2^r \| \beta)$. There might be grounds here to criticize bringing up an over elaborate mathematical tool as the task could be approached simply by defining a set partition (concerning the S_i referred to in section 2) , but the students claimed that the introduction of the equivalence relation helped them to initialize and sustain their argumentation in a mathematical framework. The disadvantage was that doing this involved relatively sophisticated symbolism. In part (a), the presentation produced by the group essentially constituted a proof. However, there were weaknesses in the handling of symbolism; for example, once an element of a space was related to a class, another time a system of indexing is introduced that is not explicitly explained and not used further. All such instances constituted rather isolated and inessential flaws. But their appearance suggested mental processing supported by the symbol system, whereas the latter did not fully constrain the former. In part (b), this tendency continues, but its effect becomes far more serious. The main problem on the technical side is that the single indexing introduced on line 6 needed a system of double indexing on both k and j. However, in spite of this serious flaw in control of symbolism, the students' mental processing was entirely functional, and even quite impressive. Another aspect of the presentation is the difficulty for the reader how to interpret line 4. Only when line 8 is reached the reader can guess the relative roles of the symbols. Even then, exactly what the symbolism refers to remains shifting and inconsistent, hence a proof is not articulated.

CONCLUSION

We have argued and illustrated that convincing argument producing correct answers can fail to constitute proof. We claim that a major problem lies in full articulation; the reader of the presentation has to make assumptions or interpretations of what is written down. A proof does not have the margin for such surmising. The two examples related in the paper might seem very different in character; the first not entering a strict mathematical framework, whereas the second is fully accommodated within one. However, we have found that there are aspects of mental thinking that seem common to both. Also, students seem to have difficulty in designing the lay out of their final proof attempt for the convenience of the reader. In particular, the students tended to give highly terse argumentation in such a way that suggests they believed that the object of a proof is parsimony itself to epitomize their personal thinking rather than an efficient channel of precise communication. These points suggest that some existing educational models bringing out a duality in informal and formal reasoning in proof production, such as semantic and syntactic thinking proposed by Weber &

Alcock (2004), have to be refined. A seemingly syntactic argument may have significant semantic overtones, and vice versa.

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HABERMAS' CONSTRUCT OF RATIONAL BEHAVIOUR AS A COMPREHENSIVE FRAME FOR RESEARCH ON THE TEACHING AND LEARNING OF PROOF

Francesca Morselli and Paolo Boero

Dipartimento di Matematica, Università di Genova, Italy

We present our adaptation of Habermas' construct of "rational behaviour" as a way of framing the teaching and learning of proof, including some cognitive and cultural aspects, which have been dealt with in mathematics education research in the last two decades. The frame is shown to be useful in the discussion of two examples at tertiary level and for some related research developments.

PROVING AS A RATIONAL BEHAVIOUR

This paper refers to the following point of the Discussion Document:

> What theoretical frameworks and methodologies are helpful in understanding the development of proof from primary to tertiary education, and how are these frameworks useful in teaching?

Balacheff (1982) points out that the teaching of proofs and theorems should have the double aim of making students understand what a proof is, and learn to produce it. Accordingly, we think that proof should be dealt with in mathematics education by considering both the object aspect (a product that must meet the epistemic and communicative requirements established in today mathematics - or in school mathematics) and the process aspect (a special case of problem solving: a process intentionally aimed at a proof as product). We have tried (Boero, 2006; Morselli, 2007) to match these considerations with Habermas' elaboration about rationality in discursive practices; we will present here a unified synthesis of our previous work and two directions for its development. Habermas (2003, ch.2) distinguishes three inter-related components of a rational behaviour: the *epistemic component* (inherent in the control of the propositions and their enchaining), the *teleological component* (inherent in the conscious choice of tools to achieve the goal of the activity) and the *communicative component* (inherent in the conscious choice of suitable means of communication within a given community). With an eye to Habermas' elaboration, in the discursive practice of proving we can identify: an *epistemic* aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of "theorem" by Mariotti & al. (1997) as the system consisting of a statement, a proof, derived according to shared inference rules from axioms and other theorems, and a reference theory); a *teleological* aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product; a *communicative* aspect, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture.

Our point is that considering proof and proving according to Habermas' construct may provide the researcher with a comprehensive frame within which: to situate a lot of research work performed in the last two decades; to analyze students' difficulties concerning theorems and proofs (see the two examples in the next Section); and to discuss some related relevant issues and possible implications for the teaching of theorems and proof (see the last Section).

If we are interested in the epistemic rationality side, i.e. in the analysis of proofs and theorems as objects, mathematics education literature offers some historical analyses (like Arsac, 1988) and surveys of epistemological perspectives (like Arzarello, 2007): they help to understand how theorems and proofs have been originated and have been considered in different historical periods and how, even in the last decades, there is no shared agreement about what makes proof a "mathematical proof" (cf. Habermas' comment about the historically and socially situated character of epistemic rationality). Concerning the ways mathematical proof and theorems are (or should be) introduced in school as "objects", several results and perspectives have been produced, according to different epistemological perspectives and focus of analyses. In particular, De Villiers (1990), Hanna (1990), Hanna & Barbeau (2008) discuss the functions that mathematical proofs and theorems play within mathematics and advocate that the same functions should be highlighted when presenting proof in the classroom, in order to motivate students to proof and allow them to understand its importance. By referring proof to the model of formal derivation, Duval (2007) focuses on the distance between mathematical proof and ordinary argumentation; he also considers how to make students aware of that distance and able to manage the construction and control of a deductive chain. Harel (2008) uses the DNR construct to frame the classification of students' proof schemes (they concern proof as a final product). We note that, in terms of Habermas' components of rationality, Harel' s ritual and non-referential symbolic proof schemes may be attributed to the dominance of the communicative aspect, with lacks inherent in the epistemic component (cf. Harel' s N, "intellectual Necessity").

Concerning the proving process, some analyses of its relationships with arguing and conjecturing suggest possible ways to enable students to manage the teleological rationality. In particular, Boero, Douek & Ferrari (2008) focus on the existence of common features between arguing, on one side, and proving processes on the other, and present some activities (from grade I on), based on those commonalities, that may prepare students to develop effective proving processes. Garuti, Boero & Lemut (1996) suggest the possibility of smoothing the school approach to mathematical proof through unified tasks of conjecturing and proving for suitable theorems (those for which the same arguments produced in the conjecturing phase can be used in the proving phase: "cognitive unity"). However Pedemonte (2007) shows how in some cases of "cognitive unity", students meet difficulties inherent in the lack of "structural continuity" (when they have to move from creative ways of finding good reasons for the validity of a statement, to their organization in a deductive chain and an acceptable proof): this study suggests to consider the relationships between teleologic, epistemic and communicative rationality (see last Section).

TWO EXAMPLES OF ANALYSIS WITHIN THE FRAME

Morselli (2007) investigated the conjecturing and proving processes carried out by different groups of university students (7 first year and 11 third year mathematics students, 29 third year students preparing to become primary school teachers). The students were given the following problem: *What can you tell about the divisors of two consecutive numbers? Motivate your answer in general.* Different proofs can be carried out at different mathematical levels (by exploiting divisibility, or properties of the remainder, or algebraic tools). The students worked out the problem individually, writing down their process of solution (including all the attempts done); afterwards, students were asked to reconstruct their process and comment it. The a posteriori interviews were audio-recorded. In (Morselli, 2007) several examples of individual solutions and related interviews are provided, and in particular it is shown how students' failures or mistakes were due to lacks in some aspects of rationality and/or the dominance of one aspect over the others. Due to space constraints, we will consider here only some pieces of two very similar examples, concerning students that are preparing to teach in primary school, in order to show how Habermas' construct works as a tool for in-depth analyses, and introduce the Discussion.

Example 1: Monica

Monica considers two couples of numbers: 14, 15 and 24, 25. By listing the divisors, she discovers that "Two consecutive numbers are odd and even, hence only the even number will be divided by 2". Afterwards, she lists the divisors of 6 and 7 and writes: "Even numbers may have both odd and even divisors". After a check on 19 and 20, she writes the discovered property, followed by its proof:

> Property: two consecutive numbers have only one common divisor, the number 1. In order to prove it, I can start saying that two consecutive numbers certainly cannot have common divisors that are even, since odd numbers cannot be divided by an even number. They also cannot have common divisors different from *1*, because between the two numbers there is only one unit; if a number is divisible by *3*, the next number that is divisible by *3* will be greater by *3* units, and not by only one unit. Since *3* is the first odd number after *1*, there are no other numbers that can work as divisors of two consecutive numbers.

Monica carries out a reasoning intentionally aimed (teleological aspect): first, at the production of a good conjecture; then, at its proof. Proof steps are justified one by one (epistemic aspect) and communicated with appropriate technical expressions (communicative aspect). The only lack in terms of rationality concerns the short-cut in the last part of the proof: Monica realizes that something similar to what happens with 3 (the next multiple is "greater by three units") shall happen *a fortiori* with the other odd numbers that are bigger than 3 ("Since 3 is the first odd number after 1"), but she does not make it explicit. Her awareness (cf.

epistemic rationality) is not communicated in the due, explicit mathematical form (lack of communicative rationality). Monica's a posteriori comments on her text confirm the analysis:

> M: (...) and then I have thought that 3 was the first odd number after 1 and so if 3 does not enter there, also the bigger ones do not enter there [from the previous text, "there" means: between two consecutive numbers on the number line].
>
> Res.: to make more general what you said with 3, what would you write now?
>
> M: ehm... I have tried to go beyond the specific case of 3, but I do not know if I have succeeded in it.

Example 2: Caterina

> Starting from the fact that two consecutive numbers are always one odd and one even, we may conclude that the two numbers cannot be both divided by an even number. Afterwards, we focus on odd divisors; we start from 1, and we know that all numbers may be divided by 1; the second one is 3. We have two consecutive numbers, then the difference between them is 1, then they will not be multiples of 3, since it will be impossible to divide both of them by a number bigger than 1.

Caterina is able to justify all the explicit steps of her reasoning (epistemic aspect), she develops a goal-oriented reasoning (teleological aspect) and illustrates her process with appropriate technical expressions (communicative aspect). Differently from Monica, in spite of a good intuition there is a lack in her reasoning: divisors greater than 3 are not considered. A posteriori, after having seen also the production of her colleagues, Caterina comments:

> My reasoning is not mistaken: indeed, I reach the conclusion giving a general explanation, saying that, since there is no more than one unit between the two numbers, the only common divisor is 1. Nevertheless, I can not create a mathematical rule. Observing the other solutions, I think that the correct rule is the following: along the number line we note that a multiple of 2 occurs every two numbers, a multiple of 3 occurs every three numbers, hence a multiple of N occurs every N numbers. Then, two consecutive numbers have only 1 as common divisor.

From the objective point of view of epistemic rationality, Caterina's argument was not complete, and in her comment she reveals not to be aware of it. From her subjective point of view, Caterina is convinced to have found a cogent reason for the validity of the conjecture ("not mistaken reasoning", "general explanation"), thus to have achieved her goal (teleological rationality). Some colleagues' solutions induces her to reflect on the lack of a "mathematical rule"; however from her comment it seems that this lack is not considered by her as a lack in the reasoning but as a lack in the mathematical communication.

DISCUSSION: TOWARDS FURTHER DEVELOPMENTS

The previous analysis suggests investigating what are the relationships between epistemic rationality, communicative rationality and teleological rationality in the case of proof and proving. We note that in the historical development of mathematics, subjective evidence (or even mathematicians' shared opinion of evidence) revealed to be fallacious in some cases, when new, more compelling communication rules obliged mathematicians to make some steps of reasoning (in particular, those concerning definitions: see Lakatos, 1976) fully explicit. From the educational point of view, while it is easy (for instance, by comparison with other solutions) to intervene on Monica by helping her to make what she thinks more explicit (according to her need: see her comments), the intervention on Caterina is much more delicate: how to make her aware that the "mathematical rule" is not only a matter of conventional, more complete communication, but also a matter of objective, cogent arguing involving the goal to achieve (an exhaustive argument)? And how to exploit texts that are complete (communicative aspect) in order to develop the need of an exhaustive argument (epistemic aspect), but at the same how to avoid that the necessities inherent in the communicative aspect prevail over the epistemic aspect (cf Harel' s "ritual proof schemes")? A direction for productive educational developments might consist in the elaboration of a suitable meta-mathematical discourse (see Morselli, 2007) for students (including an appropriate vocabulary), as well as in the choice of suitable tasks that reveal how intuitive evidence not developed into an explicit, detailed justification sometimes results in fallacious conclusions.

These considerations raise another problem: Habermas' construct offers only the possibility to evaluate a production process and its written or oral products, while in mathematics education we need also to consider a long term "enculturation" process. We are working now on the articulation between a cultural perspective to frame this process (see Morselli, 2007) and tools of analysis derived from Habermas' elaboration on rationality. We think that is necessary to deal with mathematics as a multifaceted culture (Hatano & Wertsch, 2001) evolving through the history, which includes different kinds of activities and different levels of awareness, explicitness and voluntary use of notions (thus different levels of "scientific" mastery, according to the Vygotskian distinction between common knowledge and scientific knowledge). Within mathematics, the "culture of theorems" is the complex system of conscious systematic knowledge, activities and communication rules, which concerns the processes of conjecturing and proving as well as their final products. It is in this cultural perspective that we can describe the approach to theorems and proving in school as a process of scientific "enculturation" consisting in the development of a special kind of rational behavior (Habermas), the one presented in this paper.

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RESORTING TO NON EUCLIDEAN PLANE GEOMETRIES TO DEVELOP DEDUCTIVE REASONING AN ONTO-SEMIOTIC APPROACH

Teresa Neto, Ana Breda, Nilza Costa and Juan D. Godinho

University of Aveiro Portugal, University of Granada Spain

The main idea behind this work is the study of the potential of resorting to other models of Plane Geometry (e.g. Hyperbolic Geometry, Taxicab Geometry) to help students to progress towards a proper/better understanding of what a mathematical proof is about. A teaching experiment carried out with students of 15 to 17 years-old attending the 10th and11thgrade (the two first years of secondary school) of a Portuguese school. The experience started in the10th grade and lasted in the 11thgrade. Our main focus is the analysis of primary and secondary relationships of geometric objects involved in argumentation and proof (in the sense of Godino et al. and Gutiérrez et al.) activated by the students during production of arguments.

Recent research in the onto-semiotic approach to mathematics knowledge and instruction has highlighted that the systems of practices and its configurations are proposed as theoretical tools to describe mathematical knowledge, in its double version: personal and institutional (see Godino et al., 2007). Following these ideas, this researchers refer that for a finer analysis of the mathematical activity it is necessary to take into account six types of primary entities: Problem situation; Language (e.g., terms, expressions, notations, graphs) in its various registers (e.g., written, oral, sign language); Concepts (approached through definitions or descriptions); Propositions (statements on concepts); Procedures (e.g., algorithms, operations, calculation techniques); Arguments (statements used to validate or explain the propositions and procedures, of deductive nature or another type). These six objects relate to each other by epistemic (networks of institutional objects) and cognitive configurations (networks of personal objects).

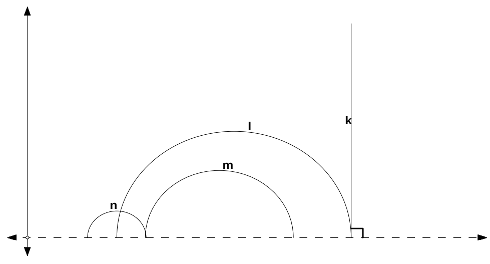
Considering an entity as being primary is not an absolute question but rather a relative one, since we are dealing with functional entities in contexts of use. The contextual attributes signaled by these researchers are: Personal/institutional - The personal cognition is the result of thought and action of the individual subject confronted by a class of problems, whereas institutional cognition is the result of dialogue, understanding and regulation within a group of individuals who make up a community of practices; Ostensive / non-ostensive – The ostensive attribute refers to the representation of a non-ostensive object, that is to say, of an object that cannot be shown to another. The classification between ostensive and non-ostensive depends on the contexts of use. Diagram, graphics and symbols are examples of objects with ostensive attributes, perforated cubes and plane sections are examples of objects with non-ostensive attributes; Expression / content (antecedent and consequent of any semiotic function) - The relationship is established by means of semiotic functions, understood as a relationship between an antecedent (expression, signifier) and a consequent (content, signified or meaning) established by a subject (person or institution) according to a specific criterion or code of correspondence; Extensive / intensive (specific / general) – This duality is used to explain one of the basic characteristics of mathematical activity, namely generalization. This duality allows for the centre of attention to be the dialectics between the particular and the general, which is undoubtedly a key issue in the construction and application of mathematical knowledge; Unitary / systemic – In certain circumstances, mathematical objects participate as unitary entities, in others, they should be taken as the decomposition of others so that they can be studied.

A considerable number of researchers have been researching the nature of argumentation and types of proof (e.g. Harel and Sowder, 2007; Marrades and Gutiérrez 2000). In this theoretical frame, the following question is of particular interest: How can other models of Plane Geometry, other than the Euclidean one, help Secondary School students to develop deductive reasoning?

Two levels of accomplishment were set up for this work. The first one, being in a classroom environment with a class of 20 students (15-16 years of age) in 10th grade – Secondary School, from the social economics field in the 2004/2005 school year. The second level, situated on the study of the individual cognitive trajectories of two students (both girls 16 years of age) from the mentioned class, during their 11th grade (2005/2006 school year) which, even though it focused on the same questions as those defined for the class, allowed for a more detailed level of analysis. The empirical study in the second stage of the study was developed in an extra-classroom scenario in sessions of small work groups that ran in parallel to the mathematics class. In particular, we are concerned with the student's ability in argumentation and proof. The mediation offered by other model of plane geometry, other than Euclidean one, in the conceptualisation of meaning for parallelism concept and the methods of proof (e.g., method of proof by contraction) is very important for the cognitive aspects of proof. Following is the epistemic configuration and the cognitive configuration and trajectory of one of the problems proposed in the study process. The problem is written below:

The following diagram represents various hyperbolic lines (l, m, n and k) on the Poincaré half-plane, defined respectively by the conditions:

l: $(x - 7)^2 + y^2 = 16 \wedge y > 0$
m: $(x – 6,5)^2 + y^2 = 6,25 \wedge y > 0$
n: $(x - 3)^2 + y^2 = 1 \wedge y > 0$
k: $x = 11 \wedge y > 0$



Indicate if there are two parallel lines and two non-parallel lines. Justify.

Episode 1: After reading and analyzing the drawing supplied, the following dialogue took place:

> X. Teach', is the definition of parallels the same? ; Teacher: Yes, the definition is the same;
>
> X. Then, two lines, no matter how far they are prolonged, never intercept; Y. These are not parallel (referring to l and n); X. But these two are (referring to l and m); Y. But they're not parallel…; X. How do you know?; Y. Oh, you can see… the distance from here to here and from here to here … (referring to the Euclidean distance between the two semi-circumferences, representative of the hyperbolic lines in question);X. But the distance doesn't have to be the same. It does, when they are parallel, this distance from here to here is always the same as from here to here and from here to here… (pointing at lines l and m). Isn't it?

Students are silent while observing the drawing; Student Y identified the value of the radius in hyperbolic lines l, m and n and noted it down next to the drawing.

> Y. Oh Teach', I have a question. Is it two lines or two straight lines? ; Teacher: Two lines. We had already decided that in hyperbolic geometry we speak of lines; Y. It's just that these don't intercept but they're not parallel either… (referring to l and m) the distance from here to here is not the same as from here to here; Teacher: Why do you say they are not parallel?; Y. Because the distance from here to here is not the same as from here to here; Teacher: Are you thinking of Euclidean geometry?; Y. Aha! Then they can be parallel …;X. Two parallels are l and m…aren't they?; Y. This here asks for two…

Episode 2: Setting up the justification. The following dialogue took place:

> X. You're only going to give one example…;Y. Yes…;X. I think we should first supply the more obvious ones let's try other interpretations… (the coordinates of the centers) The centers are seven, zero and six and a half, zero…and if you check, it's correct.

Next, and after the teacher's request, each student explained the reasoning set up by reading the respective solution.

> X. In Poincaré geometry, the definition of parallelism being the same as in Euclidean geometry, we can verify that m and l are parallel, since these lines never intercept and l and n are non-parallel since they intercept at one point; Y. Two lines are said to be parallel in any geometry when their interception is an empty set. So m is parallel to l and l is not parallel to n.; Teacher: It seems you all consider m and l to be parallel and that m, n and l, k and l, n are not parallel. Why?; X. Since the image …;Teacher: And couldn't you present a more convincing argument?; X. We can…we just need to know how (laughed); Teacher: In analytical geometry, when you wanted to determine the intersection of, for example, the straight lines of equation y equals two x plus four and y equals minus x plus two, how did you do it?; X. We would do the system and we'd have the point….

The students then adopted an analytical approach to justify the answer put forward. Student Y resorted to the resolution of systems to verify the relationship of parallelism between lines k-l and l-n. When student X determined the point of interception of lines l and k, the following dialogue took place:

> X. Teach', this gives us a very weird point…I must have this wrong!; Teacher: And why is it weird?; X. Well, because it gives eleven, zero …;Teacher: And why is it weird?; X. Because the eleven should be farther up (student laughed);Y. It's not the eleven, it's the x;X. Oh, of course it is! Ok, I was seeing this backwards; Teacher: So, is it acceptable now?; Y. Yes, it is …;X. No: Y. Yes …eleven is: X. Alright…but y has to be greater than zero; it can't be zero; Y. But they intercept in one point…; X. That's right…but it's not valid because y has to be greater than zero; Teacher: So what do you conclude?; X and Y. So the only parallel ones here are l with m; Y. (Lines) m and n are also non-parallels because they intercept in point two, zero.

After solving the two compound systems of equations, the following dialogue took place:

> Y. That definition of parallelism, when we say no matter how far they are prolonged, is wrong for circumferences because take a look at these; X. I see what you mean…;Y. We don't have to say no matter how far they are prolonged. […]; X. (Lines) l and n are the only ones that do not intercept.

Note that student X uses the designation of straight lines and not lines, she follows the definition of parallelism associated to the existence of intersection and no longer associates parallelism to the initial expression "[…] no matter how far they are prolonged, they never meet.[…]"

As for the procedures adopted, student X's choice for the algebraic one is evident. In spite of this student visualizing point B, of interception of lines m and n, she resolves a system and indicates coordinates of that point, with figures rounded off to the hundredths. The algebraization of the problem helped clarify likely doubts on the parallelism of some lines. It seems that the visualization of the drawing did not induce wrong reasoning. The justification put forward is based on the previous procedures and had a deductive nature, where the specific examples were used to support the organization of the justifications – thought-out experimentation. Student Y used graph and algebra languages, as aids in identifying parallel and non-parallel lines. The drawing supplied in the exposition comprises an aid in identifying parallel and non-parallel lines. The situation put forward aimed at strengthening visualization and valuing the role of the Poincaré half-plane definition in justifying the indication of parallel and non-parallel lines. Algebraic language aids in clarifying likely doubts on the parallelism of some lines, such as lines l and k. The problem also gave rise to the approach of concepts, properties (e.g., definition of parallel lines in an abstract geometry). The justification was of the conceptual type, based on the definitions of the Poincaré half-plane and of parallel lines. The sequence of procedures adopted by the students was visualisation – reasoning. But could visualisation, in this case, have induced wrong reasoning? Visualization, in the ascending phase of problem resolution, gave rise to the intuition of some parallel lines (e.g., n and m) which in reality were not. In fact, the relationships of parallelism between the lines given in the problem statement was not intuitive, it was not obvious and they were accepted based on carrying out a more formal verification (resorting to the resolution of systems, resorting to the Poincaré half-plane definition…).

Next we expose an interpretation centered on the student´s arguments applying the contextual attributes: Ostensive – non-ostensive- Student X, used points A and B to mark, respectively, the intersection of lines l,n and m,n. Nevertheless, it seems to us that she felt the need to determine the coordinates of the points to recognize the non-ostensive (non-parallel lines and parallel lines). Therefore, the ostensive objects brought forward in presenting the solution to the problem were the representation of points A and B in the drawing supplied in the exposition and the systems of the respective conditions which define the hyperbolic lines in question. Student Y used: the "//" notation (ostensive) to refer to the relationship of parallelism (non-ostensive) between lines; the algebraic language and the symbol (if…then…) when joining sentences; Extensive – Intensive- Student X used the condition given in the statement as support to identify the centers of the semi-circumferences. The definition given in the beginning "Parallelism – when two lines, no matter how far thy are prolonged, never intersect" is adopted by the student for hyperbolic geometry, which she designates as Poincaré geometry. However, in the solution of the problem, she only refers to the existence or not of intersection. Student Y started by writing: Two lines are said to be parallel (in any geometry) when their intersection is an empty set. In other words, she thought of the definition of parallel lines and only then she focus on the extensive objects represented in the problem statement; Institutional – personal: If, on the one hand, visualisation is revealed to be a means to provide a solution to the problem, on the other, the more recent experiences of these students in the scope of parallelism of lines, in Euclidean geometry, was carried out according to an analytical approach and by resorting to the resolution of equation systems. Therefore, at personal cognition level, the problem situation generated the following conflicts in terms of defining parallel lines: Student Y used the ostensive of parallel lines of Euclidean geometry, in the context of hyperbolic geometry (according to episode 1). Student X presented a definition of parallel lines right in the beginning of the written solution (drawing …) where she refers "…no matter how far they are prolonged, they never intersect" and confronted by the maladjustment of this definition – by student Y – she does not present any arguments; Unitary – systemic.

The analysis carried out by both students displays different aspects. Student X feels the need to break down the exposition, recording the coordinates of the centers of the semi-circumferences and the points of intersection of lines l, n and m, n. Student Y, upon breaking down the exposition, records the value of the radii of the mentioned semi-circumferences and focuses on the distance between them. In student X's case, she refers to the only "straight lines" that are not parallel and then states: "All the others are // between themselves because they never intersect since y=0 does not belong to the half-plane". In student Y's case, the conclusion includes reference to the relationship of parallelism between the lines two by two; Expression – content - The problem situation induces the definition of parallel lines in a context of hyperbolic geometry. The students revealed a command of algebraic calculus but in terms of the language, student X seems not to be familiar with some issues of hyperbolic geometry language.

The justification they present is of a conceptual nature – based on the definition of parallel lines in an abstract geometry, formulation of properties (Properties of the relationship of parallelism) and on algebraic calculus (symbolic calculus). The justification is based on the resolution of systems of equations, on the use of formalized symbolic expressions. The evolution from an ascending phase, characterized by empirical activity, to a descending phase, in which the students produce deductive justification, was clear.

The problems proposed created conflicts between an intuitive interpretation and formal argumentation. The resolution of these conflicts allowed for an evolution of knowledge and argumentative skills (e.g., the role attributed to definitions).

The study suggests that a diversified geometric approach, through various models of plane geometry, promoted a different understanding of the processes leading to the deductive reasoning.

Mathematical argumentation can be better understood and assessed if we are aware that the arguments are interconnected with the primary and secondary objects defined in the onto-semiotic focus of mathematical cognition.

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‘BECAUSE THIS IS HOW MATHEMATICIANS WORK!’ ‘PICTURES’ AND THE CREATIVE FUZZINESS OF THE DIDACTICAL CONTRACT AT UNIVERSITY LEVEL

Elena Nardi

University of East Anglia, UK

Students often have a turbulent relationship with visualisation: they have difficulty with linking visual with other representations; they are reluctant to visualise; and, they are ambivalent about the acceptability of visual representations. Here I draw on a study that engaged university mathematicians in reflection on student learning and pedagogical practice in order to explore their perspectives on the role of visualisation in student learning – especially in the light of how they employ visualisation in their own mathematical practice. The emergent perspective is of a clarified didactical contract, in which students are encouraged to emulate the flexible ways in which mathematicians to-and-fro between analytical rigour and often visually-based intuition.

INTRODUCTION

In recent years mathematics education researchers have been demonstrating increasing interest in studies of teacher thinking – both pedagogical and mathematical (Artigue & Kilpatrick, 2008). At university level this rise in interest has been even more accentuated: in a field that was, until recently, mostly preoccupied with studies of student learning of advanced mathematical topics – see (Holton, 2001) for an international overview – several studies have been focusing on mathematicians’ epistemological (e.g. Burton, 2004) and pedagogical (e.g. Jaworski, 2002) perspectives and practices. As Michèle Artigue and Jeremy Kilpatrick stressed in their ICME11 plenary session (ibid) an increasing number of mathematics education researchers are preoccupied with the relationship between how teachers think about and engage with mathematics and how they perceive student learning and teach.

In this paper I draw on a study which engaged university mathematicians with reflection on their students’ learning and their pedagogical practices (Nardi, 2007; Iannone & Nardi, 2005) in order to discuss their perspectives on a matter which has been generating increasingly intensified debate: the role of visualisation in student learning. Within the community of mathematics central to this debate has been ‘whether, or to what extent, visual representation can be used, not only as evidence or inspiration for a mathematical statement, but also in its justification’ (Hannah & Sidoli, 2007, p73). As recent works suggest (e.g. Giaquinto, 2007) whether visual representations need to be treated as adjuncts to proofs, as an integral part of proof or as proofs themselves remains a point of contention – and a point of poignant relevance also for mathematics educators (Presmeg, 2006).

A STUDY OF MATHEMATICIANS' PEDAGOGICAL PERSPECTIVES

The evidence base of the study on which this paper draws consists of **focused group interviews with mathematicians** of varying experience and backgrounds from across the UK. Its analyses, carried out through the narrative method of *re-storying* (Clandinin & Connelly, 2000) have been presented, primarily in (Nardi, 2007), in the format of **a dialogue between two fictional, yet entirely data-grounded characters, a mathematician and a researcher in mathematics education**. The study is the latest in a series of studies (1992-2004), conducted by the author and her colleagues at the University of East Anglia and Oxford. These studies:

- explored Year 1 and 2 students' learning (tutorial observations, interviews and written work)
- engaged lecturers in reflection upon learning issues and pedagogical practice (individual and focused group interviews).

The focused group interviews conducted in the course of the latest study lasted about half-a-day. Discussion in the interviews was triggered by *Student Data Samples* based on the findings of the previous studies. These were samples of students' written work, interview transcripts and observation protocols collected during (overall typical in the UK) Year 1 introductory courses in Analysis / Calculus, Linear Algebra and Group Theory.

The findings of the study were arranged in accordance with the following themes:

1. *Student learning*

- students' mathematical reasoning; in particular their conceptualisation of the necessity for proof and their enactment of various proving techniques
- students' mathematical expression and their attempts to mediate mathematical meaning through words, symbols and diagrams
- students' encounter with fundamental concepts of advanced mathematics – Functions (across Analysis, Linear Algebra and Group Theory) and Limits

2. *University-level mathematics pedagogy*
3. *The often fragile relationship between the communities of mathematics and mathematics education; conditions for collaboration.*

The discussion in this paper draws on data in 1 and 2; in particular I draw on the analyses that led to (Nardi, 2007) to discuss the interviewed mathematicians' perspectives on **students' attitudes towards visualisation**, the **role of visualisation** in student learning and the pedagogical role of the mathematician in fostering a **fluent interplay between analytical rigour and often visually-based intuitive insight**. Unless otherwise stated page numbers refer to pages in (Nardi, ibid.), mostly the episodes in p139-150, p195-199 and p237-247.

STUDENTS' ATTITUDES AND THE ROLE OF VISUALISATION

Students often have a turbulent relationship with visual means of mathematical expression. When facing difficulty with connecting different representations (for instance, formal definitions and visual representations), they often abandon visual representations – which tend to be personal and idiosyncratic – for ones they perceive as mathematically acceptable. Here we take a look at the interviewed mathematicians' perspectives on students' attitudes towards visualisation and on the ways in which these attitudes – and ensuing behaviour – can be influenced by teaching. Most of the discussion eventually highlights the importance of building bridges between the formal and the informal, in constant negotiation with the students.

First and foremost the interviewees describe 'pictures' as efficient carriers of meaning. They then note that students' appreciation of this efficiency is often hindered by their ambivalence about whether 'pictures' are 'mathematics' or not.

> 'Students often mistrust pictures as *not mathematics* – they see mathematics as being about writing down long sequences of symbols, not drawing pictures – and they also seem to have developed limited geometric intuition perhaps since their school years. I assume that, because intuition is very difficult to examine in a written paper, in a way it is written out of the teaching experience, sadly. And, by implication, out of the students' experience.', *p139*

Evidence of this ambivalence can be found in the range of students' reactions to tasks where consulting a visual representation can be beneficial – see p195-199 for an example of such a task: there the student responses vary from 'using no pictures' to 'resorting to an unhelpful picture' and 'not benefiting from the inclusion of a potentially helpful picture' (p140).

Overall the interviewees' discourse regarding the role of visualisation in student learning revolved around the following four axes:

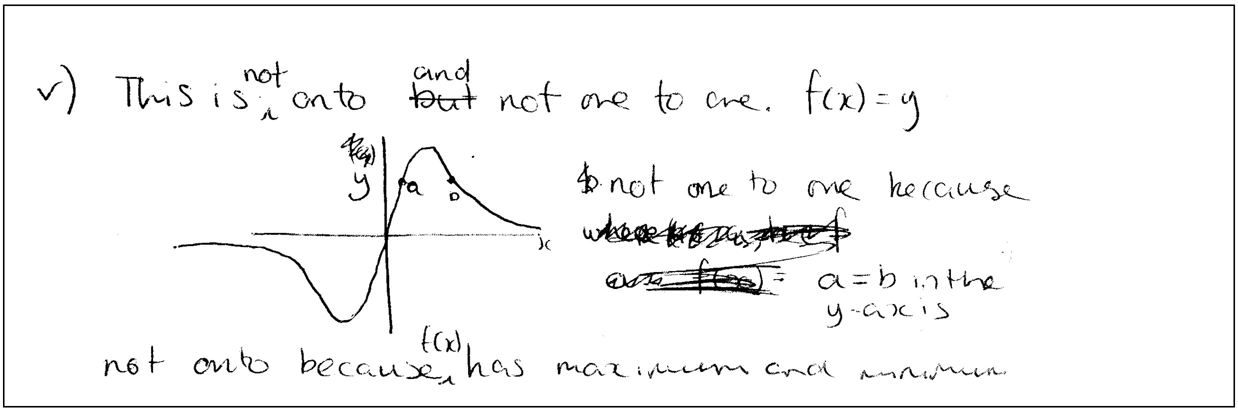
- **Usefulness** of visual representations: firm and unequivocal ('Graphs are good ways to communicate mathematical thought', p143)
- The usefulness of **educational technology**, e.g. graphic calculators: caution and concern ('Calculators are nothing more than a useful source of quick illustrations', p143)
- Students' varying **degrees of reliance on graphs** (both in terms of frequency and quality)
- The **potentially creative fuzziness of the 'didactical contract'** at university level with regard to the role of visualisation

In what follows I focus on the last two.

The premise of the discussion that follows is a question in which students were invited to explore whether certain functions from ***R*** to ***R*** were one-to-one and onto ($sinx+cosx$, $7x+3$, e^x, x^3, $x/(1-x^2)$).

The interviewees highlighted two issues when commenting on the varying degrees of the students' reliance on graphs:

- absence of transition from picture to wording. E.g.

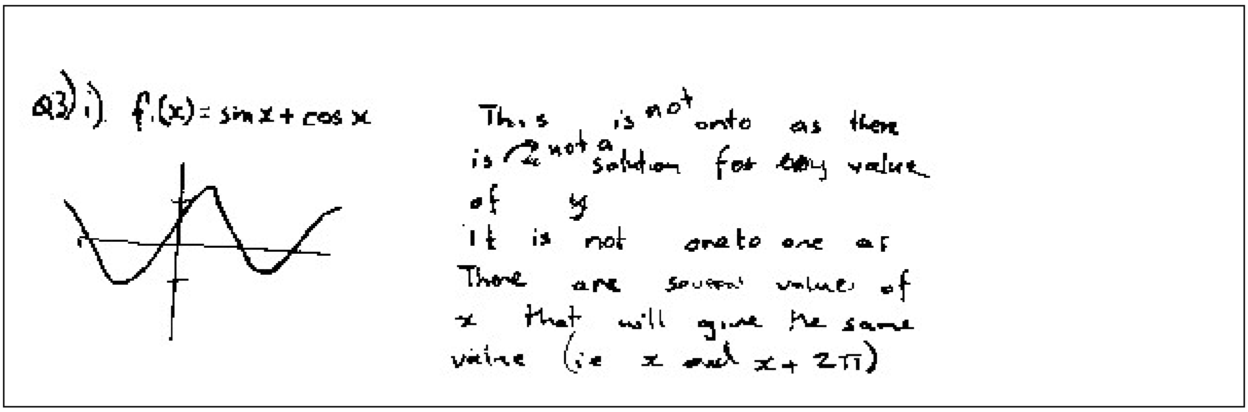


Student WD

'I am concerned about the answer being provided before the graph is produced but I also observe that the answer has been modified on the way – which may mean the graph did play some part after all in the student's decision making. If the student had drawn a line through points *a* and *b*, I would be a bit more convinced that the student is actually building the argument from what they see in the graph. I am also disappointed by the absence of a transition from the picture to some appropriate words and with the use of $a=b$ to denote that points *a* and *b* on the curve have same *y*. What a use of the equals sign! In this sense… *[see next quote]*', p144

- absence of construction evidence. E.g.

'… I am more sympathetic to Student LW…



Student LW

'…who may need the Intermediate Value Theorem to complete the argument in part (i) – the IVT is true after all –, the picture is almost perfect, all the shifting etc. is there, but this is still an incomplete answer. Still there is no construction evidence.', p144

In a nutshell the interviewees' views can be summarised as follows:

- A picture provides evidence, not proof:

'… the fact, for example, that, if a function has a maximum, it cannot be onto is immediately graspable from the graph. However some unpacking is still necessary in order to provide a full justification of the claim.', p144

- Pictures are *natural, not obligatory* elements of mathematical thinking:

'…I do not wish to see this placing value on starting with a diagram giving the students a false sense of obligation to do so, another hurdle to get over. I want them to think of doing so as a totally natural procedure to follow but also do it correctly.', p144

- Pictures are 'a third type of language':

'…used almost as a third type of language, where the other two are words and symbols, as an extension of the students' power to understand. […] I would like to see students make a sophisticated use of this power and be alert to the potential [of pictures] to be misleading too.', p145

From the above a new, or rather clarified, didactical contract (Brousseau, 1997) emerges – a contract that tries to address the 'classic problem with pictures' as part of the more general problem 'with the murky ground of using mathematics that has not been proved yet' (p149**).** In this contract students are allowed:

'…at this stage to use the graphs for something more than simply identifying the answer because after all they **allowed to use** all sorts of other **facts** – the uniqueness of cubic roots is one of those facts – **that have not been formally established yet**. So if the IVT is implicit in their finding the answer by looking at the graph, then let that be!**'**, p145

'…to use the ingredients for proving a claim **and then, at some later stage, spending some time on establishing those ingredients formally**. So prove that e^x is injective via the IVT and then later on prove the IVT. This to me is fine as long as I know that all along I have been leaving some business-to-be-finished on the side. That kind of rigour is fine with me.', p146

But they are also required to make use of the power that visualisation allows them in the aforementioned 'sophisticated' way:

'I am really keen on seeing some **evidence of thinking**, not just seeing on a graph. Some actual calculation of the maximum and the minimum, not just some pointing at a graph sketched on the basis of what is on a calculator's screen. I want them to be able to produce an accurate, elaborate graph and I want them to see the use of the calculator as a **privilege that allows them easier access to this elaboration** and as a privilege they ought to learn how to make the most of. That is much more convincing of their understanding.', p146

THE CREATIVE FUZZINESS OF THE DIDACTICAL CONTRACT

From the preceding discussion emerges a pedagogical responsibility: to foster an appreciation of the richness and creativity in certain, often very personal, visual representations:

> '…students end up believing that they need to belong exclusively to one of the two camps, the informal or the formal, and they do not understand that they need to learn how to move comfortably between them. Because in fact this is how mathematicians work! […] I am a total believer in the Aristotelian *no soul thinks without mental images.* In our teaching we ought to communicate this aspect of our thinking and inculcate it in the students. Bring these pictures, these informal toolboxes to the overt conscious, make students aware of them and help them build their own', p237

Fostering this appreciation implies revealing to the students the flexible ways in which mathematicians themselves work, especially with regard to:

> '…[the] constant tension within pure mathematics: that you want to use these methods and occasionally you need a theory to come along and make them valid. And you need these means, diagrams etc., so badly!', p238

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ENCULTURATION TO PROOF: A PRAGMATIC AND THEORETICAL INVESTIGATION

Susan Nickerson, Chris Rasmussen

San Diego State University, U.S.A.

We report on an investigation in a transition-to-proof course of undergraduate students' epistemological shifts in mathematical argumentation and identify pedagogical factors that foster and/or constrain students' ability to create mathematical arguments. We view proof as a social process in which participants of a learning community move from peripheral participation to more full members of a community engaged in the mathematical activity of proving. This report analyzes the impact of a pedagogical intervention that modifies the classic format of a two-column proof and reifies the presentation of a proof and the questions asked by the reader of the proof. We summarize our analysis in a framework that characterizes enculturation to proof.

INTRODUCTION

Cobb (2000) describes three recent profound shifts in the field of mathematics education. The first shift is described as one from students as information processors to students acting purposefully in a mathematical reality they are constructing. A second shift is described as a view of students' mathematical activity embedded within evolving classroom microculture and the larger cultural sphere. The third shift is described as one from theory informing practice to a view of theory and practice guiding each other.

Our work in the tertiary theme of teaching proof reflects all three emergent trends in the following ways. First, our perspective on learning proof is that it is not just a cognitive endeavor. We characterize learning as both a process of individual construction and a process of enculturation . Second, from our perspective, individual student activity is seen to be located within broader systems of activity and the norms constituted in class are reflexively related to shifts in beliefs about mathematics (Yackel & Rasmussen, 2002). Third, we engage in the development of a theoretical framing of students' enculturation into proof, grounded in classroom settings. In particular, our experience as teachers of a transition to higher mathematics class with an innovative pedagogical intervention evolved into a systematic study of the intervention.

The current research literature offers a number of different frameworks for classifying students' justifications (e.g., Balacheff, 1988; Chazan, 1993; Harel & Sowder, 1998; Healy & Hoyles, 2000) and it points to a number of difficulties that students have in creating proofs (e.g., Tall, 1991; Selden & Selden, 2003, 2008; Weber, 2001). Part of the difficulty with proving involves the need for the learner's increased awareness of and sensitivity to disciplinary norms and what statements in an argument need to be justified.

We view proof as a social process in which participants of a learning community move from peripheral participation to more full members of a community engaged in the mathematical activity of proving (Lave & Wenger, 1991). This report analyzes the impact on students of a pedagogical intervention to help answer the question: How can we characterize the enculturation process as students become more central members in the practice of proof?

SETTING & PARTICIPANTS

We are instructors of a transition-to-proof course intended for students who are interested in teaching secondary school mathematics. A primary goal of the course is for students to develop expertise at writing proofs and solving problems. In this course, we take a view of proof as a convincing argument that answers the question ‘Why?” Thus, the primary function of proof for the students in our undergraduate educational setting is verification and explanation (Hanna, 2000). Euclidean and non-Euclidean geometry is the content area in which we develop reasoning and communication.

The text, Henderson and Taimina’s, *Experiencing Geometry: Euclidean and Non-Euclidean with History* (3rd Edition) consists of a series of problems that ask students to make conjectures and then to justify the conjecture. A goal of this course is that students will develop personally meaningful solutions to problems and then communicate their mathematical thinking and activity to others. The structure of the class meetings typically involved working in small groups of four on challenging problems and presenting their progress on these problems. The class developed an expectation that students will also be expected to question and comment on each other’s offerings and preliminary presentations. These questions and comments were usually in the form of questions posed to the presenter of the proof and his or her group.

The first author developed a pedagogical intervention intended to raise awareness of and sensitivity to the need to query support for statements made in mathematical arguments by drawing a line next to the proof and recording the questions asked of the proof creator. This intervention was a modification of the classic format of a two-column proof, which facilitates the creation and evaluation of a proof by making explicit statements and reasons (Herbst, 2002). This modified two-column format reifies the presentation of a proof and the questions asked by the writer or a reader of the proof. The pedagogical innovation emerged in the semester prior to the study and the instructor intentionally enacted the pedagogical innovation in a subsequent semester. We view the modified two-column proof format as a boundary object (Star & Greismeier, 1989), that is, an interface between the students of the class and the instructor as an experienced member of the community into which they were becoming more central participants.

This study was conducted with undergraduates at a large, urban university in the United States. The participants were pre-service secondary teachers enrolled in an upper-division mathematics course intended to be an introduction to proof in upper-division classes. Of the 24 students (13 males and 11 females) in the class, 7 agreed to participate in individual interviews conducted at the end of the semester. The interviewees were 5 males and 2 females. They were all upper-division mathematics students, though four, as seniors, had participated in other upper-division mathematics classes; three were juniors and 'new' to proof.

DATA

The data sources for the study drew from instructor's reflective journal, copies of artifacts collected during the semester, and videorecorded individual interviews. Specifically, the data corpus consisted of the following:

• Transcripts of post-semester clinical interviews with 7 students;

• The instructor/researcher's record of instructional decisions, which sometimes included accounts and interpretations of classroom events, as well as rationales for instructional design decisions;

• Captured collective work—for example, overhead transparencies, which include a group's convincing argument and a record of the class' questions in response

• Students' relevant written work including responses to exam questions and selected homework that entails creating and critiquing proofs

Seven subjects participated in post-semester semi-structured, task-based interviews (Goldin, 2000). The interviews were designed to reveal in this particular setting how students engaged in the practice of proof and students were prompted to reflect on how they engaged in proof or rather how the activities and practices of the classroom community might have contributed to their engagement in practice of proof. Three tasks and reflection on the proof creation and proof critiquing process comprised the interview protocol.

The first task entailed asking an interviewee to construct a proof for a novel problem. In the second task, the interviewee was then presented with a student's convincing argument for the conjecture posed in the first task. Thus, the interviewee was asked to critique another's proof of the task in which he or she had just constructed. For the third task, a problem the students had previously undertaken as homework was represented with another student's solution. In the second and third task, the task had the modified two-column format (that is, the pedagogical intervention students were familiar with from class) labeled Convincing Argument and accompanying column for Questions.

After completion of the three tasks, the interviewer asked questions the students to reflect first on the process in which they had just engaged as they created convincing arguments and second, as they had critiqued others' proofs, and finally, on the practice as situated in activity of the classroom community.

METHODS

Our initial goal for the analysis was to uncover the diversity of ways in which students were reasoning about purported proofs, their ability to construct proofs, and their perspective on the modified two-column format. We therefore engaged in what Strauss and Corbin (1990) refer to as open coding, which is the process of selecting and naming categories from the analysis of data. Specifically, we began the analysis for diversity of student reasoning by first examining the seven end-of-semester interviews. We then triangulated this analysis using the constant comparison method (Glaser & Strauss, 1967) by examining all documents collected during the semester that provided additional information on student use of the modified two-column format. These documents included copies of overheads that the teacher and her students produced in class, copies of student homework, and copies of student exams.

We then engaged in the process of making explicit connections between categories and sub-categories. This step is what Strauss and Corbin (1990) refer to as axial coding. The aim of this step was to put our analysis together in new ways. Specifically, we came to see our analysis as a paradigm case of the process of enculturation into mathematical proof. As Strauss and Corbin (1990) argue, a researcher's ability to see an analysis in new ways stems largely from his or her theoretical sensitivity. Sources of theoretical sensitivity include the research literature, professional experience, and personal experience

RESULTS

Our analysis revealed critical aspects of students' positioning in the transition along the continuum of newcomer to more central participant. From the analysis emerged a framework of three dimensions along which we describe the transition of students from newcomer to more central participants. The central dimensions consist of the *Manner* in which a learner engages with proof, the *Criteria* that a learner brings to bear on proof and the *Positioning* of self with respect to proof. Each of these dimensions contains several distinguishing features that differentiate newcomer from more central participant in the practice of proof. Moveover, these dimensions can be used to describe both activity as proof creator and proof critiquer. Due to proposal space constraints we illustrate briefly here using examples from the interviews of students we characterize as being more central participants or newcomers to the practice of proof.

A student who had some experience with proof prior to and concurrently while participating in this class carefully considered what was to be proven and what was to be proved and took on the role of skeptic in proof creation. As a proof critiquer, his *manner* was one in which he tended to review the proof first holistically, paying attention to the structure of the proof and playing the believing game initially. In contrast, a student who was a relative newcomer tended to read proofs to be personally convinced. After creating a proof, he was asked to critique another student's proof and tended to read the proof to be personally convinced. He said, "They pretty much followed everything I did. That's why I didn't question it….I don't see why I would question it."

The *criteria* a learner brings to bear varies in terms of the attention to and certainty about disciplinary norms. These can include, for example, whether in a proof critique, they can rely upon the geometric picture as evidence and the expectations for the efficiency and elegance. As one student suggested: “I guess the proof is the mathematical way of writing a story. You know, you are trying to tell somebody something. But you have to do it in a certain way with a smooth flow as opposed to just [jumping around].”

Finally, we frame a difference in the individual positioned as a tentative questioner versus one who takes ownership of the critique, who reads for the purpose of understanding someone else’s point of view and speaks as if they were members of a larger group. We see these three dimensions of manner, criteria, and positioning as derived from the reflexive relationship between community in which they participate and the individual. In the larger sense, the social norms are reflexively related to the beliefs and values of a proof creator and critiquer. The criteria one brings to bear on proof are related to the socio-mathematical norms negotiated within the classroom community. The manner in which they engage in the activity of creating and critiquing proofs is related to classroom math practices.

CONCLUSIONS

We propose that our analysis outlines indicators of the journey that students take as they become more central members of practice of creating and critiquing proofs. Our work contributes pragmatically to the pedagogy that fosters enculturation in ways that that are commensurate with the discipline (Weber, 2002). Although not evidenced in our brief illustration on the results, our analysis also showed that the pedagogical innovation played a role in shifts in the positioning or identity as a mathematics teacher. We further propose that our framing as enculturation to proof as human and social activity offers an alternative lens which complements the extensive research on the cognitive aspects of understanding students’ challenges with proof. We are currently examining the usefulness of our pedagogical intervention and our enculturation into proof framework in other transition level courses.

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LEARNING TO PROVE: ENCULTURATION OR…?

**Patricia Perry, Carmen Samper, Leonor Camargo,
Óscar Molina, and Armando Echeverry**

Universidad Pedagógica Nacional, Bogotá, Colombia

Empirical evidence coming from a curriculum innovation experience that we have been implementing in the Universidad Pedagógica Nacional (Colombia), in a plane geometry course for secondary mathematics pre-service teachers, allows us to affirm that learning to prove, more than enculturation into mathematicians' practices, is participation in proving activity within the community of mathematical discourse.

AN EXPERIENCE FOCUSED ON LEARNING TO PROVE

Our contribution is linked to the curriculum innovation experience that we have been implementing since 2004 in the framework of a pre-service program, for high school mathematics teachers, that includes a high percentage of credits in mathematics formation. The experience takes place in an 80 hour plane geometry course, 2nd of the 6 courses in the area of geometry. Students' ages are between 18 and 21. Since upon entering the University, their geometry content knowledge, and know-how, and their mathematical argumentation experience are minimal, in their first geometry course the approach to geometric objects is informal. The intention is to provide experiences that help students construct or amplify their geometric background, and improve their disposition and preparation for commitment in the next course.

The plane geometry course's goal is to create opportunities to learn to prove that should affect students' conception of proof not only from a mathematical point of view but also from a didactical one. Besides learning to build deductive chains, we expect students to recognize the role of proofs as a resource for understanding and arguing and as a fundamental activity in mathematics tasks.

As we pursue the course's purpose, we defy traditional university mathematics teaching practice: we embark in a **collective construction** of an axiomatic system related to points, lines and planes, angles, properties of triangles and quadrilaterals. To achieve this enterprise, teacher and students participate jointly in mathematical activity, articulating practices such as defining geometric objects, empirically exploring problems, formulating and verifying conjectures, and writing deductive arguments. It is through the questions and tasks proposed to the students by the teacher, that the course is developed; that is, the geometric content treated in the class doesn't originate from a textbook, nor is it presented by the teacher. This situation can appear unusual and surprising if one asks how students can participate in the creation of mathematical discourse, unknown to them; it is possible due, principally, to three reasons: the instrumental mediation of a dynamic geometry program (Cabri), a clearly delimited reference framework, and the teacher's management of the class, coherent with the course's purpose.

DAWNING OF PROVING ACTIVITY: THREE PRACTICES

We are aware that a practice entails not only the actions through which it materializes but also repertories, work routines, values, interests, resources for negotiating meanings, etc. (Wenger, 1998). Even so, due to lack of space, we sketch below three mathematical practices, focusing primarily on the actions.

Analyzing a definition. When a term appears, in a question or task set by the teacher or in a student answer, which will be part of the specialized vocabulary, including it in the axiomatic system requires a precise definition that will be elaborated jointly by all the class members. They are terms that the students have an intuitive idea about and, therefore, can make a graphic representation and verbalize a statement that becomes the first version of the definition. Whether it coincides or not with the definition that will be institutionalized, the teacher leads a process which includes examining the coherence between the given verbal statement and its graphic representation, graphically presenting cases that should be excluded from the definition and are being included or vice versa. The analysis appeals to questions like "What if such a condition is excluded?", "Why is it required?", "Do these statements define the same object?", or specific questions related to the object itself. For example, after a student's definition for *segment*: "The set formed by points *A, B* and those between *A* and *B*", was accepted by his classmates, the teacher focused their attention to the characteristics of this geometric object with questions like: "Is $\overline{AB}$ a subset of some line?", "Which one?", "How do we know?". Answering the questions involved the class community in the collective production of a proof, product of the following considerations: (i) $\overline{AB}$ has more than two points and therefore the inclusion of $\overline{AB}$ in $\overleftrightarrow{AB}$ can not be justified by alluding to the fact that *A* and *B* belong to the segment and to the line; since $\overline{AB}$ has at least a point *C* different from *A* and *B*, it is indispensable to show that this point is also an element of the line through *A* and *B*; (ii) points *A, B* and *C* of $\overline{AB}$ are collinear since betweeness, which characterizes a segment, includes this condition; (iii) the line that contains *A, B* and *C* is the same one determined by *A* and *B* because two points determine a unique line. That proving activity took place in the 6th class through a conversation guided by the teacher, conformed in all by 150 interventions, 70 by the teacher and 80 of 12 of the 21 students that constituted the group.

Enunciating propositions. Some of the propositions proved in the course guarantee the existence of a geometric object. Initially, they are conjectures, suggested by the students as answers to a problem, worked on in small groups with a dynamic geometry program, exploring possible constructions of the object whose existence must be proven. However, students' answers usually are not expressed as a conditional statement or, if so, a condition that should be part of the antecedent is not included, or antecedent and consequent are interchanged. A public revision of the statements to determine whether it must be reformulated is essential. Given Cabri's mediation in the process, the revision is centered on determining whether correspondence exists between what was done with Cabri and what the conjecture states, bringing out the given conditions and determining which consequences result.

The following episode, which took place in the 21st session of the course, illustrates characteristics of the above practice. As response to the given problem, Group 1 and Group 2 formulated their conjectures.

Problem	Conjectures formulated
Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be opposite rays and $\overrightarrow{AD}$ another ray. Is it possible to determine a point *E*, in the same half plane in which *D* is found, for which $\angle BAD$ is complementary to $\angle CAE$?	If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays and having $\overrightarrow{AD}$, then there exists a point *E*, in the same half plane in which *D* is found, such that $m\angle EAD = 90$ and $\angle EAC$ and $\angle DAB$ are complementary. [Group 1]
	If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, $\angle BAD$ is acute, $E \notin \text{int}\angle BAD$ and $m\angle DAE = 90$, then $\angle CAE$ y $\angle BAD$ are complementary. [Group 2]

In their report, Group 1 presented the details of the Cabri construction and exploration carried out; the group constructed the figure as required: they constructed $\angle EAC$ complementary to $\angle DAB$, dragged to vary their amplitude, noticed that the amplitude of $\angle EAD$ remained invariant, measured and found it to be 90°. The correspondence construction-conjecture was studied through teacher questions which lead students to focus on specific aspects that aided determining whether the correspondence existed or not; she asked questions like "Did the group construct rays $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ as required by the problem and just with that noticed that an angle was right and all the rest?", "Besides constructing the three rays, did they construct a fourth ray that satisfied a certain condition?", and "Was the construction of $\overrightarrow{AE}$ carried out to construct $\angle EAC$ or $\angle EAD$?". When the answers to these questions were discussed, students realized that the group had constructed $\angle CAE$ to be complementary to $\angle BAD$ and therefore obtained the existence of $\angle EAD$ which turned out to be a right angle. The analysis showed that the Group 1's conjecture didn't coincide with the construction and information extracted. Reformulating it was necessary, since the class was convinced of the regularity evidenced by the empiric experience.

Having the reformulated statement, its relation with the conjecture given by Group 2 was examined. For this, the teacher posed the following question to the other members of the class: "Suppose correspondence between construction and conjecture exists. How do you imagine the construction process was?". In the analysis, hypothesis and conclusion were identified, thereof explicitly setting the four conditions, included in the antecedent of the conditional that must have been constructed to obtain what the consequent expresses. It was then realized that second conjecture was almost the reciprocal of the reformulated first one.

Submitting a proof for consideration. The students start to participate actively in the construction and evaluation of proofs from the beginning of the course, as the following example illustrates. Students were asked to prove: *Given a line and a point not on it, there is exactly one plane containing both of them*. After allotting time for students to reformulate the statement as a conditional and devise a plan for the proof, Ana, voluntarily, writes on the board the proof she and Juan produced; students were asked to be vigilant so as to approve it or not Ana's proposal. She reformulated the statement as: "If $\overleftrightarrow{AB}$ exists, and a point F, that doesn't belong to $\overleftrightarrow{AB}$, then there exists a plane α such that $\overleftrightarrow{AB}$ union F is contained in α." Juan immediately intervened to point out that the plane is unique. Ana wrote the steps of the argument as she verbalized it, as shown:

	Statement	**Justification and steps involved**	
1	$\overleftrightarrow{AB}$ exists.	Given.	
2	Point F exists that doesn't belong to $\overleftrightarrow{AB}$.	Given.	
3	Points A and B exist that belong to $\overleftrightarrow{AB}$.	Line theorem. (Every line has at least two points.)	1
4	A, B, F are non-collinear.		2, 3
5	There exists exactly one plane α.	Plane postulate. (Three non-collinear points determine exactly one plane.)	4

Once finished, Germán objects on how Ana mentions the line in her first step because there she was "assuring the existence of the two points… if I declare them from the beginning, I am giving the existence of those points". He proposed writing this statement as "m is a line". Juan intervened, indicating that Ana and Germán were saying the same thing and that the issue was simply one of notation. His counterargument was: "Well, the way I see things, she is giving the existence of the points because when she says 'line *AB*' she is mentioning where the line passes. It is very different to say line m because it doesn't indicate where the line passes". Daniel, although expressing agreement with Germán, saw no problem with Ana's notation, and commented that it was possible to write a better statement: "It is better to use m or n, any name, and then apply the theorem to obtain the two points". The teacher intervened to analyze the situation, indicating that the issue is just notation because the existence of the two points was not taken as given but was deduced from the line's existence; to illustrate the issue more clearly, she said: "If we change the expression 'line *AB*' for m, notice that practically nothing changes in the argument; we would only have to change 'line *AB*' for m' in step 3". However, she pointed out that "it might be more elegant to express it as Germán and Daniel suggest". Once the discussion was finished, attention was set on what Daniel labeled as a "trivial" point, a step of the proof that was lacking: "I think saying the line is in the plane is missing", which lead to Ignacio´s question "Don't we need to mention the Flatness Postulate (if two points of a line lie in a plane then the line is in the same plane) to say the line is in the plane?", which led to modifying the proof. This episode took place in the 22nd session of the course, through a mathematical dialogue that included 26 interventions, 10 of the teacher and the rest, of 6 students.

DISCUSSION

What do the above examples say about what learning to prove in our course is? Firstly, and although no details of the interaction were included in the sketches, we think that it's possible to envision on it the student participation in the proving activity through which the course is developed; *proving activity* refers to all the actions involved in the formulation of conjectures and the production of deductive justifications based on the axiomatic system constructed by the class community. This characteristic feature can be associated with the idea that *learning to prove* is a process through which students acquire more capability to participate in proving activity in a genuine (i.e., voluntarily assuming their role in achieving the enterprise set in the course), autonomous (i.e., activating their resources to justify their own interventions and to understand those given by other members of the class community), and relevant form (i.e., make related contributions that are useful even if erroneous). Since participation occurs in a community which begins to form as soon as the course begins, and is made up, with respect to learning to prove, by apprentices and only one expert, the teacher, we don't see the process as an enculturation one. We understand that the concept, *enculturation,* has to do principally with a way of knowing, proper of a cultural group and linked to dispositions, forms of acting, beliefs and values that characterize the group: specifically, enculturation is the process through which a person acquires a group's culture due to his interaction with the group's members and observation of their interactions as they carry our their practices. Instead, we see as more appropriate the idea of formation of a *community of practice* (Wenger, 1998).

About the first example, we underline that the conceptualization process gives rise to proofs that justify the answers to questions posed. In those cases, the function of proof is not to validate or verify a conjecture so as to incorporate it in the axiomatic system; instead, it is to help understand the implications that the analyzed statement has, function recognized as important by mathematicians. In the second example, we bring out, not only the type of task that focuses attention on the existence proof of an object through a characterization that permits its construction, but also, the emphasis placed on the comprehension of conditionality and its expression as a statement. Students' expertise with the notation and specialized vocabulary must be remarked on. In the third example, related to the practice of submitting a proof produced by one of its members to the community's criticism, and making relevant criticism, undoubtedly reflects one of the most important practices of mathematicians. Also, we can point out not only the deductive axiomatic character that the arguments presented as proofs possess but also the rigor with which we seek to work, reflected in the issues students pay attention to when they comment a fellow student's production. We consider their preoccupation for controlling the unconscious action of using in the justification that which is being justified very valuable.

The above remarks allow us to argue that the community of practice conformed worries about and undertakes mathematical issues related to proof from a Euclidean geometry point of view. That makes our community of practice fitting in perfectly into what Ben-Zvi and Sfard (2007) consider as a community of mathematical discourse. For them, *discourse* is a type of communication, established historically, that congregates a human group and segregates it from other groups; the membership in the wider community of discourse is achieved through participation in communicational activities of any collective that practices this discourse, no matter what its size is; and to belong to the same discourse community, individuals don't have to face one another and don't need to actually communicate. We think that considering the community of practice conformed in our course as a micro-culture of the community of mathematical discourse expresses the fact that mathematics is present when making curricular decisions. Therefore, although we don't see the process of learning to prove as enculturation into the practices of mathematicians, we are interested in, and think we achieve it to some measure, student acquisition of dispositions characteristic of mathematicians as, for example: (i) preference of the if-then format to express propositions and the use of a particular generic to make deductive reasoning agile; (ii) controlled use of graphic representations, with clearly established conventions, to support the statements that conform the final proof; (iii) careful use of terms and notation for geometric objects; (iv) exclusive recursion to the axiomatic-deductive system for the justifications of the statements in a proof; (v) acceptance of the convenience of a detailed deductive process; (vi) belief that proving activity involves exploring, conjecturing, searching ideas for a justification, producing a proof based exclusively on the theory constructed and submitting the production to criticism.

How are these dispositions developed in the students? Without doubt the type of tasks in which students systematically participate, from the beginning of the course, the collaborative work between students, the teacher's role as expert of the community with whom students interact, the instrumental mediation of dynamic geometry are decisive factors in the formation of such dispositions.

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ASSIGNING MATHEMATICS TASKS VERSUS PROVIDING PRE-FABRICATED MATHEMATICS IN ORDER TO SUPPORT LEARNING TO PROVE

Patricia Perry, Leonor Camargo, Carmen Samper, Óscar Molina y Armando Echeverry

Universidad Pedagógica Nacional, Bogotá, Colombia

We present types of mathematics tasks that we propose to our students —future high school mathematics teachers— in a geometry course whose objective is learning to prove and whose enterprise is collectively building an axiomatic system for a portion of plane geometry. We pursue the achievement of the course objective by involving students in different types of tasks instead of providing them with pre-fabricated mathematics.

INTRODUCTION

As a result of a curriculum innovation process (Perry, Samper, Camargo, Echeverry, & Molina, 2006), that we have been implementing and adjusting since 2004, a plane geometry course for pre-service mathematics teachers was transformed from centering on the direct teaching of geometric content to focusing on learning to prove. At present, the general course objective is that students learn to prove, widen their vision about proof and its fundamental role in mathematics activity, and recognize proof as an explicative and argumentative resource for mathematics discourse. The course's enterprise is the collective construction of an axiomatic system for a portion of plane geometry theory that includes as themes: relations between points, lines, planes, angles, properties of triangles, congruency of triangles and quadrilaterals. There are drastic changes in the content management: content is not presented as something pre-fabricated and, therefore, neither the teacher nor a textbook are the source for the content that is studied; neither is the usual definition-theorems-application exercises sequence privileged. A great amount of the propositions that are proven are formulated, by the class community, as conjectures that arise from student productions when they solve the tasks the teacher proposes. Practically all the proofs carried out are done by the students with teacher support, in a greater or lesser degree. Some propositions are enounced and proven the instant they are recognized as indispensable to complete another proof that is being constructed. Definitions are introduced to satisfy a manifested necessity to determine exactly which geometric object is being considered; to define it, the starting point is the student's concept image, and then the careful analysis of the role each condition mentioned in the definition has. How have we been able to bring such a change in the class functioning? With respect to curriculum at the class level, there are various factors, that articulated, have made this change viable: the use of a dynamic geometry program always available in the class; the group or individual student work and the collective work as a community in different moments and with different purposes; the norms that regulate the use of the dynamic geometry program, the interaction in class and what is accepted as a correct proof; the teacher's role in managing the content; and the mathematics tasks in which we involve the students.

In this paper, we present the types of tasks through which the course is developed. This way, we give a partial answer to the question: "how do we involve students in the deductive systematization of some parts of mathematics, both in defining specific concepts and in axiomatizing a piece of mathematics?". Since the experience on which our contribution is based occurs at university level, this article is well placed in the seventh theme of ICMI Study 19.

BRIEF PRECISIONS

For us, *proving activity* includes two processes, not necessarily independent or separate. The first process consists of actions that sustain the production of a conjecture; these actions generally begin with the exploration of a situation to seek regularities, followed by the formulation of conjectures and the respective verification that the geometric fact enounced is true. Hereafter, the actions of the second process are concentrated on the search and organization of ideas that will become a *proof*. This last term refers to an argument of deductive nature based on a reference axiomatic system of which the proven statement can be part of.

Learning to prove is a process through which students acquire more capability to participate in proving activity in a genuine (i.e., voluntarily assuming their role in achieving the enterprise set in the course), autonomous (i.e., activating their resources to justify their own interventions and to understand those given by other members of the class community), and relevant form (i.e., make related contributions that are useful even if erroneous). Learning to prove in our course implies a great quantity of aspects that we group into three classes: (i) those related to the proving procedure itself (e.g., the use of conditionals in valid reasoning schemes, construction of a deductive chain that leads from the hypothesis to the thesis); (ii) those related to the proof within the framework of a reference axiomatic system (e.g., distinguishing the different types of propositions that conform an axiomatic system); (iii) those proper to proofs in geometry (e.g., visualization of figures on which proofs rest, the use of figures to obtain information, auxiliary constructions).

TYPES OF TASKS USED TO SUPPORT LEARNING TO PROVE

Related to the procedure of proving

Type 1. *Determine whether a specific set of postulates, definitions or theorems permit validating a given proposition*. With this type of task, precision is begun in the course about what is proving and how a proof is done. For example, students are asked whether the postulate that establishes the correspondence between points on the line and real numbers, conformed by two reciprocal conditionals, permits assuring the truth of the proposition: *Every line has at least two points*. Carrying out this task is an opportunity for students to start realizing how the reasoning scheme *modus ponendo ponens* is used, and how a deductive sequence of propositions is conformed that permits going necessarily from the hypothesis to the desired conclusion.

Type 2. *Starting from a plan or ordered sequence of key statements to prove a given proposition, write a complete proof of the proposition*. This type of task, proposed principally at the beginning of the course, requires that students include the missing sufficient conditions of every conditional that is involved in the given sequence or plan, the theoretic justification for each statement, indicating which numbered statements intervene when obtaining the partial conclusions that make up the proof. For example, the following plan is presented: "If 0 is the coordinate of F and b the coordinate of G, $b > 0$, I look for the point H on the line for which the coordinate is $5b$. This way, G is between F and H", that must be used to write a complete proof of the proposition *For each pair of points F and G on a line, there exists a point H on the line such that G is between F and H.* With the plan, students are given a guide that should conduce them through a suitable path for the proof and the delegated work intends to concentrate their attention on details such as: which are the given premises in the proposition that must be proved, which postulate, theorem or definition guides the proof and which statements must be made to be able to use it.

Type 3. *Critically examine a proof written on the blackboard by one or two students.* Although students know their interventions in class are always possible and desired, on occasions, the responsibility of accepting or not a proof is delegated explicitly to them. This type of task compels recognizing key issues, generally problematic, which have been highlighted throughout the course. For example, the use of an element of the reference axiomatic system as warrant for a conclusion when not all the sufficient conditions of the respective conditional are on hand; the existence of an object is justified through the corresponding definition; the inclusion of statements that are not used in a proof or a sequence of statements that could be replaced by a proposition that has already been incorporated in the axiomatic system, which makes a proof longer than it should be.

Type 4. *Generate an ordered sequence of key statements that outline a route for the proof of a proposition*. This occurs either when a conjecture is generated as a solution to a construction problem or when, especially towards the end of the course, a theorem is proved because its proof follows easily from another. For example, the following problem is presented: *Given three non collinear points A, B and C, determine, if possible, a point D such that* $\overline{AB}$ *and* $\overline{CD}$ *bisect each other*. As part of the solution, the student must describe in detail his construction process and validate each step within the reference axiomatic system, which becomes a resource to outline the proof. In this type of task, students are asked to enounce the proposition in the if-then format, and occasionally, to give a synthetic formulation as a mean to give sense to the geometric fact treated. In the example, the statement is *Three non collinear points determine two segments which bisect each other.*

Related to the proof within the framework of a reference axiomatic system

Type 1. *Produce a diagram of dependency relations between the different propositions that make up a portion of the axiomatic system related to a specific topic.* In certain moments of the development of the theory, students are involved in the revision of what has been done related to a specific topic with the purpose of reconstructing the network of the propositions incorporated into the system, signaling, for each proposition, those it depends on and those that depend on it. This type of task fosters, on one hand, discriminating between postulates, theorems and definitions, and on the other, recognizing not the relation between hypothesis and thesis of a conditional but relations between the proposition and the rest of the theory.

Type 2. *Decide if a proposition, product of an exploration or search for statements that are required to complete a proof, is going to become a postulate, definition or theorem of the axiomatic system.* This type of task contributes to the discrimination of postulates, definitions and theorems that make up the system, and to establish the possibility of their use in proofs. For example, searching for a way to prove that vertical angles are congruent, a student suggested the possibility of affirming that the sum of the measurements of two angles that form a linear pair is 180°. Since this was not yet an element of the system, it was discussed whether it should be assumed as a postulate, definition or theorem. Trying to decide if it could be a definition, the class community noticed that its reciprocal was not true, and therefore discarded that possibility. To decide if the statement could be a theorem, they looked for propositions in the axiomatic system so far developed that could lead to concluding that the sum of the measurements of the angles was 180°, parting from having angles that form a linear pair; since none were found, they discarded this option. Finally, and given that the proposition was fundamental for the proof in question, and for future proofs, they decided to incorporate it as a postulate.

Type 3. *Produce a set of propositions and prove them, establishing dependency relations between them, thus forming a portion of the axiomatic system relative to a particular topic.* This type of task is initiated by proposing one or more open-ended problems that demand students' involvement in an exploratory activity, with a dynamic geometry program, that must lead to the formulation of a conjecture. Once all conjectures have been communicated, these are revised to determine conviction with respect to its truthfulness, examine whether its enunciation is clear and complete, and if necessary, carry out the pertinent modifications; moreover, during this revision process, the required definitions are elaborated. Then, with the indispensable teacher support to establish the sequence in which the conjectures are to be proven, students either produce a plan to construct the proof —that each one must finish as homework— or they collectively construct the proof. For example, in the first version of the course, problems like "Determine the quadrilaterals for which one diagonal bisects the other diagonal", "What happens in a triangle, with the segment that joins the midpoints of two sides?", and "In a quadrilateral, we choose midpoints of opposite sides or of adjacent sides. What can be said about the segment that joins them?", gave rise to the definition of parallelogram, kite, isosceles trapezoid, among others; and propositions like *In a kite, the diagonals are perpendicular and only one bisects the other, If a quadrilateral has one pair of opposite sides that are both congruent and parallel, then it is a parallelogram, The length of the segment that joins midpoints of the non-parallel sides of a trapezoid is equal to the half sum of the lengths of the bases.* Not all the theorems proved arose as conjectures from the initial exploration; some were generated during the proof of another theorem as a needed proposition to complete the proof that was being done, and others appeared when asked whether the reciprocal of the theorem was or not true.

Related to proper issues of proving in geometry

Type 1. *Obtain or use information that a graphic representation on paper or product of a dynamic geometry construction provides.* With this type of task we expect students to use the graphic representations of the objects, involved in a statement, to find useful geometric relations, but discriminating between information that can be considered true about the figure and that which is not. In a paper representation, for example, complying with norms established (system of symbols, of conventions), the task of carefully examining the figure that represented vertical angles, to find geometric relations that would permit proving they were congruent, gave the clue needed for the proof. Since the only acceptable information that could be deduced from the figure was betweenness of points, the students had to justify the existence of two pairs of opposite rays and thereof of linear pair angles. On the other hand, in dynamic geometry, the identification of invariance or the variance of certain properties by dragging became a fundamental element for discovering new properties; discard others or establish which properties depend on others. For example, students investigated the position of $\overrightarrow{BK}$ for which the bisectors of angles *KBA* and *KBC*, that compose a linear pair, form the angle with greatest measure, and realized that such bisectors always form a right angle, because in any position of $\overrightarrow{BK}$ two pairs of congruent angles are formed whose sum is 180°. So they concluded that the measure of the angle determined by the bisectors is 90°.

Type 2. *Find an appropriate auxiliary construction that directs a proof process.* A type of task particularly frequent in the fifth version of the course consists in finding the auxiliary construction that can be useful to enlarge the set of propositions to be used in a proof. To carry out the task, the teacher organizes the proposed constructions and the class analyses the benefits of one or another, leading to the appropriate one. For example, to prove that two right triangles *ABC* and *DEF* are congruent, given that their hypotenuse and a leg are congruent, a student's first idea was to construct a triangle that shared a leg with ΔABC that also had two congruent sides with it, to be able to use the known congruency criteria, but he never referred to ΔDEF. The teacher pointed out, as an important idea, the construction of a triangle "stuck to" another triangle. Then another student suggested constructing a ΔGHI congruent to ΔABC with triangles *GHI* and *DEF* sharing the congruent leg; this way, he expected to use the transitive property to prove that $\Delta ABC \cong \Delta DEF$. The teacher explained that this proposal was better than the first, but could not be used because it was impossible to justify the betweenness property of some points. Finally, another student proposed constructing ΔGHI as suggested by his classmate, but using the non-congruent leg. This way the inconvenience presented previously was overcome and they found the how to carry out the proof.

Type 3. *Recognize and use certain figures of the axiomatic system as resource to find a way to prove a proposition.* In a portion of the axiomatic system associated to angles, triangles and quadrilaterals, there are some geometric figures that become fundamental pieces of the proof process because their properties are a source for the use of elements of the system. Identifying or constructing, in a given figure, an isosceles triangle, two congruent triangles, the external angle of a triangle or a parallelogram is part of the required expertise to guarantee properties that lead to the desired conclusion. For example, in the proof of the congruency of two right triangles with hypotenuse and a leg congruent, students take advantage of their knowledge of isosceles triangles and congruency criteria to carry out the proof.

FINAL REMARKS

The types of tasks described exemplify the effort carried out in planning the class to genuinely involve students in the collective construction of the axiomatic system. Due to lack of space, we can not amplify the information about how the teacher manages the tasks, essential element to obtain from them the greatest benefit in the generation of a participative climate. The sensitivity to find in student's expressions and ideas the source to key propositions for the system and the path towards their proof, since these are not necessarily exposed in the appropriate language, joined to flexible thinking capable of sacrificing organization and rigor, proper of an advanced mathematical discourse, in pro of favoring student proving activity is a determining aspect of the success of this curriculum innovation.

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AN EMPIRICAL EVALUATION OF STRUCTURED DERIVATIONS IN HIGH SCHOOL MATHEMATICS

Mia Peltomaki and Ralph-Johan Back

University of Turku and Åbo Akademi University, Finland

Structured derivations is a new method for presenting mathematical proofs and derivations. It is based on a systematic and standardized way of describing mathematical arguments, and uses basic logic to structure the derivation and to justify the derivation steps. We have studied the use of structured derivations in high school in two successive controlled studies. The results indicate that using structured derivations gives a marked performance improvement over traditional teaching methods. We describe the structured derivations method, the set up of our study and our main results.

BACKGROUND

One of the cornerstones of mathematical reasoning is mathematical proof. However, proofs are considered difficult and consequently today's high school curricula typically mention proof only in connection with geometry. Strong arguments have been presented in favor of more training in rigorous reasoning (Hanna & Jahnke 1993, Hoyles 1997). Mathematical proofs are based on logic and logical notation, but using logic in proofs is usually not taught systematically in high schools today. Proofs in high school are therefore informal and not uniform. Where logic is taught, it is seen as a separate object of study, rather than as a tool to be used when solving mathematical problems.

Writing solutions to mathematics problems in an unstructured and informal way makes it hard for students to know when a problem has been acceptably solved. It also makes it difficult to look at solutions afterwards, study them and discuss them. Only in connection with a few specific problem areas (typically algebraic simplification and equation solving) is a more uniform format for writing solutions at hand. However, even then it is often not clear to students, e.g., why deriving 0=1 from an equation means that the equation has no solutions.

Structured derivations provide an alternative approach to teaching mathematics, based on systematic proofs and derivation and the explicit use of logical notation and logical inference rules. Structured derivations have been developed by Back and von Wright (Back et al, 1998; Back & von Wright, 1999; Back & von Wright, 2006; Back et al, 2008), first as a way for presenting proofs in programming logic, and later adapted to provide a practical approach to presenting proofs and derivations in high school mathematics. Structured derivations are a further development of the *calculational proof method* originally developed E.W. Dijkstra and his colleagues (see Dijkstra, 2002, for a summary and motivation). Structured derivations add a mechanism for doing subderivations and for handling assumptions in proofs to calculational proofs. Structured derivations can be seen as a combination of Dijkstra like calculational proofs and Gentzen like backward chaining proofs.

We have been experimenting with using structured derivations for teaching mathematics in high school, with very encouraging results. It seems that the standardized format provided by structured derivations helps the students in constructing a proof and in checking that their proof is correct, without being too formal and/or intimidating to be useful in practice.

We start below with a short overview of structured derivations, before we proceed to describe two large empirical studies that we have carried out to evaluate the use of our approach in teaching mathematics in high school.

STRUCTURED DERIVATIONS BY EXAMPLE

We illustrate structured derivations with a simple example: solve the equation $(x-1)(x^2+1)=0$. The solution is as follows:

$\bullet$ $(x-1)(x^2+1)=0$

$\equiv$ {zero product rule: $ab=0 \equiv a=0 \vee b=0$}

$x-1=0 \vee x^2+1=0$

$\equiv$ { add 1 to both sides of left disjunct}

$x=1 \vee x^2+1=0$

$\equiv$ {add -1 to both sides in right disjunct }

$x=1 \vee x^2=-1$

$\equiv$ {a square is never negative}

$x=1 \vee \text{False}$

$\equiv$ {disjunction rule}

$x=1$

The original equation is transformed in a sequence of equivalence preserving steps to the solution "x=1". Each step in the derivation consists of two terms, a relation and an explicit justification for why the first term is related to the second one in the indicated way. In this case, the terms are Boolean formulas, and the relation is equivalence between the terms.

This example does not show a number of important features in structured derivations, such as the possibility to present derivations at different levels of detail using subderivations, and the use of assumptions in proofs. These are not the focus of this paper, so we have chosen not to present them here. For information on subderivations and a more detailed introduction to the format, please see the articles by Back et al.

It is important that each step in the solution is justified. The final product will then contain a documentation of the thinking that the student was engaged in while completing the derivation, as opposed to the implicit reasoning mentioned by Dreyfus (1999) and Leron (1983). The explicated thinking facilitates reading and debugging both for students and teachers. It also leaves a more explicit documentation of the teacher's explanation of an example, making it easier for students to catch up later on issues they did not understand during the lectures.

Moreover, the defined format gives students a standardized model for how solutions and proofs are to be written. This can aid in removing the confusion that may result from teachers and books presenting different formats for the same thing (Dreyfus, 1999). A clear and familiar format also has the potential to function as mental support, giving students belief in their own skills to solve the problem, and the satisfaction of being able to check for themselves that they have indeed produced a correct solution. The use of subderivations renders the format suitable for new types of assignments and self-study material, as examples can be made self-explanatory at different levels of detail.

EMPIRICAL STUDY

Our purpose was to test whether teaching mathematics using structured derivations in high school (upper secondary education) would improve the students learning, as compared to teaching mathematics in the traditional way. High schools in Finland are 3-4 years and the students are 16 – 19 years old. Mathematics is taught at two levels, standard and advanced. Mathematics at the advanced level is in practice a pre-requisite for studying Science, Engineering, Medicine and Business Administration at University level. The advanced level is therefore quite popular, and is taken on average by 40 % of the students. There are altogether 10 compulsory mathematics courses on the advanced level, as well as some optional courses. High school ends with a national matriculation exam in mathematics, which is taken by almost all advanced level students.

We carried out our empirical studies at Kupittaa High School in Turku, Finland. This school offers extra courses in IT at high school level (programming and telecommunication courses). We carried out two 3-year empirical studies at this high school, the first 2001 – 2004, and the second 2002 – 2005.

Both studies were organized in the same way. The students starting high school were divided into three groups. First there was a test group consisting of those students who wanted to take some extra courses in IT. The remaining students were divided into two groups, a control group that was chosen so that its starting situation would be as similar as possible to the test group, and a third group consisting of all the remaining students. The third group did not participate in the study. The students had the final say on which groups to join, so it was not possible to make the test and the control groups exactly similar. The test and control groups were of approximately same size.

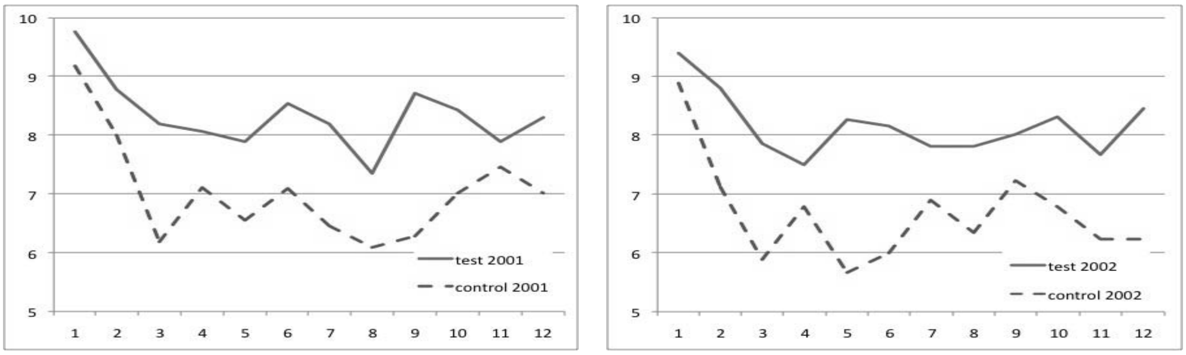
Groups had lectures and exams at exactly the same time, they followed the same curriculum, and they had exactly the same exams. The test group was taught all mathematics courses using structured derivations, while the control group was taught in the traditional way. The exception was the course in Geometry, which was taught in the traditional way also in the test group. The test group and the control group had different teachers.

The study measured the average performance of the test and the control groups on all ten mathematics courses, as well as on the final matriculation exam.

MAIN RESULTS

Graph 1 shows the performance of the students on the individual math courses, (a) shows the 2001 – 2004 study and (b) the 2002 – 2005 study. Courses are graded on a scale from 4 – 10, where 4 is not passed, 5 is the lowest grade (barely pass) and 10 is the best grade (excellent). The results of the test groups are shown as a solid (blue) line, while the results of the control groups are shown as a dashed (red) line.

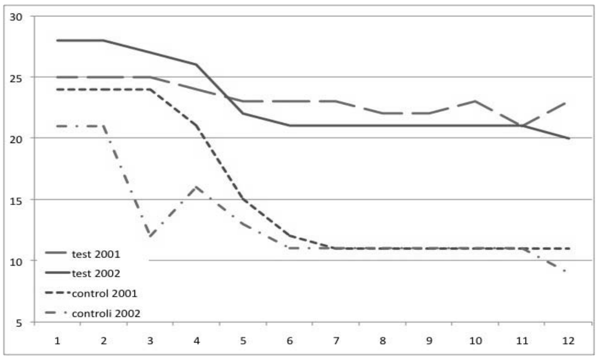
The diagram shows first the average grade in mathematics for each group when entering high school (x=1). Then the average grade for each group in each of the ten compulsory mathematics courses is shown (x= 2, 3,..., 11). Lastly, the average for each group in the national matriculation exam is shown (x = 12). The data for the two groups show in each study only those students who completed all 10 courses and took the final matriculation exam.

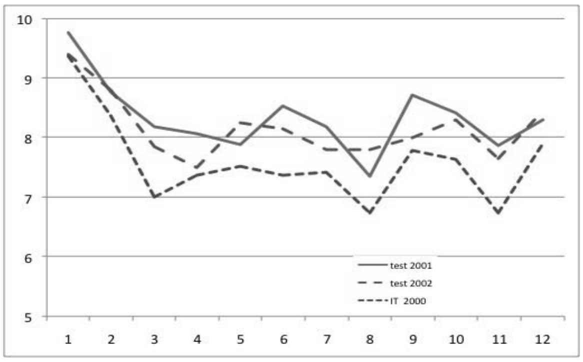


Graph 1: (a) Study 2001 – 2004 (b) Study 2002 – 2005

Both studies show that the test group performs markedly better than the control group in all courses, as well as in the final matriculation exam. This can be partly explained by the initial difference in the entrance scores for the two groups. However, the test group performance is much better than what one would expect from the initial difference in entrance grades alone. This is also supported by a more detailed analysis of the data.

The strength of the test group in both studies is further emphasized by the difference in attendance in the two groups (Graph 2 (a)). Each study followed the students throughout their three years at high school. The students have always an option to drop out of the group, either by moving from advanced level mathematics to standard level, or by performing badly in exams so that they need to retake courses and thus cannot anymore follow the same program as the rest of the test or control group.





Graph 2: (a) Attendants in both studies (b) Comparing IT-groups 2000-2003, 2001-2004 and 2002-2005

The graph shows that there is almost no dropout in the test group. The situation is quite different for the control group. The dropout rate is much higher, less than half of the students in the control group actually finished and took the matriculation exam in due time.

DISCUSSION

The main difference between the test group and the control group is the method of teaching: the test group uses structured derivations and the control group uses traditional teaching methods. But there are, of course, also other differences that could explain the results: the test groups have a somewhat higher average entry grade in mathematics than the control groups, the groups are taught by different teachers, and the students in the test group are there because they preferred to take IT related courses to some other courses. We can check whether these other differences can explain the results, by comparing the results of the same teacher teaching the group of students interested in IT that enrolled one year earlier, in 2000, and wrote their matriculation exam in 2003 (Graph 2 (b)). The selection criteria are thus the same for this group and for the two test groups (2001-2004 and 2002-2005). These two groups also happen to have exactly the same average entry grades in mathematics.

We see that the groups using structured derivations still outperform the group using traditional teaching methods. The difference is not as great as in the earlier comparisons, but it is still quite noticeable. We interpret this as showing that part of the difference between the test and the control groups in the earlier experiments can be attributed to the difference between teachers and entry grades, but not all. A marked difference in favor of the test group remains, indicating that the use of structured derivations really does improve mathematics learning for high school students.

We can also statistically compare the test group of 2002 to the IT group of 2000, because these two groups have the same average entry grades, same selection criteria were used, and the teacher was the same in both groups. A two sample t-test shows that the differences between the course averages in the two groups is statistically significant in 7 cases out of 10 ($0.01 < p < 0.1$, depending on the course). The difference in the matriculation exam is also statistically significant ($p < 0.1$). Two courses where no statistically significant difference was found were Geometry (which was taught in a traditional way also in the test group) and Integrals. A characteristic of the latter course is that it uses a calculational style of reasoning which is not that far away from the structured derivations method.

CONCLUDING REMARKS

The results seem to validate our hypothesis that the use of structured derivations does indeed improve the mathematics performance of high school students. The structured derivations approach to teaching mathematics seems very promising, with a potential for achieving marked improvement in learning results in high school.

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CONSIDERATIONS ABOUT PROOF IN SCHOOL MATHEMATICS AND IN TEACHER DEVELOPMENT PROGRAMMES

Ruy César Pietropaolo and Tânia M. M. Campos

Bandeirante University of São Paulo, Brazil

ABSTRACT: THE AIM OF THIS ARTICLE IS TO PRESENT THE CONCEPTIONS OF A GROUP OF 14 BRAZILIAN SCHOOL TEACHERS REGARDING THE ROLE OF PROVING IN MATHEMATICS TEACHERS EDUCATION COURSES IN THE SPHERE OF A GENERAL LEARNING AND, MORE PARTICULARLY, TAKING INTO ACCOUNT THE FACT THAT THIS SUBJECT MIGHT CONSTITUTE A FUTURE OBJECT OF STUDY. THUS, WE ARE GUIDED BY THE FOLLOWING QUESTION THAT EMERGES FROM THE RESEARCH STUDY: ¨WHAT ARE THE IMPLICATIONS FOR IN-SERVICE TEACHERS COURSES OF CARRYING OUT A PIECE OF WORK WITH PROOF IN SECONDARY-SCHOOL1 CLASSES¨?

1. CHARACTERISING THE RESEARCH PROBLEM

It is well-known that there is a good deal of international research about the introduction of argumentation and proof in secondary school. However, this subject has not been debated much in Brazil, in particular with regard to related areas in teacher education courses. Research among the community of Brazilian Mathematics Educators in this area is comparatively rare and we are thus facing a world that is still unexplored either by research or practice.

Some educational specialists argue that we should not undervalue the role of argumentation and proofs in students´ learning and support the idea of including this subject in the school curricula. Ball et al. (2000) and Dreyfus (2000) are examples of this and in their studies they support this position in the opening lines of their respective abstracts:

> Proof is central to mathematics and as such should be a key component of mathematics education. This emphasis can be justified not only because proof is at the heart of mathematical practice, but also because it is an essential tool for promoting mathematical understanding (Ball et al., 2000, IX ICME)
>
> Proof is at the heart of mathematics, and is considered central in many high school curricula (Dreyfus, 2000).

This attitude in favor of proving in school curricula can doubtless be explained by the attention that has been given to it in recent years, by several researchers.

[1] From 11 to 18 years.

Schoenfeld (1994), for example, argues that demonstrations are not something that can be taken away from Mathematics as it occurs in a lot of teaching programmes. In his view, proving is an essential feature of practice and communication in Mathematics.

In Brazil, the National Curricular Parameters for secondary-school (PCN) recommend, albeit in a rather half-hearted way, that from secondary-school education towards, there should be work involving argumentation and proof. In contrast, Brazilian Mathematics educators, judging from the limited number of research studies on the subject, seem to pay little attention to this issue.

This position might be corroborated by Pietropaolo (2000) in the analysis of the position expressed by Mathematics educators about PCN for secondary school, when trying to put in evidence their agreements and disagreements with regard to the theoretical conceptions of those guidelines. In this research study, it was found that less than 10% of readers made any reference to this subject in their critical analyses and only 4.8% questioned the position of the PCN concerning their work on argumentation and proof in secondary-school. This occour because they believed that those guidelines did not clearly support a revival of this matter and the policy had been abandoned in programmes of the 1980s. The adherents of this view supported the idea of effective work with proof in the last school years of secondary-school, since this question is fully discussed in the initial teacher education.

It is well-known that there is a big difference between recommendations set out in official programmes and the curricula actually adopted in the schools. Even so, as far as work with proof is concerned, the failure of the curricula to be explicit about the matter, allows one to conjecture that it does not form a part of everyday Mathematics classes.

Following a survey that was undertaken with teachers from in-service teachers courses in the public network of São Paulo in 2001, we were able to conjecture that the initial and in-service programmes for Mathematics teachers had not attached importance to teaching skills in proof. This conjecture is strengthened by the results of The National Examination of Courses: the performance of the students in the last year of their initial teacher course was unsatisfactory in dissertational questions, especially those with instructions requiring the student to ¨prove¨, ¨demonstrate¨ and ¨justify¨, even so the students from the universities who, overall, achieved the best results.

As an example, we can refer to the performance of prospective teachers in two dissertational questions proposed in 2001, which involved tasks that are usually carried out in the 7th and/or 8th year of secondary-school. One question aimed to assess if the future teachers knew how to proof the Bhaskara Formula, used to solve a quadratic equation. The results obtained were surprising: on a scale of 0 to 100, Brazilian average was 8.6. In the other one, students were asked to proof a theorem about the diagonals of a rhombus and the results were even worse: 4.4 in a scale of 0 to 100.

In the light of this situation, we sought to examine the opinions of some teachers, with strong Mathematics knowledge and whose teaching practice includes some sort of work involving argumentation and proof, about the need to introduce this content into the curricula of secondary-school and how far they had access to them. We also investigated the implications of this innovation for the curricula that governed the initial teacher education.

2. THE STUDY

For the purposes of this study, we carried out 14 interviews with Mathematics teachers from secondary-schools which had control over the subject-matter used at this school level. The teachers undertook to provide evidence of the opportunities to include argumentation and proof in the secondary-school Mathematics curricula and the work that had to be carried out in the initial teacher education so that they would have greater skills in planning and controlling learning situations in this area. During the interviews, the teachers were also asked to examine the proofs prepared by the students, using the same models employed by Healy and Hoyles (1998, 2000) and Dreyfus (2000) which respectively dealt with the opinions of students and teachers about proofs in Mathematics. This study also relied on the work of Knuth (2002).

Reading the testimony of the interviewees allowed us to identify what we call the ´units of meaning´, or rather, the particular comments which were most significant in enabling us to provide aspects of the research questions for discussion. In other words, these units were extracts from the remarks of the interviewees and were of great potential significance in the view of the researcher.

3. SUMMARY OF POINTS OF AGREEMENT

The conclusions of our investigation with regard to the role of proofs in initial and in-service teacher education can be outlined as follows:

- ✓ *The notions and beliefs that teachers have about their work with proofs at secondary-school act as obstacles to implementing innovatory ideas.*

These teachers do not believe that teaching proof can be a potential means of enhancing doing Mathematics in classroom because this work would be restricted to a few students. In their view, the work with proof – when there is a lot of it - should only be restricted to semi-formal proofs. They explained their opinion by referring to their memories of experiences with demonstrations (almost always unsuccessful), when they were students in secondary-school or in initial teacher course.

However, the analysis that they conducted of work carried out by students (which was shown to them during the interviews), revealed to us that there was a state of tension: their remarks swung between accepting and classifying an ¨empirical proof¨ as excellent and creative, and rejecting it on the grounds that it was not really a mathematical proof, or rather, it was not a rigorous proof. This was the case with all the teachers: when analyzing the work carried out by another student, the teacher shifted his position from praising the ¨experimental qualities¨ to, the next moment, again rejecting this kind of diligence. This tension was noticeable in the testimony too. It can definitely be accounted for by the beliefs and notions of the teachers. This is because while they have been shaped by the conceptions of Maths professionals throughout their lives, they are, at the same time, teachers who are involved in practice – they know how to recognize the diligence of a student and value it. All those who are teachers are delighted to see the proofs; they regard most of them as useful and creative, even though they are not acceptable.

- ✓ *The inclusion of* proof in initial and in-service teacher education should be undertaken both in the list of substantive items of knowledge and in the list *of pedagogical and curricular types of knowledge.*

Our interviewees set out various explanations to show the importance of undertaking rigorous proofs in initial teacher courses subjects. They were needed to learn more Mathematics and were essential when doing or communicating Mathematics, as well as being an essential component of the culture of this area of knowledge. The participants in this research study thought that in the initial teacher course, a student should learn how to demonstrate, even if she was not going to carry out proofs in the classroom in the future. This was because the teacher has to acquire knowledge beyond what she is going to teach – what is sometimes described as ¨a supplementary stock¨.

The interviewees believed that a Mathematics teacher, who knows how to demonstrate theorems and formulae regarding the subject-matter they are going to teach, can share leadership with their peers. As well as this, the subjects of the research described the status accorded by the teachers to the initial teacher course who laid stress on the value of rigorous proofs: it is an excellent course regardless of whether or not it prepares this student for a teaching career.

There was also a consensus about the way that future teachers should study proofs in the initial teacher course; they should experience situations analogous to those they were going to share with their students. As they all thought that the proof in secondary-school should only be understood in its broadest sense, this understanding should also be adopted in initial teacher course subjects. This was not just because the students were going to teach in the future but is also of value, from a learning perspective, to demonstrate and thus acquire logical-deductive powers of reasoning.

This summary of goals, objectives and methods shows that in initial teacher courses in Mathematics, the proof should be regarded as:

- an instrument, which is involved in various subjects in the course (to test the validity, explain, refute, outline theories) and as an important subject in forging links between mathematical topics (historical problems, the connections between different subject areas) or else from the perspective of understanding and looking at concepts and procedures in greater depth;
- a characteristic and indispensable feature of Mathematics, a syntactical element, not confined to any particular subject-matter but rather, regarded from the perspective of a cross-section of disciplines. Yet at any given moment, it – the proof – will constitute subject in itself. (Throughout history, there have been a number of discussions about the kinds of proof that are accepted by mathematicians, the language, the terms employed, the features of axiomatic systems, notions of logic, and alterations in the notion of rigor);
- a subject that will become a constituent part of teaching or more precisely, a part of its pedagogical and curricular perspective (specific aims, examples and counter-examples, analogies, representations, problem situations that need to be given validity, the results of research from a didactic perspective, didactic series).

4. FINAL WORD

In our interviewees' testimony about the inclusion of proofs in initial and in-service teacher courses, some explanations were made, regarding a type of knowledge that is classified as ¨substantive knowledge¨ (Shulman, 1986). In our view, knowing proofs from the perspective of substantive knowledge means that the teacher must possess a sufficient amount of knowledge to allow him to have intellectual autonomy over the subject. This autonomy means, for example, not only knowing the demonstrations of the theorems and formulae, which will be employed in the future but also having the ability to select and organizes these theorems and knowing their respective applications. It implies knowing how to distinguish between what is of major or secondary importance. It requires, above all, that one knows how to set up problems from the demonstrations in a way that can combine them with the subject that is being undertaken. To achieve this, it must act as a mediator between the historically produced knowledge and the kind of knowledge which will be adapted by the students.

In other words, they will be able to extend the range the proofs that they explain (Hanna, 1990). A demonstration that proves and a demonstration that explains are both legitimate forms of demonstration. The difference is that a demonstration that proves only shows *that* a given theorem is true, while a demonstration that explains also shows *why* the theorem is true. In the view of Hanna, not every demonstration has the power to explain and he warns that abandoning demonstrations that validate in favor of those that explain, will not make the curriculum less satisfactory in reflecting acceptable practical mathematics.

Thus, our research shows that demonstrations in initial and in-service teacher courses should be given greater scope than has been given, nowadays. This greater prominence can be achieved if the courses do not make use of proofs just to learn more Mathematics or with the aim of developing mathematics reasoning skills, but could be applied from a didactic, curricular and historical perspective. One possibility would be, for example, to reflect on the ¨evolution¨ of mathematical thinking, which includes the notion that demonstration is something indispensable to Mathematics.

Finally, it should be stressed that our discussion about what constitutes the teachers knowledge of proofs in three areas – substantive knowledge of content, pedagogical knowledge and curricular knowledge – is at an extremely important even decisive, moment of history, with regard to initial training courses, though obviously not in a decisive phase. We know that training a teacher to undertake a professional activity is a process that entails several stages – both forwards and backwards – and that in the last analysis, it is always – or nearly always - incomplete.

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STUDENTS' UNDERSTANDING AND USE OF LOGIC IN EVALUATION OF PROOFS ABOUT CONVERGENCE

Kyeong Hah Roh

Arizona State University, USA

This study investigated what aspects of logic undergraduate students struggle with in evaluating arguments as valid proofs. Students in an introductory real analysis course discussed if arguments were valid as proofs of the convergence of a sequence. Many students had difficulties with constructing and using a negation of a statement in the given arguments. In addition, these students did not understand that the order of variables determined dependence and independence between these variables, and that reversing the order of these variables in a statement would result in the change of the meaning of the statement.

INTRODUCTION

Research in mathematics education indicates that students have difficulties in comprehension and evaluation of mathematical arguments (Mamona-Downs & Downs, 2005; Martin & Harel, 1989; Selden & Selden, 2003; Selden & Selden, 1995). In particular, many students regard an argument as a valid proof as far as each component of the argument was valid (Alcock & Weber, 2005).

There are conjectures on factors that influence students' comprehending and evaluating arguments in relation to logic. First, students may not understand or use rules of logic in mathematically conventional ways. Even though students have been trained in mathematics, they seem to have difficulties in applying proof strategies such as Modus Tollens (Inglis & Simpson, 2004, 2007) or proof by contrapositive (Stylianides, Stylianides, & Philippou, 2004). In addition, they unlikely present or accept deductive arguments (Harel & Sowder, 1998; Hoyles & Kuchemann, 2003).

Second, students' difficulties seem to be related with their understanding and their use of implications which are frequently used in definitions, theorems, and proofs. In particular, many students determine the truth value of a conditional statement only with true antecedent (Duran-Guerrier, 2003), and such a tendency may cause students' confusions between conditional statements and other compound statements (Roh, 2008; Zandieh & Knapp, 2007).

Finally, undergraduate students' difficulties with comprehension and evaluation of mathematical arguments may be due to their difficulties in understanding and using mathematical statements involving multiple quantifiers. For instance, students tend to consider an EA statement "there exists y such that for all x, $P(x, y)$" as the same as an AE statement "for all x, there exists y such that $P(x, y)$" (Dubinsky & Yiparaki, 2000). Also, they seem to face difficulties in understanding the dependence rule in the AE statement, that is, the value of y is determined depending on the value x (Duran-Guerrier, 2005; Roh, 2005).

The purpose of the study examined undergraduate students' comprehension of arguments and their evaluation of the validity of the arguments as mathematical proofs. In particular, this study addresses the following research question: What aspects of logic students struggle with during their evaluation of arguments? This paper analyzes a class episode in which a group of students worked to evaluate arguments about the convergence of a sequence.

DESCRIPTION OF THE STUDY

This study was conducted as part of a larger study from a semester long teaching experiment in an introductory real analysis course at a public university in the United States in 2007. The subjects of this study were students who were majoring mathematical sciences or secondary mathematics education. Data consisted of videotape recordings of all class sessions, office hour sessions and task-based interviews, and copies of students' written work. During class sessions, students worked in small groups with proper guidance from their instructor. The main feature of the group activities were to conjecture, to construct a proof for their conjecture, and to evaluate if arguments given by the instructor were valid as mathematical proofs.

This study focused on one day in which students evaluated if given arguments about the convergence of a sequence were mathematically valid proofs. Prior to the class day, students had already experienced with conjecturing the convergence of various types of sequences as well as with constructing proofs of the convergence of the sequences by using the following definition of the convergence of a sequence:

> A sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number L if for any $\varepsilon > 0$, there exists a positive integer N such that for all $n > N$, $|a_n - L| < \varepsilon$.

In the class day that this paper is focusing on, the students were given a sequence $a_n = 1/n$ for any positive integer n along with the following two arguments:

> <u>Argument 1</u>: We claim that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to 0. Let $\varepsilon = 1/N$ for $N \in \mathbb{N}$. Then $\varepsilon > 0$. For all $n > N$, $|a_n - 0| = \left|1/n - 0\right| = 1/n < 1/N = \varepsilon$. Therefore, the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0.
>
> <u>Argument 2</u>: We claim that the sequence $\{a_n\}_{n=1}^{\infty}$ is not convergent to 0 because for all $N \in \mathbb{N}$, let $n = N + 1$. Then $n > N$. Choose $\varepsilon = 1/(n+1) > 0$. Then $|a_n - 0| = 1/n > 1/(n+1) = \varepsilon$. Therefore, there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $n > N$ such that $|a_n - 0| \geq \varepsilon$. Therefore, $\{a_n\}_{n=1}^{\infty}$ does not converge to 0.

Students were then asked to examine if each of these arguments were valid as mathematical proofs.

In fact, in both arguments given to students, the values of ε were determined depending on the values of N whereas ε is independent of N in the definition of convergence. Therefore, both arguments include a flaw in using the definition of convergence of a sequence. This paper gives illustrative examples to demonstrate the main struggles that a group of students, Stacy, Megan, Mat, and Sophie, had while they evaluated these arguments.

RESULTS

The students began their discussion with the convergence of the sequence $\{1/n\}_{n=1}^{\infty}$, and they all determined this sequence to be convergent to 0. Some students among them considered an argument as a valid proof as far as it concluded the sequence to be convergent to 0. For instance, Stacy and Megan first observed that Argument 1 made the correct conclusion to the convergence of the sequence as did they. Therefore, these students directly accepted Argument 1 as a valid proof, and did not make any further examination on Argument 1. On the other hand, they observed that Argument 2 concluded the sequence not to converge to 0. Since they believed that the sequence was convergent to 0, Stacy and Megan determined that that Argument 2 was not a valid mathematical proof.

Stacy: Okay, well, [the argument] Number 2 is obviously not right. Because [the argument] Number 2 is saying that the sequence does not converge to 0.
Megan: [Laughs] Yeah.

These students continued to examine if Argument 2 contained any other flaw in its assertion. It was noted that they did not understand how to recruit the definition of convergence in proving the sequence not to converge to 0. As seen below, Sophie observed that Argument 2 showed $|a_n - 0| \geq \varepsilon$, but she believed that proofs about the convergence should prove the expression "$|a_n - 0| < \varepsilon$" instead of proving the expression "$|a_n - 0| \geq \varepsilon$". Stacy also argued that that Argument 2 should have examined the absolute value $|a_n - 0|$ for *all* $n > N$ instead of examining $|a_n - 0|$ for *some* value of n by choosing $N+1$ for n. These comments, made by Sophie and Stacy, indicate that they did not recognize that the negation of the statement "for all $n > N$, $|a_n - L| < \varepsilon$" in the definition of convergence of a sequence to be recruited in Argument 2.

Unlike Sophie and Stacy, Mat seemed to recognize that the negation of the statement "for all $n > N$, $|a_n - L| < \varepsilon$" was used in Argument 2. To be precise, Mat pointed out that the statement "there exists $n > N$ such that $|a_n - 0| \geq \varepsilon$" in Argument 2 was the negation of the expression "for all $n > N$, $|a_n - L| < \varepsilon$" in the definition of convergence of a sequence, and, therefore, the proof for this expression in Argument 2 was valid. However, Sophie and Stacy did not understand what Mat meant by when he was saying Argument 2 "did the negation" until he pointed out the negation of the universal quantifier is the existential quantifier.

Sophie: Isn't the definition that this [$|a_n - 0|$] is supposed to be less than ε?
Mat: They did the negation, that's what this… They did the negation of this [the definition of convergence].
Stacy: Okay, well, I think here's one of the problems. This [Argument 2] says "the sequence does not converge to 0, because for all *n* in the natural numbers, let $n = N+1$…." They're choosing the *n*, but it's supposed to be

for all *n*. And we're choosing ε [in Argument 2], because it's going like here and then here.

Mat: Hold on. They're doing the negation. So they're doing the opposite of "for all $\varepsilon > 0$", as we all know and love, there exists N [which is] an element of etc.", right? Greater than *n*? Well, here when you negate that [for all], we have "exists". [The negation of] "for all" [is] "exists." So they're choosing, they have to choose ε and you have to choose n.

After listening Mat's analysis on Argument 2, Sophie understood that the negation of the statement "for all $n > N$, $|a_n - 0| < \varepsilon$" was used in inferring the conclusion of Argument 2, that is, the sequence is not convergent to 0. Sophie was then becoming unsure that what made Argument 2 as invalid in showing its conclusion, and she seemed to experience a cognitive dissonance. On the other hand, Stacy and Mat pointed that Argument 2 concluded the sequence not to converge to 0 and it made Argument 2 incorrect. Their assertion seemed to show that they evaluated components of Argument 2, but did not infer its conclusion by inferring the previous statements on Argument 2.

Sophie: So then why is that [Argument 2] wrong? I mean what makes that proof wrong? I mean it does converge to 0, so…

Stacy: It says here "therefore $\{1/n\}_{n=1}^{\infty}$ does not converge to 0."

Mat: Right. The conclusion is wrong.

Mat and Stacy continued to evaluate Argument 2 only by the validity of its components. When they examined the assertion "choose $\varepsilon = 1/(n+1) > 0$" in Argument 2, they insisted that the statement was valid as far as a positive number was assigned to ε. Since n was a natural number, so was $1/(n+1)$. Mat and Stacy then came to the conclusion that the chosen value of ε should be positive in the case of assigning its value as $1/(n+1)$. On the other hand, Sophie focused on examining if ε *can actually be chosen* as $1/(n+1)$. In fact, the variable ε should not depend on the index N, whereas N is determined depending on ε. Such a relationship between ε and N is implicitly indicated according to the order describing ε and N in the definition of convergence of a sequence. However, no students in this group seemed to recognize that, in Argument 2, the variable ε was not independent of the variable N, and that it made the argument invalid.

Sophie: What about the "choose $\varepsilon = 1/(n+1) > 0$"?

Stacy: Which [$\varepsilon > 0$] is true, that's not the…

Sophie: But can you choose that? I'm looking at this thing ["choose $\varepsilon = 1/(n+1)$"] right here. Is that a statement you can actually make?

Stacy: I don't see why not.

Mat: If it [ε]'s greater than 0, [you can choose ε].

Sophie: You can?

Mat: As far as I know. Well, wait a minute. If you choose *n* to be greater than *N*, I don't know why you couldn't.

According to the above excerpt, these students seemed think that the value of ε can be chosen depending on the value of N in the definition of convergence. Such a misunderstanding can be stated as follows:

(1) For any $N \in \mathbb{N}$, there exists $\varepsilon > 0$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$.

However, every bounded sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the statement (1) for any real number L; therefore, understanding the statement (1) as a sufficient and necessary condition for convergence of a sequence results in the assertion that every bounded sequence is convergent to any real number. For instance, the oscillating divergent sequence $a_n = \{1 + (-1)^n\} / 2$, for any $n \in \mathbb{N}$, satisfies $|a_n - 0| \leq 1$. Hence, for any $n \in \mathbb{N}$, we can choose $\varepsilon = 2$ to obtain the relation "for all $n \geq N$, $|a_n - L| < \varepsilon$" for $L = 0$. On the other hand, convergent sequences satisfy the negation of the statement (1) as seen in Argument 2. Hence applying the statement (1), a sequence can be determined to be both convergent and divergent simultaneously. In fact, as seen above, many students in this study did not grasp this independence rule of ε. Rather, they accepted the reversal of the order between ε and N in the definition of the convergence of a sequence.

CONCLUDING REMARKS

A primary cause of students' difficulties in comprehension and evaluation of arguments about the convergence of a sequence was their lack of logic with multiple quantifiers. In particular, many were not able to use a negation of a statement with multiple quantifiers. In addition, they were not able to use the independence rule between ε and N in the definition of the convergence of a sequence. Unfamiliarity of statements with multiple quantifiers may be a cause of student difficulties in evaluating mathematical argument at the tertiary level. However, understanding and using multiple quantified statements are fundamental in undergraduate mathematics. In particular, changing the syntactic structure of a statement by reversing the order of variables makes subtle but prevalent semantic differences in meaning.

This study adds to the literature by describing students' evaluation of arguments involving multiple quantifiers. It has been known that students have difficulties in understanding the dependence of N on ε in the definition of the convergence of a sequence (Duran-Guerrier, 2005; Roh, 2005). This study suggests that only the dependence rule of N on ε but also the independence rule of ε from N may also be difficult for students to grasp unless it is particularly emphasized in instruction.

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“IS THAT A PROOF?”: USING VIDEO TO TEACH AND LEARN HOW TO PROVE AT THE UNIVERSITY LEVEL

Manya Raman, Jim Sandefur, Geoffrey Birky, Connie Campbell, Kay Somers

Umeå University, Sweden; Georgetown University, USA; Millsaps College, USA; Moravian College, USA

This paper reports a design experiment using videocases to help teach university level students how to construct mathematical proofs. In our third iteration, we have found it useful to identify three “moments” in the production of a proof: one that gives cause for believing the truth of a claim, one that indicates how a proof could be constructed, and one that formalizes the argument, logically connecting given information to the conclusion. We suspect that these three “moments” are familiar to mathematicians, as each one gives a recognizable peak of satisfaction, but they are not always articulated in the classroom. As a result, students might learn to see these three “moments” as more disconnected than they often are.

INTRODUCTION

Design experiments are becoming increasingly common, at elementary, secondary, and even tertiary education, as researchers and teachers try to find theoretically grounded answers to real problems in the classroom[1]. While the potential of merging theory and practice is quite alluring for many reasons, the practical and conceptual realities of doing so remain challenging. This paper describes part of one design experiment aimed at improving the teaching of proof at the university level as an example of how theory and practice can sometimes meet in a mutually productive way.

Using methodology outlined by Cobb, et al (2003), our design is iterative, interventionist, and theory-oriented. It involves gathering and indexing of longitudinal data from a number of sources, including videos of classroom practice, individual and group interviews with teachers and students, journal and email records from the teachers, written records of student work, and audio and video records of behind-the-scenes discussion among the research team. Like Cobb, et al, we see this design experiment as a “crucible for the generation and testing of theory.” It is the tangible pressures of classroom realities that provide a needed spark for the theory to develop and crystallize, and one of the goals of this

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[1] A longer version of this paper will include a review of design experiments with focus on using video to teach upper level mathematics. We omit here for space reasons.

paper is to make part of that process visible to both research and practitioner communities.

In the third iteration of our design experiment, we now have a fairly stable set of curricular materials, which include (1) carefully edited videos of students working on proofs that many other students find difficult, and (2) materials to help teachers use these videos, both for their own understanding of student thinking and for classroom use. These materials have been tested in four colleges in the United States in the context of "Introduction to Proof" courses that are used to bridge between the computation-driven lower division courses and the theory-driven upper division courses[2].

THEORETICAL TOOLS

In the process of observing videos for potential use in the classroom, we have identified three significant "moments" in creating a proof[3] (not always found or used in this order.)

The first moment is the getting of a *key idea*, an idea that gives a sense of "now I believe it". The key idea is actually a property of the proof, but psychologically it appears as a property of an individual (we say that a particular person "has a key idea" if it appears that they grasp the key idea of a proof.) We refer to "a" key idea rather than "the" key idea, because it appears that some proofs have more than one key idea. While a key idea engenders a sense of understanding, it does not always provide a clue about how to write up a formal proof.

The second moment, is the discovery of some sort of *technical handle*, gives a sense of "now I can prove it," that is, some way to render the ideas behind a proof communicable[4]. The technical handle is sometimes used to communicate a particular key idea, but it may be based on a different key idea than the one that gives an 'aha'-feeling, or even on some sort of unformed thoughts or intuition (the feeling of 'stumbling upon'.)

The third moment is a culmination of the argument into a *standard form*, which is a correct proof written with a level of rigor appropriate for the given audience. This task involves, in some sense, logically connecting given information to the conclusion. We assume that for mathematicians the conclusion is probably in mind for most of the proving process. But for students, the theorem might

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[2] See Alcock (2008) for a similar project, focusing primarily on professional development, which has been successfully piloted in the US and UK.

[3] The identification of these moments builds on work of Hanna (1989), Raman (2003) and others. Connections to the literature will be expanded upon in a longer version.

[4] The term "technical handle" here is akin to the term "key insight" in Raman and Weber (2006). We have chosen to change the term in part because it sounded too similar to "key idea" which has a very different character, and in part because the technical aspect of this "moment" seemed central to its nature.

sometimes be lost from sight, adding a sense of confusion to their thinking processes.

THE EPISODE

The following example illustrates the presence and/or absence of these three moments as students work on the following task:

Let n be an integer. Prove that if $n \geq 3$ then $n^3 > (n+1)^2$.

Students were videotaped working on this task in the presence of the research team. After they worked on the task, students were asked questions about their thinking. Afterwards, the research team watched and discussed the videos. We were drawn to one part of the proof process that turned out to be a genuine mystery—an episode, near the beginning, in which the students generate what the faculty identify as a correct proof, but what the students, at least at some level, do not recognize as one.

Details: In the first two minutes of working on this task, the students came up first with an argument that the professors identified as a key idea of the proof, namely that a cubic function grows faster than a quadratic. Rather than trying to formalize this idea, the students switched to an algebraic approach, what we label as a technical handle, to try to get to a proof. They wrote $n^3 > n^2 + 2n + 1$ which they manipulated into $n(n^2 - n - 2) > 1$ and then $(n\text{-}2)(n+1) > 1/n$.

The students then noticed that if $n \geq 3$ then the terms on the left are both positive integers so the product is a positive integer. And since n is an integer greater than two, the right hand side is going to be between 0 and 1. They wrote these observations as

if $n \geq 3$ (line break) $n\text{-}2 > 0$ (line break) $n+1 > 0$ (line break) $0 \leq 1/n \leq 1$

and seemed quite pleased with their reasoning, one student nodding and smiling as the other one wrote the last line.

S2: Yeah.

S1: This is if n is greater than 3, if n is greater than or equal to 3.

S2: Yeah…. Cool.

At this point in the live proof-writing, the three professors were convinced that the students had a proof. They believed that "all" the students needed was a reordering of their argument. To show $n^3 > n^2 + 2n + 1$, it suffices to show $(n\text{-}2)(n+1) > 1/n$, which one can establish by showing that the left-hand side is a positive integer while the right is between 0 and 1.

However, it turned out that the students, despite being pleased with their argument were less than sure that they were near a formal proof. A professor asked the students "Is that a proof?" and S1 replied, "That's what I'm trying to figure out." As the students moved to now write up the proof, they switched to a new track, trying a proof by contraposition, which ended up turning into a confusing case analysis in which they tried to prove the converse of the contrapositive and investigated many irrelevant cases.

AN EVOLVING EXPLANATION

That students can come so close to a proof without recognizing it is probably familiar to most experienced teachers[5]. *Why* the students were not able to recognize that they are so close is another, more difficult, question. Here we show how looking at the three "moments" of the proof, described above, allows us to compare what the students did in this problem with an idealized version of what faculty might have done.

The moments are represented graphically in Figure 1 below, with the blue line representing the "ideal" (professor-like) proving process[6], and the red line (broken to indicate different solution strategies) representing the students'. The marks m_i indicate the points in the proof at which different moments are achieved: m_1 for the key idea, which both faculty and students achieved (though the students may not realize this), m_2 for the technical handle (which students in this case see as disconnected from their key idea), and m_3 for the organization of the key idea and/or technical handle into a clear, deductive argument (which in this case the students never reach.)

Specifically, m_1 is recognizing that cubic functions grow faster than quadratic ones. m_2 is choosing an algebraic approach, factoring the polynomials before and after the inequality sign. We label this as a technical handle even though the students do not know from the beginning where this might lead. m_3 is connecting the assumption that $n \geq 3$ with the conclusion that $n^3 > (n+1)^2$. In this case, the students never reached m_3, and in fact—during their attempt to write a formally accepted proof, they seem to lose sight of what they are proving.

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[5] Another example can be found in Schoenfeld (1985) where two geometry students have what the researcher is convinced is a correct "proof" but when asked to write it up, they draw two columns and abandon all their previous work.

[6] In creating this "idealized" version of a proof, we depict a continuity between the key idea and the technical handle, although we realize in practice that many proofs are made without the author being able to connect the two. The question about whether there exists such a connection, even if it has not been found, is an open one. We also realize that the process of proof development is not linear, even for an able mathematician, in many cases. This picture points out more the over-all trajectory of the proof, with minor false-paths ruled out. Further the heights of the peaks could vary.

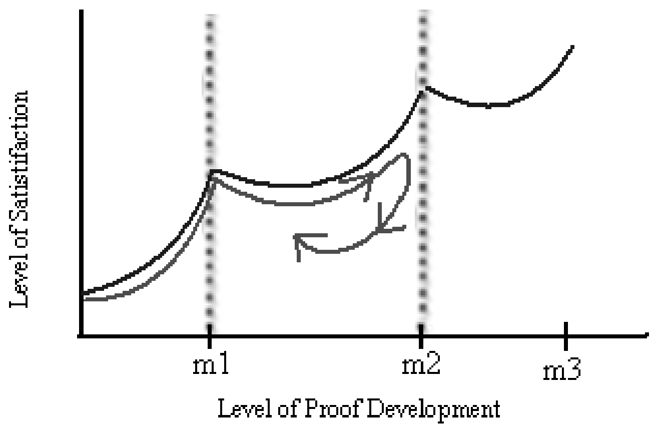

Figure 1: Comparing student (red) and faculty (blue) proof strategies

In the episode above, the students find two key ideas: one that cubics grow faster than quadratics, and another, after students have written $(n\text{-}2)(n\text{+}1) > 1/n$, that the right-hand term is trapped between 0 and 1 while the left grows indefinitely. Neither of these ideas gets developed into a formal proof. The curved line between m_1 and m_2 represents how students move towards a technical handle and end up at the second key idea.

The crucial distinctions between the "ideal" graph and the "student" graph are the breaks at m_1 (students do not try to connect their key idea to a technical handle) and m_2 (students lose sight of the conclusion and end up trying to prove a converse.) Our data indicate that these breaks are not merely cognitive—it isn't that the students do not have the mathematical knowledge to write a proof, since they articulate the essence of the proof after three minutes. The problem is epistemological—they don't seem to understand the geography of the terrain. Expecting discontinuity between a more intuitive argument and a more formal one, the students practically abandon their near-perfect proof for something that appears to them more acceptable as a formal proof[7].

Of course it is not always possible to connect key ideas to a technical handle, or to render a technical handle into a complete proof. But what distinguishes the faculty from the students is that the faculty are *aware* that this connection is possible, and might even be preferable given that sometimes it takes little work—in this case a simple reordering of the algebraic argument would suffice for a proof. As one professor in the study said:

> "It became clear that to formalize meant something different to them and to us. To us, formalize seemed to mean 'simply clean up the details'. To them, it seemed to mean 'consider rules of logic and consciously use one'."

[7] They actually arrived at what would have been a second technical handle, except their work was incorrect.

Recognizing the difference between *radical* jumps that need to be made to move mathematical thinking forward and *local* jumps that allow one to delicately transform almost rigorous arguments into rigorous ones might be an essential difference that mathematics teachers can learn to recognize, diagnose, and communicate to their students.

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THE ALGORITHMIC AND DIALECTIC ASPECTS IN PROOF AND PROVING

Man-Keung Siu

The University of Hong Kong, Hong Kong SAR, China

Through some examples we discuss the algorithmic and dialectic aspects in proof and proving, with a pedagogical implication on emphasizing both aspects which complement and supplement each other in this mathematical activity.

In 1973 Peter Henrici coined the terms "algorithmic mathematics" and "dialectic mathematics" and discussed the desirable equilibrium of these two polarities (Henrici, 1974; see also Davis & Hersh, 1980, Chapter 4). According to Henrici, "*Dialectic mathematics* is a rigorously logical science, where statements are either true or false, and where objects with specified properties either do or do not exist. *Algorithmic mathematics* is a tool for solving problems. Here we are concerned not only with the existence of a mathematical object, but also with the credentials of its existence. *Dialectic* mathematics is an intellectual game played according to rules about which there is a high degree of consensus. The rules of the game of *algorithmic* mathematics may vary according to the urgency of the problem on hand. (…) *Dialectic* mathematics invites contemplation. *Algorithmic* mathematics invites action. *Dialectic* mathematics generates insight. *Algorithmic* mathematics generates results." (Henrici, 1974, p.80) In a lecture given in Crete in July of 2002 I borrowed these two terms and attempted to synthesize the two aspects from a pedagogical viewpoint with illustrative examples gleaned from mathematical developments in Eastern and Western cultures throughout history. In this note I reiterate this theme with a focus on proof and proving and discuss how the two aspects complement and supplement each other in this mathematical activity. A few examples are taken from the 2002 lecture, the text of which remains unpublished. We would not go into a culture-related aspect, for which readers can read (Siu, 2008).

Readers are asked to bear with a more liberal usage of the word "algorithm" used here, namely, any well-defined sequence of operations to be performed in solving a problem, *not* necessarily involving branching upon decision or looping with iteration. Following (Chabert et al, 1994/1999, p.455; McNaughton, 1982) we mainly require: (i) "The algorithm is a procedure which is carried out step by step"; (ii) "whatever the entry data, the execution of the algorithm will terminate after a finite number of steps."

The first example is a clay tablet dating from the 18th century B.C., on which was inscribed a square and its two diagonals with numbers (in cuneiform expressed in the sexagesimal system) 30 on one side and 1.4142129... and 42.426388... on one diagonal. There is no mistaking its meaning, namely, the calculation of the square root of 2 and hence the length of the diagonal of a square with side of length 30.

Some historians of mathematics believe that the ancient Babylonians worked out the square root of 2 by a rather natural algorithm based on the following principle. Suppose x is a guess which is too small (respectively too large), then $2/x$ will be a guess which is too large (respectively too small). Hence, their average $0.5(x + 2/x)$ is a better guess. We can phrase this procedure as a piece of "algorithmic mathematics" in solving the equation $X^2 - 2 = 0$: Set $x_1 = 1$ and $x_{n+1} = 0.5(x_n + 2/x_n)$ for $n \geq 1$. Stop when x_n achieves a specified degree of accuracy.

It is instructive to draw a picture (see Figure 1) to see what is happening. The picture embodies a piece of "dialectic mathematics" which justifies the procedure: ξ is a root of $X = f(X)$ and ξ is in $I = [a, b]$. Let f and f' be continuous on I and $|f'(x)| \leq K < 1$ for all x in I. If x_1 is in I and $x_{n+1} = f(x_n)$ for $n \geq 1$, then $\lim_{n\to\infty} x_n = \xi$.

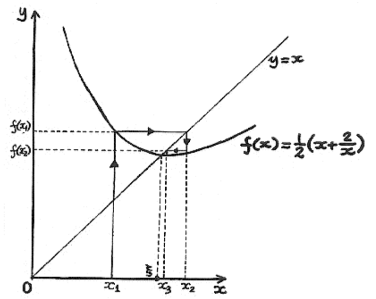

Figure 1

"Algorithmic mathematics" abounds in the ancient mathematical literature. Concerning the extraction of square root Problem 12 in Chapter 4 of the Chinese mathematical classics *Jiuzhang Suanshu* [*Nine Chapters On the Mathematical Art*] (Shen et al, 1999), compiled between 100B.C. and 100A.D., asks: "Now given an area 55225 [square] *bu*. Tell: what is the side of the square?"

The method given in the book offers an algorithm that yields in this case the digit 2, then 3, then 5 making up the answer $\sqrt{55225} = 235$. Commentaries by Liu Hui in the mid 3rd century gave a geometric explanation (see Figure 2) in which integers $a \in \{0, 100, 200, \ldots, 900\}$, $b \in \{0, 10, 20, \ldots, 90\}$, $c \in \{0, 1, 2, \ldots, 9\}$ are found such that $(a + b + c)^2 = 55225$.

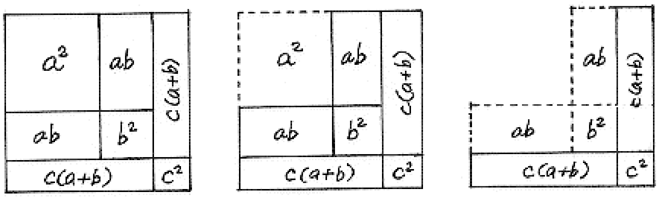

Figure 2

A suitable modification of this algorithm for extracting square root gives rise to an algorithm for solving a quadratic equation, which is explained through a typical example like Problem 20 in Chapter 9 of *Jiuzhang Suanshu* that amounts essentially to solving the equation $X^2 + 34X = 71000$.

The same type of quadratic equations was studied by the Islamic mathematician Muhammad ibn Mūsā Al-Khwarizmi in his famous treatise *Al-kitāb al-muhtasar* *fĩ hisab al-jabr wa-l-muqābala* [*The Condensed Book On the Calculation of Restoration And Reduction*] around 825A.D. The algorithm exhibits a different flavour from the Chinese method in that a closed formula is given. Expressed in modern terminology, the formula for a root x of $X^2 + bX = c$ is what we see in a school textbook. Just as in the Chinese literature, the "algorithmic mathematics" is accompanied by "dialectic mathematics" in the form of a geometric argument.

Let us get back to the equation $X^2 - 2 = 0$. On the algorithmic side we have exhibited a constructive process through the iteration $x_{n+1} = 0.5(x_n + 2/x_n)$ which enables us to get a solution within a demanded accuracy. On the dialectic side we can guarantee the existence of a solution based on the Intermediate Value Theorem applied to the continuous function $f(x) = X^2 - 2$ on the closed interval [1, 2]. The two strands intertwine to produce further results in different areas of mathematics, be they computational results in numerical analysis or theoretic results in algebra, analysis or geometry. At the same time the problem is generalized to algebraic equations of higher degree. On the algorithmic side there is the work of Qin Jiushao who solved equations up to the tenth degree in his 1247 treatise, which is equivalent to the algorithm devised by William George Horner in 1819. On the dialectic side there is the Fundamental Theorem of Algebra and the search of a closed formula for the roots, the latter problem leading to group theory and field theory in abstract algebra. In recent decades, there has been much research on the constructive aspect of the Fundamental Theorem of Algebra, which is a swing back to the algorithmic side.

Thus we see that it is not necessary and is actually harmful to the development of mathematics to separate strictly "algorithmic mathematics" and "dialectic mathematics". Traditionally it is held that Western mathematics, developed from that of the ancient Greeks, is dialectic, while Eastern mathematics, developed from that of the ancient Egyptians, Babylonians, Chinese and Indians, is algorithmic. As a broadbrush statement this thesis has an element of truth in it, but under more refined examination it is an over-simplification. See for example (Chemla, 1996).

We look at a second example, the Chinese Remainder Theorem. The source of the result, and thence its name, is a well-known problem in *Sunzi Suanjing* [*Master Sun's Mathematical Manual*], compiled in the 4th century, that amounts to solving, in modern terminology, the system of simultaneous linear congruence equations

$$x \equiv 2 \pmod 3, \quad x \equiv 3 \pmod 5, \quad x \equiv 2 \pmod 7.$$

The name "Chinese Remainder Theorem" (CRT) is explicitly mentioned in (Zariski & Samuel, 1958, p.279), referring to Theorem 17 about a property of a Dedekind domain, with a footnote that reads: "A rule for the solution of simultaneous linear congruences, essentially equivalent with Theorem 17 in the case of the ring J of integers, was found by Chinese calendar makers between the fourth and the seventh centuries A.D. It was used for finding the common periods to several cycles of astronomical phenomena."

In many textbooks on abstract algebra the CRT is phrased in the ring of integers $\mathbb{Z}$ as an isomorphism between the quotient ring $\mathbb{Z}/M_1 \ldots M_n \mathbb{Z}$ and the product $\mathbb{Z}/M_1\mathbb{Z}\times\cdots\times\mathbb{Z}/M_n\mathbb{Z}$ where M_i , M_j are relatively prime integers for distinct i, j. A more general version in the context of a commutative ring with unity R guarantees an isomorphism between $R/I_1 \cap\cdots\cap I_n$ and $R/I_1 \times\cdots\times R/I_n$ where $I_1,\ldots,I_n$ are ideals with $I_i + I_j = R$ for distinct i, j. Readers will readily provide their own "dialectic" proof of the CRT.

In a series of articles published in the Shanghai newspaper *North-China Herald* titled "Jottings on the science of the Chinese" the British missionary Alexander Wylie of the mid 19th century referred to the famous Chinese mathematician Qin Jiushao (Tsin Keu Chaou), who compiled in 1247 the treatise *Shushu Jiuzhang* [*Mathematical Treatise in Nine Sections*] and introduced the technique "*Da Yan* (or *Ta-yen*, meaning the Great Extension) art of searching for unity".

Let us phrase this technique in modern terminology to illustrate the algorithmic thinking embodied therein. The system of simultaneous congruence equation is

$$x \equiv A_1 \pmod{M_1}, \quad x \equiv A_2 \pmod{M_2}, \ \ldots, \ x \equiv A_n \pmod{M_n}.$$

Qin's work includes the general case when $M_1 , \ldots , M_n$ are not necessarily mutually relatively prime by arranging $m_i \mid M_i$ with $m_1 , \ldots , m_n$ mutually relatively prime and $LCM(m_1 , \ldots , m_n) = LCM(M_1 , \ldots , M_n)$. The next step in Qin's work reduces the system (in the case $M_1 , \ldots , M_n$ are mutually relatively prime) to solving separately a single congruence equation of the form $k_i b_i \equiv 1 \pmod{M_i}$. Finally, in order to solve the single equation $kb \equiv 1 \pmod{m}$ Qin uses reciprocal subtraction, equivalent to the famous Euclidean algorithm, to the equation until 1 (unity) is obtained. When the calculation is performed by manipulating counting rods on a board as in ancient times, the procedure is rather streamlined.

Within this algorithmic thinking we can discern two points of dialectic interest. The first is how one can combine information on each separate component to obtain a global solution. This feature is particularly prominent when the result is formulated in the CTR in abstract algebra. The second is the use of linear combination which affords a tool for other applications such as for curve fitting or the Strong Approximation Theorem in valuation theory.

Let me give three more examples gleaned from my own experience in learning and teaching.

(1) I vividly remember my "moment of revelation" in school algebra. One day, after working on several problems on long division of one polynomial by a linear polynomial $X - \alpha$, I was told that the tedious algorithmic work can be skipped because the same answer will fall out simply by evaluating the given polynomial at α. The proof given in the textbook was to me quite an eye-opener at the time. Familiarity with the problem through the "algorithmic mathematics" allows me to appreciate better the "dialectic mathematic" based on the Euclidean algorithm.

(2) As a pupil I came across in school algebra many homework problems which ask for writing expressions like $p^3q + pq^3$ or $5p^2 - 3pq + 5q^2$ or $p^4 + q^4$ in terms of a, b, c where p, q are the roots of $aX^2 + bX + c = 0$. It was only many years later that I came to understand why this can always be done. The underlying result is the Fundamental Theorem on Symmetric Polynomial, which has different proofs and can be formulated in a rather general context over a commutative ring with unity. It is helpful to work out one example in an algorithmic fashion to get a flavour of the dialectic proof. For instance let us try to express the polynomial

$$X_1^3X_2^2 + X_2^3X_3^2 + X_3^3X_1^2 + X_1^2X_2^3 + X_2^2X_3^3 + X_3^2X_1^3$$

in terms of $\sigma_1 = X_1 + X_2 + X_3,\ \sigma_2 = X_1X_2 + X_2X_3 + X_3X_1,\ \sigma_3 = X_1X_2X_3$. Naturally we can write the polynomial in X_1, X_2, X_3 as a polynomial in X_3 with coefficients involving X_1, X_2, i.e.

$$f(X_1, X_2, X_3) = (X_1^3X_2^2 + X_1^2X_2^3) + (X_1^3 + X_2^3)X_3^2 + (X_1^2 + X_2^2)X_3^3$$

Applying our knowledge of polynomials in X_1, X_2 (after so much working in school algebra), we arrive at

$$f(X_1, X_2, X_3) = \tau_1\tau_2^2 + (\tau_1^3 - 3\tau_1\tau_2)X_3^2 + (\tau_1^2 - 2\tau_2)X_3^3$$

where $\tau_1 = X_1 + X_2,\ \tau_2 = X_1X_2$. With some further working we can express the coefficients $\tau_1\tau_2^2, \tau_1^3 - 3\tau_1\tau_2, \tau_1^2 - 2\tau_2$ in terms of $\sigma_1, \sigma_2, \sigma_3$ and X_3 up to the second power. Substituting back to $f(X_1, X_2, X_3)$ we obtain, after some rather tedious (but worthwhile!) work,

$$f(X_1, X_2, X_3) = \sigma_1\sigma_2^2 - 2\sigma_1^2\sigma_3 - \sigma_2\sigma_3.$$

Note that suddenly all terms involving X_3 vanish and that is the answer we want! Coincidence in mathematics is rare. If there is any coincidence, it usually begs for an explanation. The explanation we seek in this case will lead us to one proof of the Fundamental Theorem on Symmetric Polynomial.

(3) The simplest type of extension field discussed in a basic course on abstract algebra is the adjunction of a single element $\alpha \in \mathbb{C}$ algebraic over the ground field $\mathbb{Q}$, that is, α is the zero of some polynomial with coefficients in $\mathbb{Q}$. The dialectic aspect involves the "finiteness" of the extension field $\mathbb{Q}(\alpha)$ viewed as a finite-dimensional vector space over $\mathbb{Q}$. It is helpful to go through some algorithmic calculation to get a feel for the "finiteness". For instance, take $\alpha = \sqrt{2}$. By knowing what $\mathbb{Q}(\alpha)$ stands for we see that a typical element in $\mathbb{Q}(\alpha)$ is of the form $(a + b\alpha) / (c + d\alpha)$ where a, b, c, d are in $\mathbb{Q}$, because any term involving a higher power of α can be ground down to a linear combination (over $\mathbb{Q}$) of 1 and α. The procedure on conjugation learnt in school allows us to simplify it further to the form $a' + b'\alpha$ where a', b' are in $\mathbb{Q}$. It is instructive to follow with a slightly more complicated example such as α equal to the square root of $1+\sqrt{3}$, in which case it is much more messy to revert the denominator as part of the numerator. This will motivate a more elegant dialectic proof modelled after the algorithmic calculation for $\alpha = \sqrt{2}$.

We now come to the pedagogical viewpoint. In looking at how the two aspects — "algorithmic mathematics" and "dialectic mathematics" — intertwine with each other, one is reminded of the *yin* and *yang* in Chinese philosophy in which the two aspects complement and supplement each other with one containing some part of the other. If that is the case, then we should not just emphasize one at the expense of the other. When we learn something new we need first to get acquainted with the new thing and to acquire sufficient feeling for it. A procedural approach helps us to prepare more solid ground to build up subsequent conceptual understanding. In turn, when we understand the concept better we will be able to handle the algorithm with more facility. This remains so even in studying the seemingly more 'theoretical' process known as proof and proving.

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ABILITY TO CONSTRUCT PROOFS AND EVALUATE ONE'S OWN CONSTRUCTIONS[1]

Gabriel J. Stylianides — *University of Pittsburgh, USA*

Andreas J. Stylianides — *University of Cambridge, UK*

In this article we focus on a group of 39 prospective elementary teachers who had rich experiences with proof and we examine their ability to (1) construct proofs and (2) evaluate their own constructions. We claim that this combined examination can offer deep insight into individuals' understanding of proof.

For proof to pervade elementary students' mathematical education, it is necessary that elementary teachers have solid understanding of this concept. Prior research showed that many prospective elementary teachers face serious difficulties understanding the differences between proofs, invalid general arguments, and empirical arguments (Goetting, 1995; Martin & Harel, 1989).[2]

The studies by Goetting (2005) and Martin and Harel (1989) examined prospective elementary teachers' understanding of proof by analyzing *prospective teachers'* (*PTs'*) evaluations of specific arguments presented to them. The evaluations took the form of responses to multiple-choice questions, which included different characterizations of the given arguments. PTs' limited understanding of proof might have been one reason for which these studies did not ask PTs to construct their own proofs. It is generally harder for individuals to construct proofs than evaluate given arguments, so it would be meaningful for a study on PTs' ability to construct proofs to use a non-typical sample.

In this article we focus on a group of 39 prospective elementary teachers who had rich experiences with proof and we examine their ability to (1) construct proofs and (2) evaluate their own constructions. We claim that the examination of this dual "construction-evaluation" activity can illuminate certain aspects of individuals' understanding of proof that tend to defy scrutiny when individuals are asked to evaluate given arguments. For example, the construction component

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[2] In this article we use the following definitions. A *general argument* for the truth of a claim denotes a sequence of assertions that refer to all cases involved in the claim. A *proof* is a valid general argument, where the term "valid" denotes that the argument is deductive and provides conclusive evidence for the truth of a claim. Although a proof is a valid general argument, a valid general argument is not necessarily a proof because further justification may be required for an assertion made in the argument that is not readily acceptable by the community where the argument is developed. An invalid general argument has some flaw in its logic. An *empirical argument* denotes an invalid argument that provides inconclusive evidence for the truth of a claim by verifying its truth in a proper subset of all cases involved in the claim.

makes it possible for individuals to provide arguments they believe qualify as proofs and that a researcher might not include in an evaluation activity. The evaluation component makes it possible for researchers to distinguish between individuals who provided, for example, an empirical argument for a claim and believed that they had a proof and others who provided the same argument but realized its limitations. So the combined examination of individuals' constructions and self-evaluations of these constructions can cast light on the degree of matching between individuals' perceptions of whether their constructions fulfilled their intended purpose to qualify as proofs (*psychological perspective*) and the mathematical classification of these constructions as empirical arguments, proofs, etc., based on our definitions of these terms (*mathematical perspective*). High degree of matching can be indicative of good understanding of proof.

METHOD

We report findings from the last research cycle of a four-year *design experiment* in an undergraduate mathematics course for prospective elementary teachers. Our goal in this design experiment was to develop, implement, and analyze the potential of instructional sequences to promote PTs' mathematical knowledge for teaching with particular attention to their knowledge about proof. This was a prerequisite course for admission to the masters level elementary teacher education program at a large American university. It was the only mathematics course in the program, so it covered a wide range of mathematical topics.

Our Approach in the Course to Promote PTs' Knowledge about Proof

We treated proof as a vehicle to sense-making and as a process that underpinned PTs' mathematical work in all topics covered in the course. We supported multiple opportunities for PTs to develop proofs, to represent them in different ways (using everyday language, algebra, or pictures), and to examine the correspondences among different representations.

In order to create and support developmental progressions in PTs' knowledge about proof, we developed instructional sequences that generated *cognitive conflict* (e.g., Swan, 1983) for PTs. Our goal was to help PTs reflect on their current understandings about proof, confront contradictions that arose in contexts where some of these understandings no longer held, and see the "intellectual need" (Harel, 1998) to develop new understandings that better approximated conventional understandings. We used two primary means to support the resolution of cognitive conflicts experienced by PTs: (1) social interactions among the class members, and (2) an active role of the instructor who was viewed as the representative of the mathematical community in the classroom. The instructor's role was not only to scaffold PTs' work and help them become (more) aware of their current understandings, but also to offer them access to conventional knowledge for which they saw the intellectual need but were unable to develop on their own (due to conceptual barriers, etc.).

After the first few sessions in the semester, the class collectively developed a list of characteristics for "good proofs" that the class would use throughout the course to make decisions about whether different arguments qualified as proofs. The lists that were generated by the PTs in both sections of the course that participated in the last cycle of our design experiment incorporated the key aspects of generality and validity in our definition of proof (cf. footnote 2).

Data

The data for the article include the written responses of 39 PTs to two tasks from the midterm and final take-home exams (figure 1). Academic honor codes required that the PTs completed the exams individually and with reference only to the course materials. The concepts involved in the two tasks were known to the PTs but the conjectures were unfamiliar to them. Also, the two tasks had similar structure and were considered to be of comparable level of difficulty.

Part 1 of Task 1 (midterm exam):
You teach fourth grade. Yesterday your students explored (on the set of whole numbers) what happens when two consecutive odd numbers are added together, and they came up with the following conjecture:

The sum of any two consecutive odd numbers is a multiple of 4.

Part 1 of Task 2 (final exam):
You teach fourth grade. Yesterday your students explored (on the set of whole numbers) different relationships with odd numbers and multiples of numbers, and they came up with the following conjecture:

If you multiply any odd number by 3 and then you add 3, you get a multiple of 6.

Part 2 of both tasks
Your students became interested in knowing whether their conjecture is true or false, and in class tomorrow you plan to help them prove the conjecture true or false.
Question 1: Is the conjecture above true or false? Prove your answer.

[Note: You need to avoid an algebraic proof because your students do not know about algebra yet. However, your students have a lot of experience in representing their ideas using pictures or everyday language.]

Question 2: Do you think you have actually produced a proof? Why or why not?

[Note: It is important that you evaluate objectively your proof. You shouldn't feel obligated to identify flaws in your proof. If you have produced a proof, you should say so in order to get the credit for this question. If, however, there are flaws in your proof, you need to identify them in order to get the credit for this question.]

Figure 1: The two tasks

The contextualization of PTs' mathematical work in a teaching situation (a class of fourth graders who do not know about algebra yet) was part of our broader goal in the course to promote and assess *mathematical knowledge for teaching*.

Analytic Method

All PTs identified the conjectures in the two tasks as true, and so our analysis of question 1 was a classification of their constructions to show the truth of the conjectures. Adopting a mathematical perspective, we coded each construction in one of the following categories: *proof* (code **M1**), *valid general argument but not a proof* (**M2**), *unsuccessful attempt for a valid general argument* (i.e., invalid or incomplete general argument) (**M3**), *empirical argument* (**M4**), and *a non-genuine argument* (i.e., response that shows minimal engagement or irrelevant response) (**M5**). For our analysis of PTs' self-evaluations of their constructions in question 2, we adopted a psychological perspective and we coded each response in one of the following categories: *claimed a proof* (code **P1**), *mixed claim* (**P2**), or *claimed not a proof* (**P3**) depending on what the PT said about whether the argument he/she provided qualified as a proof. Two researchers (one of the authors and a research assistant) coded independently all the responses across the mathematical and psychological perspectives. They compared their codes and reached consensus for all disagreements.

RESULTS AND DISCUSSION

Table 1 summarizes the distribution of PTs' responses to Task 1 (midterm exam) and Task 2 (final exam) across the codes we described earlier.

Mathematical / Psychological	M1	M2	M3	M4	M5
P1	9 (11)	5 (6)	3 (8)	4 (1)	1 (1)
P2	1 (0)	1 (0)	1 (1)	1 (0)	1 (0)
P3	0 (0)	1 (1)	5 (8)	4 (2)	2 (0)

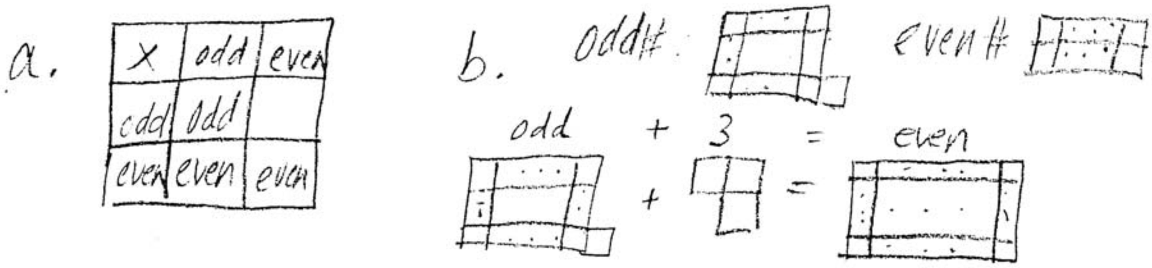
Table 1. Distribution of PTs' responses to Tasks 1 and 2 (in parentheses are the frequencies for Task 2)

If we considered only the mathematical categories for **Task 1**, we would say that 17 PTs produced proofs or valid general arguments that were not proofs (cf. M1 and M2). Of the remaining 22 PTs, almost half (nine) produced empirical arguments (cf. M4). Thus, we would likely conclude that a significant number of PTs seemed to believe that empirical arguments are proofs. Yet, if we also considered the psychological categories, we would reach a different (and more accurate) conclusion. About half (four) of the PTs who produced empirical arguments were aware that their arguments did not qualify as proofs (cf. M4P3). More broadly, half of the 22 PTs whose arguments were coded as M3, M4, or M5 showed awareness of the limitations of their arguments, recognizing that the arguments did not qualify as proofs. Only four PTs offered empirical arguments and expressed the belief that their arguments qualified as proofs (cf. M4P1).

PTs' understanding of proof became more refined by the end of the course when they completed Task 2 than when they completed Task 1. Specifically, in **Task 2** more PTs produced proofs and identified them as such (eleven in Task 2 vs. nine in Task 1), fewer PTs produced empirical arguments (three in Task 2 vs. nine in Task 1) with only one of them considering an empirical argument as a proof (vs. four in Task 1) and another one producing a non-genuine argument (vs. four in

Task 1). Next, we consider some responses given by PTs to Task 2 in order to illustrate the merit of asking PTs to evaluate their own arguments.

Sherrill's argument was coded as M3:



> 7x3=21+3=24 ✓ 11x3=33+3=36 ✓ 21x3=63+3=66 ✓
> The conjecture is true. We know from the table [see figure a above] that an odd number times an odd number will give you an odd number. By adding 3 then, the odd number becomes even and is a multiple of 6 [see figure b]. The examples I chose: 7, 11, 21 all prove true and the geometric figure illustrates every possibility.

Sherrill's argument seen in isolation from the rest of her response in Task 2 suggests limited understanding of proof. However, the combined consideration of this argument and the accurate evaluation that Sherrill gave for it (the evaluation was coded as P3) show good level of understanding:

> No, I do not think that I have produced a proof because the diagram makes it seem like all even numbers are multiples of 6, which is untrue. The geometric figure doesn't properly show how any odd number x 3 + 3 equals a multiple of six, rather it shows that any odd number x 3, + 3 equals an even number, but is not specific.

Amy and Joan both provided empirical arguments (cf. M4) for the conjecture in Task 2 by verifying its truth in five and seven cases, respectively. Yet, even though their arguments were almost identical, their understanding of proof was significantly different as reflected in their evaluations of their arguments:

> *Amy* (coded as P1): Yes, I believe I have produced a proof. I have not found an example that does not hold true to the conjecture.

> *Joan* (coded as P3): […] I have shown there is a pattern and connection between adding and multiplying 3 to odd # to get multiples of 6, but I have not proved it. I know all multiples of 6 are also multiples of 3, but I don't know how to explain or determine an all-purpose rule for all odd #s x 3 +3 = multiple of 6.

Contrary to Amy, Joan seemed to be aware of the limitations of her empirical argument and to understand that a proof needs to offer conclusive evidence for all cases involved in a claim. Also, Joan's self-evaluation illustrates the point that one reason for which PTs provided empirical arguments is that they were not able to construct better arguments, e.g., (valid) general arguments.

CONCLUDING REMARKS

Our dual focus on "construction-evaluation" activities drew attention to the important psychological phenomenon of students providing erroneous responses to mathematical tasks posed to them while being aware that their responses are incorrect. In order to offer a possible explanation for this phenomenon we use Brousseau's (1984) notion of *didactical contract*, which refers to the system of reciprocal obligations between an instructor and his/her students that are specific to the target knowledge and include issues such as the legitimacy of the tasks that the instructor poses to his/her students.

It is normal to expect that an instructor will select tasks for a summative evaluation that he/she believes the students are able to solve given the learning experiences they received. Then a blank paper from a student can indicate a breach in the didactical contract, because it can communicate messages like: the instructor did not choose an appropriate task or did not teach the material well enough for the student to be able to solve the task. Thus, if a student feels incompetent to solve correctly a task in an evaluation, the least the student can do to preserve the didactical contract is to write something in response to the task. The instructor is expected in turn to reward the student for the attempt to solve the task by giving him/her some credit, even for an incorrect response.

The phenomenon of students providing erroneous responses to mathematical tasks posed to them while being aware that their responses are incorrect has implications for instruction. For example, by encouraging students to evaluate the ideas that they propose for a task, instructors can remove the pressure from the students that, once the students offer an idea for a task, this means the students believe their idea is correct. Also, students' evaluations of their ideas can offer to instructors a deep insight into their students' understanding of a topic. For example, our study showed that an unsuccessful attempt to prove a claim does not imply that the solver has limited understanding of proof.

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REGENERATE THE PROVING EXPERIENCES: AN ATTEMPT FOR IMPROVEMENT ORIGINAL THEOREM PROOF CONSTRUCTIONS OF STUDENT TEACHERS BY USING SPIRAL VARIATION CURRICULUM

Xuhua Sun and Kahou Chan[1]

Education faculty, the University of Macao, China

This study is to explore the possible opportunities for student teachers to acquire the experience necessary to provide effective instruction about proof and proving. The 14 student teachers from the University of Macao were provided with spiral variation curriculum (one problem multiple solutions mainly stressed in this study) to prove "Mid-Point theorem of triangles". The results revealed that their own original theorem proof constructions were generally enhanced. 9 creative methods of proving this theorem were generated.

INTRODUCTION

Proof undoubtedly lies at the heart of mathematics. Proof has played a major role from the Euclidean geometry to formal mathematics. The learning of proof and proving in school mathematics would clearly depend on teachers' views about the essence of proofs, on what teachers do with their students in classrooms, on how teachers implement curricular tasks that have the potential to offer students opportunities to engage in their own original theorem proof constructions. However, Student teachers as pre-service teachers had learned most of fundamental mathematics theorems they will teach already. When they teach these theorems, they used to unconsciously recollect old proofs copied from their curriculum or past-experiences. The ready-made solutions of theorems proof previously established from past-experiences have impeded their exploration of own original proof constructions, which would enhance to shape their successful teaching practices into a proving-by-rote model in place of opportunity for development their students` original proof constructions. To teach original proof, one should first know what original proof is. The teachers' deficiency in understanding how to construct an original proof determined their inability to teach original proof construction. Even their pedagogical knowledge could not make up for their ignorance experiences of own original proof. Questions arise what we need to do necessarily to help student teachers come to enculturation into the practices of mathematicians and improvement own original proof constructions, which might support their successful teaching practices for their

[1]Thank student teachers below from the Math Interest Group ,education faculty, the university of macao, kahou Chan ;Yongxian Liang ;Zhenyu Zeng; shaobing Huang ;Jiachan Huang ;Mingfeng Liang ;Guohao Feng; Jinqing Xie; Jianzhu liang; Bowei Tan; Guohong Lin, for joining the course and providing creative methods.

students.

In this study, the student teachers were asked to prove “Mid-Point theorem of triangles” with their own original solutions, the notion of one problem multiple solutions in spiral variation curriculum (more details see Sun, 2007). The aim of design tended to change the situation that student teachers used to recollect old proof copied from their curriculum or past-experiences by regenerating their own proving experiences.

METHODOLOGY

About 14 Students from the University of Macao were asked to prove “Mid-Point Theorem” with their own methods for 2 hours and then wrote down their own methods of proving on the blackboard one by one. Each a method was named after their first names. Their drafts and the whole process videotaped were collected for further analysis. All methods in the blackboard were taken photos by a camera.

RESULTS

The results revealed the student teachers had ability for their own original theorem proof constructions. More than 10 creative methods[2] of proving this theorem were generated. The results also revealed that their own original theorem proof constructions were generally enhanced by this simple requirement of multiple original solutions, which is one of the notions in spiral variation curriculum.

Method I (清)

Because AD=BD

AE=CE

$\because \frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{2}$

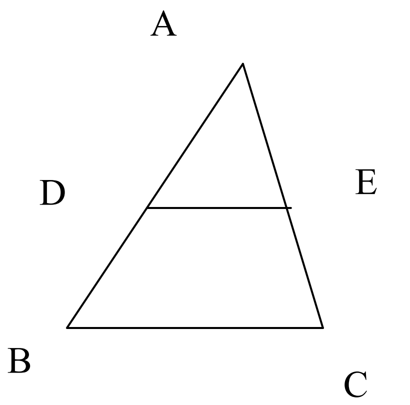
$\angle A = \angle A$

$\therefore \Delta ADE \sim \Delta ABC$

$\therefore \frac{DE}{BC} = \frac{1}{2}$

$\angle ADE = \angle ABC$

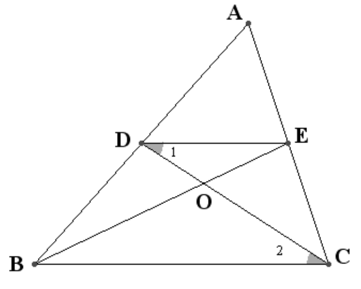
$\therefore DE // BC$



Method II (賢)

Join CD; the intersection O of the medians is the centroid of the triangle ABC, by the property of centroid, we have

$DO : OC = EO : OB = 1 : 2$



[2] The other 2 methods were deleted because there is no room for them in this paper.

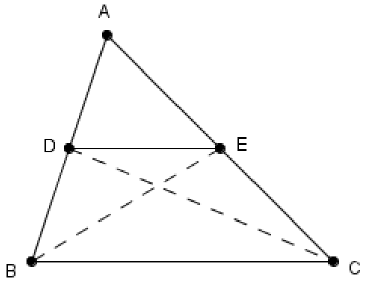
In ΔDOE and ΔCOB,

$$\begin{cases} DO:OC = EO:OB = 1:2 \\ \angle DOE = \angle COB \end{cases}$$

We have $\Delta DOE \sim \Delta COB$, then

$$\begin{cases} DE = \frac{1}{2}BC \\ \angle 1 = \angle 2 \Rightarrow DE // BC \end{cases}$$

Method III (宇)



Connect BE and CD.

∵ CD is the median of ΔABC，

∴ $S_{\Delta BCD} = \frac{1}{2}S_{\Delta ABC}$ 。

Similarly, $S_{\Delta BCE} = \frac{1}{2}S_{\Delta ABC}$ ，then $S_{\Delta BCD} = S_{\Delta BCE}$ 。

∵ ΔBCD and ΔBCE share the same base，

∴ DE // BC。

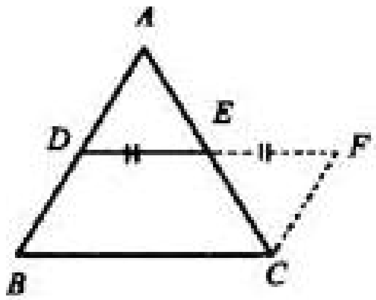
On the other hand，ED is the median of ΔABE，

Then $S_{\Delta BDE} = \frac{1}{2}S_{\Delta ABE} = \frac{1}{2}S_{\Delta BCE}$ 。

∵ DE // BC，∴ ΔBDE and ΔBCE have the same height，Thus, DE =1/2 BC。

Method IV (嬋)

Extend DE to F such that DE=EF，join CF



∵ ∠AED=∠CEF
DE=EF
AE=EC

∴ΔADE≅ΔCFE

∵∠DAE=∠FCE

∴AD//FC

And AD=FC

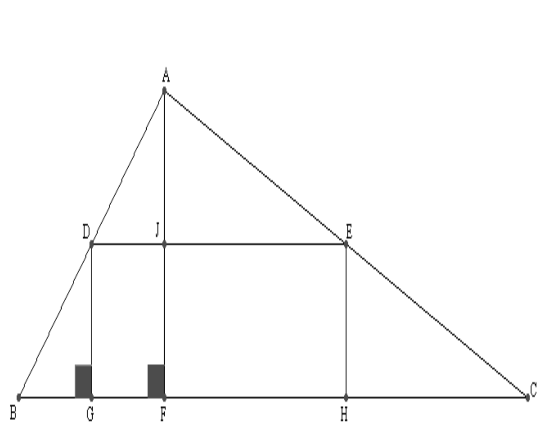
∵AD=DB, AD//DB

∴BD//FC and BD=FC

∵DE//DF, DE=1/2 DF

∴DE//BC and DE=1/2BC

Method V (峰)



Draw AF⊥BC，DG⊥BC，EH⊥BC。

∠BGD=∠BFA

DG//AF

∠BDG=∠BAF，∠B=∠B

BGD~△BFA

DG/AF=1/2，similarly，EH/AF=1/2

DG=EH，DG//EH

DGHE is a parallelogram

DE//GH，DE=GH，DE//BC，∠ADJ=∠DBG

ADJ≅△DBG (A.S.A.)

DJ=BG，similarly，JE=HC，

∵DE=GH

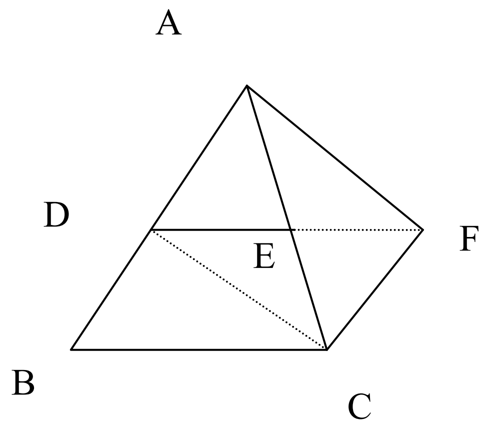
∴BG+HC+DE=BG+HC+GH

DJ+JE+DE=BC

2DE=BC

DE =1/2 BC ■

Method VI (柱)



Proof: Extend DE to F, such that EF=DE，

Join AF，FC，CD，

∵AE=CE，DE=EF，

∴ADCF is a parallelogram.

We have FC//AD, FC=AD，

∵D is the mid point of AB，

∴BD=AD=FC, BD// FC，we have AD//FC

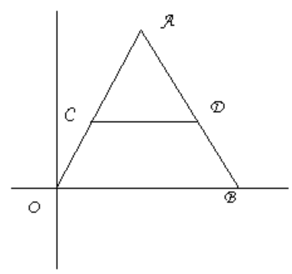
∴BCFD is a parallelogram，

We have DF//BC, DF=BC

∵ DE =1/2 DF，

∴DE =1/2 BC, DF//BC

Method VII (威)



Let the coordinates of O be (0, 0), A (x, y), B (b, 0); and the mid-points C, D of OA and AB are $\left(\frac{x}{2},\frac{y}{2}\right)$ and $\left(\frac{x+b}{2},\frac{y}{2}\right)$ respectively.

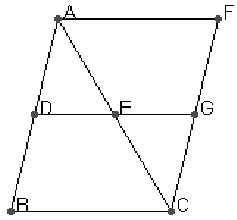
Then $OB=\sqrt{(b-0)^2+(0-0)^2}=b$ and

$$CD=\sqrt{\left(\frac{x+b}{2}-\frac{x}{2}\right)^2+\left(\frac{y}{2}-\frac{y}{2}\right)^2}=\frac{b}{2}$$

Hence, we have CD=1/2 OB

Also, as C and D have the same y-coordinate, we have CD//OB.

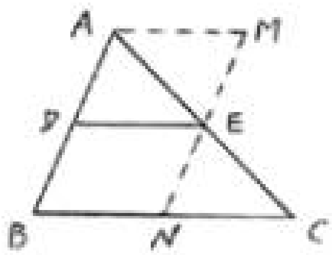
Method VIII (斌)



Construct CF and AF from C and A respectively, such that CF//AB and AF//BC.
Let G be the mid-point of CF
As quadrilateral ABCF is a parallelogram, we have AB=FC, AD=BD, AD=FG.
Then we have quadrilateral ADGF is a parallelogram.
Similarly, we have quadrilateral BDGC is a parallelogram.
Thus, AF//BC//DG and DG=BC=AF.
As ∠AFD=∠CFG, AF=FC, AD=CG
We have △ADF≅△CGF ⇒ DF=FG ⇒ DF=1/2 DG =1/2BC

Method IX (浩)

Draw MN//AB and AM//BC such that line MN passes through point E
Then we have AMNB is a parallelogram.



∵AM//BN
∴AM//BC and ∠AME=∠CNE
∵∠AEM=∠CEN and AE=EC
∴ΔAME ≅△CNE
We have AM=NC, ME=EN, AB=MN and
∵DENB and AMED are parallelograms,
∴DE//AM//BC and AM=DE=BN=NC ⇒ ED=1/2 BC

IMPLICATION

The results revealed the student teachers had integrated their mathematics knowledge and regenerated 9 simple and creative methods, different from the curriculum material provided. The results also indicated that their own original theorem proof constructions were generally enhanced by this simple requirement of multiple original solutions, which is one of the notions in spiral variation curriculum (Sun，2007).

In fact, spiral variation curriculum is not a new course but a curriculum model stressing variations (Bianshi[3]), which are identified as an important element of learning / teaching mathematics in China by some researchers, educators, and teachers in recent years (Gu, Huang & Marton, 2004; Marton & Booth, 1997; Huang, 2002; Nie, 2003; Sun, Wong & Lam 2005, Sun, 2006; Wong, Lam, & Sun, 2006; Sun，2007; Wong, 2007). The curriculum model specially filtered and rationalized problem variations with multiple conceptions connection or multiple

[3] "*Bianshi*" is written as "變式" in Chinese, with "*Bian*" meaning "changing" and "*shi*" meaning "form". "*Bianshi*" can be translated liberally as "variation" in English in most studies.

solutions connection[4] from Chinese own teaching experience from its own mathematics curriculum practice , was tried out in 21 classes at the primary schools in Hong Kong, The effect of curriculum are significant (Sun, 2007; Wong, 2007). In this study we noted that problem variations with multiple solutions connection in spiral variation curriculum (one problem multiple solutions mainly stressed in this study) just successfully helped students reconstruct their own solution system by regenerating own proving experience.

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4 The former one is called One problem multiple changes i.e.“yiti duobian ” , “ 一题多变” in Chinese, varying conditions, conclusions. The latter variation within problem is called as one problem multiple solutions i.e. “yiti duojie” “一题多解” in Chinese , varying solutions.

RENEW THE PROVING EXPERIENCES: AN EXPERIMENT FOR ENHANCEMENT TRAPEZOID AREA FORMULA PROOF CONSTRUCTIONS OF STUDENT TEACHERS BY "ONE PROBLEM MUTIPLE SOLUTIONS"

Xuhua Sun

Education faculty, University of Macao, China

This study is to explore the possible opportunities for student teachers to acquire the experience necessary to provide effective instruction about proof and proving. The 14 student teachers from the University of Macao were required to prove "area formula of trapezoid" with "one problem multiple solutions". The results revealed that their own original proof constructions were generally enhanced. Twelve creative methods of proving this formula were generated. The notion of one problem multiple solutions should be uplifted into a curriculum design framework guiding our teaching practice was discussed.

INTRODUCTION

Proof undoubtedly lies at the heart of mathematics. Proof has played a major role in mathematics. The learning of proof and proving in school mathematics would clearly depend on teachers' views about the essence of proofs, on what teachers do with their students in classrooms that have the potential to offer students opportunities to engage in their own original proof constructions. However, Student teachers as pre-service teachers had learned most of fundamental mathematics formulas and theorems they will teach latter. When they teach these formulas and theorems, they used to unconsciously recollect old proofs copied from their textbooks or past-experiences. The ready-made solutions of formulas and theorems proof have impeded their exploration of own original proof constructions, which would shape their proof teaching practices into superficial and imitate proving in place of opportunity for development their students` original proof constructions. In other hand, there is a pressure on textbooks to be self-contained (so students do not have to ask the teacher many questions) by providing guiding questions, which have also impeded their exploration of own original proof (Lithner, 2003). However, to teach original proof, one should first know what original proof is. The teachers' deficiency in understanding how to construct own original proof determined their inability to teach original proof construction and would have no real practice in teaching original proving . Even their pedagogical knowledge could not make up for their ignorance experiences of own original proof.

"Area formula of trapezoid" is a basic formula to calculate the area of trapezoid. Most of curriculum materials heavily focus on memorizing by rote and mechanically applying the formula, rather than own original proof in most counties. It should eventually have impeded the building proper conceptions of math learning in the long run.

It is impressive to note that the method in the USA textbook(Bolster, Boyer, Butts, & Cavanagh , 1996, page 350) presented one justifying methods alone by illustration (The two same trapezoids are reorganized into a parallelogram) However, Chinese textbook (Mathematics textbook developer group for elementary schools, 2003, p.88) presented three justifying methods by illustration (The trapezoid is separated into two triangles; the trapezoid is separated into a triangles and a parallelogram; the two same trapezoids are reorganized into a parallelogram.) It seems that Chinese textbook was better at using “one problem with multiple solutions” than the US counterpart in this case. The prior study (Sun, 2007) found that “one problem multiple solutions” is widespread, and well known, in China but still far from uplift into a curriculum design framework guiding our teaching practice. “One problem multiple solutions” could be regarded as an effective tool to guide students to explore own methods. We wonder whether student teachers may improve own proving of trapezoid formula proof constructions by “one problem multiple solutions”. In this study, the student teachers were asked to prove “area formula of trapezoid” with one problem multiple solutions. The aim of design tended to change their habit to recollect old proof copied from their curriculum or past-experiences by regenerating their own proving experiences.

Methodology

About 14 Students from University of Macao were asked to prove “area formula of trapezoid” with their own methods for 2 hours and then wrote down their own methods of proving on the blackboard one by one. Each a method was named after their first names. Their drafts and the whole process videotaped were collected for further analysis. All methods in the blackboard were taken photos by a camera.

RESULTS

The results revealed the student teachers had ability for their own original theorem proof constructions. The 12 creative methods [1]below of proving this formula were regenerated.

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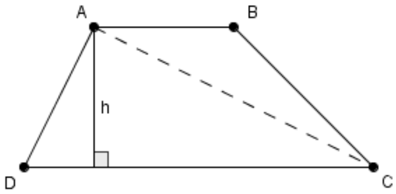
[1] I present 7 methods here due to no room for them.

1. Method of Can

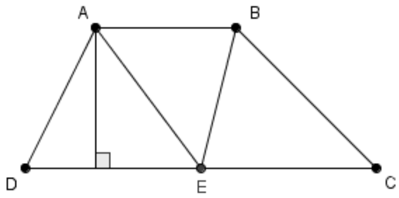


Connect AC. The triangle ΔABC and ΔACD have the same height h，

So
$$S_{ABCD} = S_{\Delta ABC} + S_{\Delta ACD} = \frac{ah}{2} + \frac{bh}{2} = \frac{(a+b)h}{2}$$

COMMENT：This is a simplest proving method among all methods presented by the textbooks of different countries.

2. Method of Bin



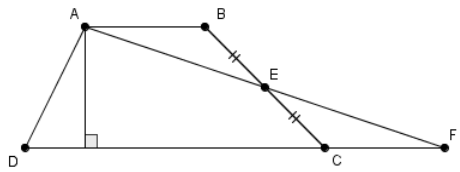
E is the mid point of CD. Connect AE and BE. So,

$$S_{ABCD} = S_{\Delta ADE} + S_{\Delta ABE} + S_{\Delta BCE}$$
$$= \frac{1}{2}\cdot\frac{b}{2}\cdot h + \frac{ah}{2} + \frac{1}{2}\cdot\frac{b}{2}\cdot h = \frac{(a+b)h}{2}$$

COMMENT：The trapezoid is divided into 3 triangles. The key point of the method is finding of midpoint, which make proving simple .Of course, any a point on the line DC is an available too.

3. Method of Zhu

E is midpoint of BC. Connect AE. F is the intersection of extended line DC and extended line AE.



$$\begin{cases} \angle ABE = \angle FCE \\ BE = CE \\ \angle BEA = \angle CEF \end{cases} \Rightarrow \Delta ABE \cong \Delta FCE \Rightarrow \begin{cases} AB = FC \\ S_{\Delta ABE} = S_{\Delta FCE} \end{cases}$$

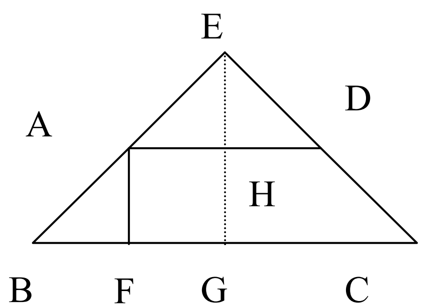
Then

$$S_{ABCD} = S_{\Delta ADF} = \frac{(a+b)h}{2}$$

COMMENT：The trapezoid is skillfully transformed into a triangle with same area by replacing ΔABE $by \Delta FCE$. It is a creative proving.

4. Method of Chan

Extend BA and DC. E is intersection of BA and CD. Draw height EG and height AF. G is the intersection of EG and BC. F is the intersection of AF and BC. Because $\mathrm{AD} // BC$，the triangle EAD is similar to the triangle EBC，



$$\frac{\mathrm{EH}}{\mathrm{EG}} = \frac{\mathrm{EH}}{\mathrm{h}+\mathrm{EH}} = \frac{\mathrm{AD}}{\mathrm{BC}} = \frac{\mathrm{a}}{\mathrm{b}}$$

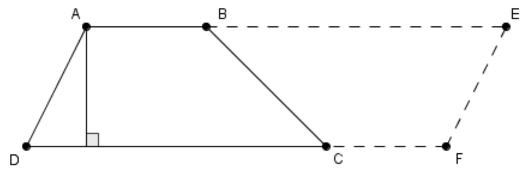
Then EH$=\dfrac{ah}{b-a}$

$$S_{ABCD} = S_{\Delta EBC} - S_{\Delta EAD}$$

$$= \frac{a(h+\mathrm{EH})}{2} - \frac{b\mathrm{EH}}{2} = \frac{(a+b)h}{2}$$

COMMENT：The trapezoid is extended into a triangle by extending its two sides. The EH was eliminated according to the property of the similar triangle.

5. Method of Xian



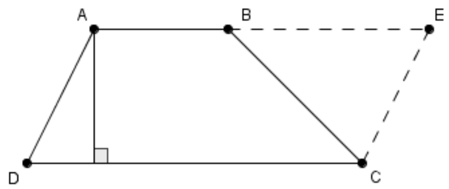
Extend AB to E，so as to $BE = CD$. Extend DC to F，

so as to $CF = AB$. Then $AE = FD$ and $AE // FD$. So $AEFD$ is a parallelogram.

$$S_{ABCD} = \frac{1}{2} S_{AEFD} = \frac{(a+b)h}{2}$$

COMMENT：The trapezoid is reorganized into a parallelogram by copying the same trapezoid.

6. Method of Feng



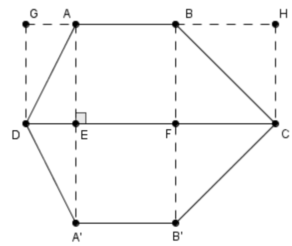
Draw $CE // DA$ such that line CE passes through point E. Then we have AECD is a parallelogram.

$$S_{ABCD} = S_{AECD} - S_{\Delta BEC}$$

$$= bh - \frac{(b-a)h}{2} = \frac{(a+b)h}{2}$$

COMMENT：The trapezoid is reorganized into a parallelogram by making a parallel line.

7. Method of Yu



Draw the symmetry points $A`$ and B' of A and B

based on symmetry axis DC

Then

$$S_{ABCD} = \frac{1}{2} S_{ABCB'A'D}$$

$$= \frac{1}{2}\left(S_{CDGH} + S_{A'B'FE}\right) = \frac{(a+b)h}{2}$$

COMMENT：The trapezoid is reorganized into 2 rectangles by making a symmetry figure.

Here are some photos about students` proving solutions presented at the classroom. (See figure 1).

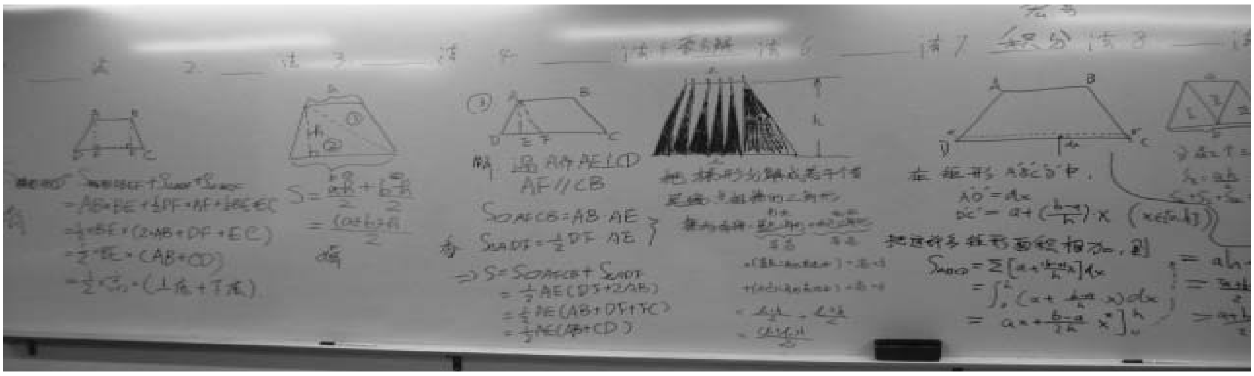


Figure 1 THE PHOTO OF STUDENTS` SOLUTIONS

IMPLICATION

Why did we stress "one problem multiple solutions"?

"one problem multiple solutions" is one of frameworks in spiral variation curriculum, specially filtered and rationalized problem variations with multiple conceptions connection or multiple solutions connection[2] from Chinese own teaching experience from its own mathematics curriculum practice, was tried out in 21 classes at the primary schools in Hong Kong, The effect of curriculum are significant (Sun, 2007; Wong, 2007).It could be traced to prior variations study, which are identified as an important element of learning / teaching mathematics in China by some researchers, educators, and teachers in recent years (Gu, Huang & Marton, 2004; Sun，2007; Wong, 2007). The study indicated that "one problem multiple solutions" (called problem variations with multiple solutions connection in spiral variation curriculum) just successfully helped student teachers renew their proof experience and further reconstruct their own solution system to some extent, which tends to display some advantages[3] in changing the habit of superficial and imitate proving , lead to long term gains, like improving interest, self-efficacy and independent analysis in the central roles of mathematics learning. The results make us realize that "one problem multiple solutions" is not fully apprehended by students, teachers, textbook writers, and perhaps also among many researchers. One reason may be that we lack exploration deep its effectiveness to extend students` method system and gain the original insights in more specific and real ways due to too familiarness.

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2 The former one is called One problem multiple variation , " 一题多变" in Chinese, varying conditions, conclusions. The latter variation is called as one problem multiple solutions i.e. "yiti duojie" "一题多解" in Chinese , varying solutions.

3 We also did other experiments of more than ten theorems and formulae by one problem multiple solutions. The whole effect is inspiring and significant.

“One problem multiple solutions” is a simple and powerful framework for guiding teaching and learning. In fact the notion of one problem multiple solutions is widespread, and well known, just like air we breathe we seldom are aware of its existence. But the notion of one problem multiple solutions is still far from uplift into a curriculum design framework guiding our teaching practice. Just to refer that practice with one problem multiple solutions will not help much, if we cannot specify a curriculum design frame and further probe its effectiveness. This case show it can take us far, and indeed it used to do.

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DYNAMIC GEOMETRY AND PROOF: THE CASES OF MECHANICS AND NON-EUCLIDEAN SPACE

Ian Stevenson

Department of Education and Professional Studies, King's College, London.

Euclidean dynamic geometry applications are extrinsically dynamic: motion is provided by the user. For intrinsically dynamic applications, motion is provided by the software. This paper examines the relationship between proof and the use of intrinsically dynamic applications for Newtonian mechanics and non-euclidean turtle geometry. Two short cases studies of learners developing their understanding of geometric ideas with each of these applications are used to discuss the nature of proof in the context of intrinsically dynamic geometry.

WHAT MAKES GEOMETRY "DYNAMIC"?

"Dynamic Geometry" (DGS) has come to refer to a specific type of digital application that models euclidean geometry, which enable learners to create, construct, and "drag" geometric objects on screen. Links between dragging and cognition (Arzarello et al., 2002), and DGS role in motivating proof (Laborde, 2000), have focused on learner's dynamism with geometric objects, rather than the dynamism of the objects themselves. DGS may be described as *extrinsically* dynamic in the sense that the source of an object's motion is the action of the learner with the mouse. By contrast, turtle geometry (Papert, 1980), for example, can be called *intrinsically* dynamic since motion, both linear and angular, is an integral part of defining a turtle's state: turtles can either move forwards or backwards and can turn either left or right. Papert (*ibid*.) describes this as syntonic or body geometry, and it may be thought of as non-euclidean or even pre-euclidean, if one adopts a Piagetian approach. What matters is that different sources of motion are linked with different kinds of geometry. This paper examines the relationship between intrinsically dynamic geometries and the notion of proof by drawing on two case studies of learners working with a simulation application for Newtonian mechanics (*Interactive Physics; www.fable.co.uk*) and a non-euclidean turtle geometry microworld (Stevenson, 2006). These two applications are chosen because they are both intrinsically dynamic, and deal with different types of geometry to DGS. The focus is on the ways in which learners understand aspects of the geometries, and how that relates to ideas about proof. The paper concludes with a discussion of the implications for proof of using digital technologies in mathematics.

CASE STUDY 1: DYNAMICS VIEWED AS GEOMETRY

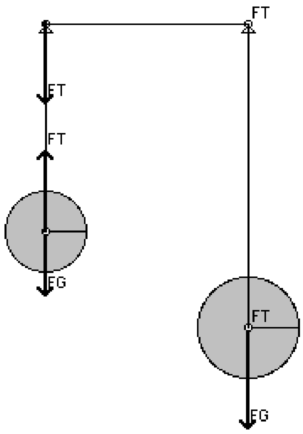
From its inception by Newton, mechanics has been closely linked with geometry both as a problem-solving tool and as a framework for describing motion. The link is explored in this case study through the work of Richard and Len, two pre-university students studying mechanics as part of their mathematics course (Stevenson, 2002). At the time of this case study they have some knowledge of

Newtonian mechanics but have not used digital technology as part of their studies, or indeed any other part of their mathematics course. Their work, shown in Figure 1, is taken from their activities in a workshop which introduced them to the software (*Interactive Physics*). The participants of the workshop were then asked to work in pairs to analyse a system of connected particles. Richard and Len were chosen at random from the twenty students in the workshop, and with their

1 (a) Screen Image	Acceleration remains constant to 1 d.p. This is proved by the formula below: (Resultant Mass / Total Mass) * 9.81” where Resultant Mass is “larger mass – smaller mass” and “Total mass” is sum of the masses 1 (b) Richard and Len’s description of their result

permission, their activities were videoed and transcribed.

Figure 1(a) Interactive Physics screen image used by the pair. 1(b) An extract from their final presentation



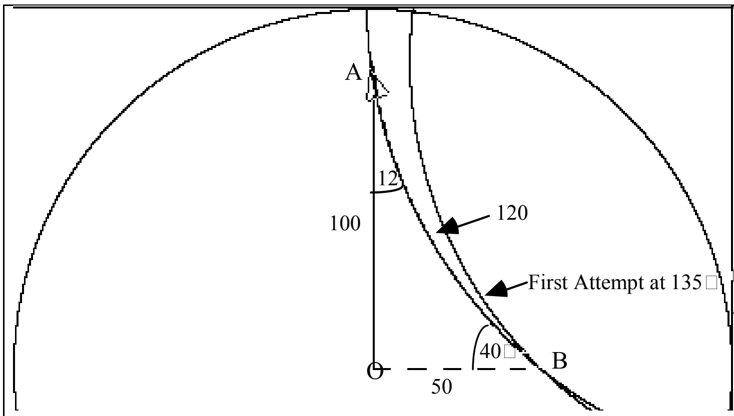
e screen Richard and Len produced for the connected r system was drawn directly on the screen using a palette of spherical/rectangular/polygonal masses, springs, inelastic vots, and pulleys. A number of measuring devices and also used. In Figure 1(a) the magnitude of each mass is a, and FG and FT refer to *Force Gravitational* on the masses, he string, respectively. These force vectors were overlaid by ts were added and connected together. Richard and Len set of the connected system and the application then computed its development in real time using the laws of motion and gravitation. Figure 1(b) is an extract from the presentation that they made at the end of the workshop to describe their findings. What is striking about Richard and Len’s approach is its difference to the type of solution usually associated with “paper-and-pencil” method. They treated the connected particles as a single system rather than splitting it into two separate systems and eliminating tension. Geometrically their result implies that they have thought about the two particles moving in different directions as one particle moving in a single direction. However, their result is neither an induction nor a deduction, although it is based on their experiments, and may best be called a “creative abduction” (Eco, 1983). It could be argued that a proof would be needed to link the results of the learners’ activities with the

Newtonian model, to demonstrate the necessity of their conclusion independently of the means by which it was generated. On the one hand this illustrates the role of digital technology as enhancing the process of generating conjectures, which then need to be established "properly" using another medium: paper and pencil. On the other hand it raises the question of whether the digital technology provides a new kind of knowledge (as Richard and Len seemed to do), and with it the need for different kinds of justification. The result has come from the actions of the learners using a digital model of Newtonian mechanics, which, in turn, is a model of the "real" world. By analogy, their result is connected to a digital model just as a paper and pencil proof must be connected to the Newtonian model. Underlying this is what might be described as a "wittgensteinian" idea of proof in which meaning is inherent in the practices of a discourse, and justification lies in "what people do" rather than appealing to a domain that lies beyond their activities. Such relativism may run contrary to the deeply attractive notion of proof as providing justified and certain knowledge, but it does highlight the relationship between knowledge and the role(s) that activities and media play in its production. Put simply, different media, motive, and *modus operandi* give different knowledge.

CASE STUDY 2: DYNAMICS PRODUCES GEOMETRY

A second sense of dynamic geometry can be found in turtle geometry, and the non-euclidean spaces that it can be used to model. Two-dimensional hyperbolic and spherical geometries can be represented in euclidean space using projections that preserve angles but not distances. Non-euclidean turtle geometry (Stevenson, 2006) provides the learner with three types of turtle corresponding to each geometry-hyperbolic, spherical and euclidean. Each turtle picks out the straight lines of a given geometry as they are projected onto the two-dimensional flat screen. In spherical geometry, the turtle "steps" get longer and it speeds up as it moves towards the edge of the screen, while for hyperbolic geometry, the steps shorten and the turtle slows down as it approaches the screen edge. Learners are also provided with physical hyperbolic and spherical surfaces to look at and handle. At first, the peculiar behaviour of the turtles on screen presents a set of perceptual and epistemological challenges for learners, but the combination of intrinsic dynamism of the turtle coupled with physical surfaces provides learners with resource to support their thinking.

To illustrate this is a short extract taken from Stevenson and Noss (1999). Sean (S) and Paul (P) are adult volunteers, training to be teachers. They had been shown how the screen images were obtained by projection, and were given a task that aimed at drawing their attention to the angle sum of a triangle in hyperbolic geometry. Starting with the horizontal OB and vertical lines OA, both in Figure 2, they had to construct a third side to make a triangle. They decided initially to choose *left* 135º at point B in Figure 2, and then used *Path* (a procedure that shows the trajectory of the turtle if it continues on it current heading) to close the triangle. This did not work so they turn the turtle left by another 5˚ and use *Path* again. This closes the triangle and they use the turtle to find the angle of 12° at A.



Sean and Paul's annotated screen shot. The semi-circle shows the "edge" of hyperbolic space at infinity. As the turtle moves towards the boundary it slows down. The arcs at B are the result of using the *Path* procedure which shows the trajectory of the turtle if it continues on it current heading

S: We haven't got 180, but it's walking a straight-line path.

P: Yeah, you've probably got to turn.

S: No, you don't have to turn. It's actually drawing a triangle on the surface.

S: The projection defies Pythagoras. No! Hang on, walking on the surface is defying it, isn't it! because we walk straight lines on the surface we just see them as curves on the projection.

Figure 2. Sean and Paul's annotated screen and their subsequent dialogue

As the extract on the right hand side of Figure 2 shows, Sean connects the movement of the Turtle with an imagined path on a hyperboloid surface. On the one hand it provides him with confidence that the angle sum is less than 180, and, on the other, confirms that it is a genuine aspect of the geometry. Central to his conclusion is the metaphor that Turtles are walking "straight paths" over a hyperbolic surface which is being projected onto the screen. Dynamism in this case comes from turtles, controlled partly by the learners' programming and partly by the turtle's movement along a path calculated in real-time.

Encapsulated in the short extract is one learner's realization of how the various elements of software and physical surface work together, and they provide him with a degree of confidence about his conclusions. However, it raises the question about the epistemological status of digital technology for the learners. Clearly this is not a chance observation, and one may speculate that Sean had a belief that he and Paul were in a rule-governed context: it was their job to "uncover" the rules. This belief may have provided a degree of "fore-sight" that shaped their subsequent interpretation. From this point of view, the learners have an expectation that their investigations do have a rationality, which they do not for the moment understand, but they will. One may tentatively conjecture that a dynamic interplay between model exploration and personal expressiveness with the software can be seen developing as the learners build up a set of connections between screen images, physical surfaces, and hyperbolic geometry (in this case). If this is case, does Sean's conclusion require a further step of justification so that it may be shared and checked by others? If so, what might this entail? Is the fact that both the digital technology and physical artefacts are "theoretical" objects in the sense of being constructed for a specific purpose in a specific pedagogical context, sufficient justification for the conclusions reached? Put another way, this is not a chance empirical observation that needs justifying, but Sean's entry into a specific set of mathematical practices and knowledge outcomes. Is a proof, therefore, an invitation to enter that digital context and reproduce Sean's actions?

WHAT MAKES A PROOF IN A DYNAMIC GEOMETRY?

What is a proof? Balacheff (2008) identifies a range of different interpretations to answer this question, which he links with differences in mathematical epistemologies, and, it could be argued, ontologies. He looks for a common terminology which straddles these interpretation, to provide a way of taking the field forward. Coupled with this is the question of digital technologies' role(s) in the process: do the technologies enhance a process that is understood or do they mediate a new kind of knowledge? An insight into this can be found in the distinction often drawn between a "drawing" on the computer screen which learners can manipulate and "figure": a mental entity. Screen images are considered to be external representations of a mental concept (Marriotti, 1997). From this point of view, dynamic geometry applications are seen as transitional objects (Papert, 1980) between physical and mental domains. It accords with a "platonic" notion of proof that involves the traversal of a hierarchy from (possibly infinite) physical drawings to the figural concept. Digital technology, in this view, can be considered as enhancing the transition process that lies at the heart of the notion of proof which it implies. In the context of pedagogy, however, dynamic geometry applications may also be considered as a resource for teaching and learning, but one which must be mediated by teachers to connect with the formal curriculum (Balacheff and Sutherland, 1999). Part of the need for this process of mediation lies in the difference between knowledge that DGS make available and what is required by paper and pencil approaches. The central point is that the multiple roles which DGS (and digital technology more generally) play vary according to the purposes to which they are put. Differences in the notion of proving, coupled with the alternative interpretations of what DGS are for, can interact in unpredictable ways with the process of learning how to understand and constructing proofs using paper and pencil technology. As the two cases illustrate, different notions of dynamism and its relationship to geometry can provide a context in which learners come to what seem like reasonable and rationale results using digital media, but which may require a considerable amount of "translation" to express them in a paper-and-pencil format. Whether the "abductive leaps" in learners' thinking, shown in the two cases, would be possible in a static media related to the different types of geometry illustrated, is an open question. What they do suggest is that use of DGS in learning to "prove" with euclidean geometry is only one of several possibilities, both in terms of the geometry available, the notion of dynamism used, and what it means to prove with digital technology. What is needed is a framework that can take account of the differences in the notion of proof identified by Balacheff, coupled with an analysis of the role(s) of digital technologies and the context in which they are used. Current work with Activity Theory looks promising from this point of view, although space does not permit a detailed discussion (Stevenson, 2008).

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PROFESSIONAL COMPETENCE OF FUTURE MATHEMATICS TEACHERS ON ARGUMENTATION AND PROOF AND HOW TO EVALUATE IT

Björn Schwarz, Gabriele Kaiser

University of Hamburg, Germany

The paper describes ways how to evaluate the complex competence of argumentation and proof and their accompanying didactical knowledge in a qualitative comparative case study. Based on a distinct theoretical frame evaluation instruments combining questionnaires and interviews with open items are developed. Methods to evaluate these data are described using the approach of qualitative content analysis. The case studies in Germany, Hong Kong and Australia yield interesting results about the competencies future mathematics teachers' have in the areas of mathematical knowledge and pedagogical content knowledge concerning argumentation and proof.

BACKGROUND AND AIM OF THE STUDY

Teacher education has already been criticised for a long time, but systematic knowledge about how teachers perform at the end of their education is almost non-existent (for an overview on the debate see Blömeke et al., 2008 and Adler et al., 2005). Even in the field that is covered by most of the existing studies – the education of mathematics teachers – research deficits have to be stated. Only recently more empirical studies on mathematics teacher education have been developed.

In an attempt to fill existing research gaps, the knowledge and beliefs of future lower secondary teachers are investigated in the study "*Mathematics Teaching in the 21st Century (MT21)",* which aims to shed light on the question how lower secondary mathematics school teachers were prepared to teach in six countries (Schmidt et al., 2007),. The IEA is presently carrying out an international comparative study on the professional knowledge of future teachers, the study "Teacher Education and Development Study – Learning to Teach Mathematics (TEDS-M)". In both studies argumentation and proof is not examined deeply, which was the reason to establish a collaborative study between researchers at universities in Germany, Hong Kong, and Australia. This study uses the theoretical framework and theoretical conceptualisation from *MT21*, but carries out qualitatively oriented detailed in-depth studies on selected topics of the professional knowledge of future teachers, namely modelling, argumentation and proof, the latter being the theme of this paper. The case study is focusing on future teachers and their first phase of teacher education.

The planning of the ICMI study focusing on proof reflects a worldwide renewal of interest in proof. In the ICMI study discussion document the question is asked: "How can we design opportunities for student teachers to acquire the knowledge (skills, understandings and dispositions) necessary to provide effective instruction about proof and proving?" Thus empirical studies are necessary to determine the competence and the nature of the knowledge (mathematical, pedagogical or pedagogical content) future teachers possess with regard to proof. In the following we will describe such an empirical study focusing mainly on methodological reflections.

THEORETICAL FRAMEWORK OF THE STUDY

The initial ideas of *MT21* are considerations about the central aspects of teachers' professional competencies as basically defined by Shulman (1986) and differentiated further by Bromme (1994). The following three knowledge domains are distinguished: Mathematical content knowledge, Pedagogical content knowledge in mathematics, General pedagogical knowledge; additionally beliefs concerning mathematics and teaching mathematics are considered.

Concerning the area of argumentation and proof we refer to specific European traditions, in which various kinds of reasoning and proofs are distinguished, especially "pre-formal proofs" and "formal proofs". These notions were elaborated by Blum and Kirsch (1991): pre-formal proof means "a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises" (Blum & Kirsch, 1991, p. 187). In the discussion document for this ICMI study similar distinctions are made.

Concerning the role of proof in mathematics teaching, Holland (1996) details the plea of Blum and Kirsch (1991) for pre-formal proofs besides formal proofs as follows: For him pre-formal proofs may be sufficient in mathematics lessons with cognitively weaker students, in other classes both kinds of proofs should be conducted. Pre-formal proofs have many advantages due to their illustrative style. In addition, pre-formal proofs contribute substantially to a deeper understanding of the discussed theorems and they place emphasis on the application-oriented, experimental and pictorial aspects of mathematics. However, their disadvantage is their incompleteness, their reference to visualisations, which require formal proofs in order to convey an appropriate image of mathematics as science to the students. The scientific advantage of formal proofs, namely their completeness, is often accompanied by a certain complexity, which may cause barriers for the students' understanding and might be time-consuming. However, there is no doubt, that treating proofs in mathematics lessons is meaningful with the aim of developing general abilities, such as heuristic abilities. The teaching of these two different kinds of proofs leads to high demands on teachers and future teachers. Teachers must possess mathematical content knowledge at a higher level of school mathematics and university level knowledge on mathematics on proof. This comprises the ability to identify different proof structures (pre-formal – formal), the ability to execute proofs on different levels and to know alternative specific mathematical proofs. Additionally, teachers should be able to recognise or to establish connections between different topic areas. To sum up: Teachers should have adequate knowledge of the above-described didactical considerations on proving as well (for details see Holland, 1996, pp. 51-58). It can be expected that in addition to being able to construct proofs, teachers will need to draw on their mathematical knowledge about the different structures of proving such as special cases or experimental 'proofs', pre-formal proofs, and formal proofs and pedagogical content knowledge when planning teaching experiences and when judging the adequacy or correctness of their, and their students' proofs in various mathematical content domains.

Based on the theoretical distinctions concerning professional knowledge of future teachers and based on the theoretical debate on proof our study was aims to answer the following questions:

- Which mathematical content knowledge and pedagogical content knowledge do future teachers acquire during their university study?
- Which connections between these two domains of knowledge can be reconstructed within these future teachers?

In order to decrease the level of complexity we consider general pedagogical knowledge and beliefs only marginally.

METHODOLOGICAL APPROACH

Based on the methodological approach of triangulation questionnaires with open questions and in-depth thematically oriented interviews were developed. The instruments are restricted to the areas of modelling, argumentation and proof, we will focus in the following on argumentation and proof. The questionnaire consists of several items that are domain-overlapping designed – as so-called 'Bridging Items'. Each of the items captures several areas of knowledge and related beliefs. Complimentary to this questionnaire an interview guide for a problem-centred guided interview was developed, which contains pre-structured and open questions (i.e., elaborating questions) on modelling, argumentation and proof. The questions are linked to the items in the questionnaire and deepen parts of the interview. The selection of the interviewees follows theoretical considerations and takes the achievements in the questionnaire into account. That means interviewees were selected according to an interesting answering pattern in the questionnaire or extraordinary high or low knowledge in one or more domains.

The evaluation of the questionnaires as well as of the interviews is carried out by means of Mayring's qualitative content analysis method (2000). We apply a method of analysis that aims at extracting a specific structure from the material by referring to predefined criteria (deductive application of categories). From there, by means of formulation of definitions, identification of typical passages from the responses as so-called anchor examples and development of coding rules, a coding manual has been constructed to be used to analyse and to code the material. For this, coding means the assignment of the material according to the evaluation categories. In addition, the method of structuring scaling (Mayring, 2000) is applied by which the material is evaluated by using scales (predominantly ordinal scales). Subsequently, quantitative analyses according to frequency or contingency can be carried out. Thus our approach can be visualised as in Figure 1.

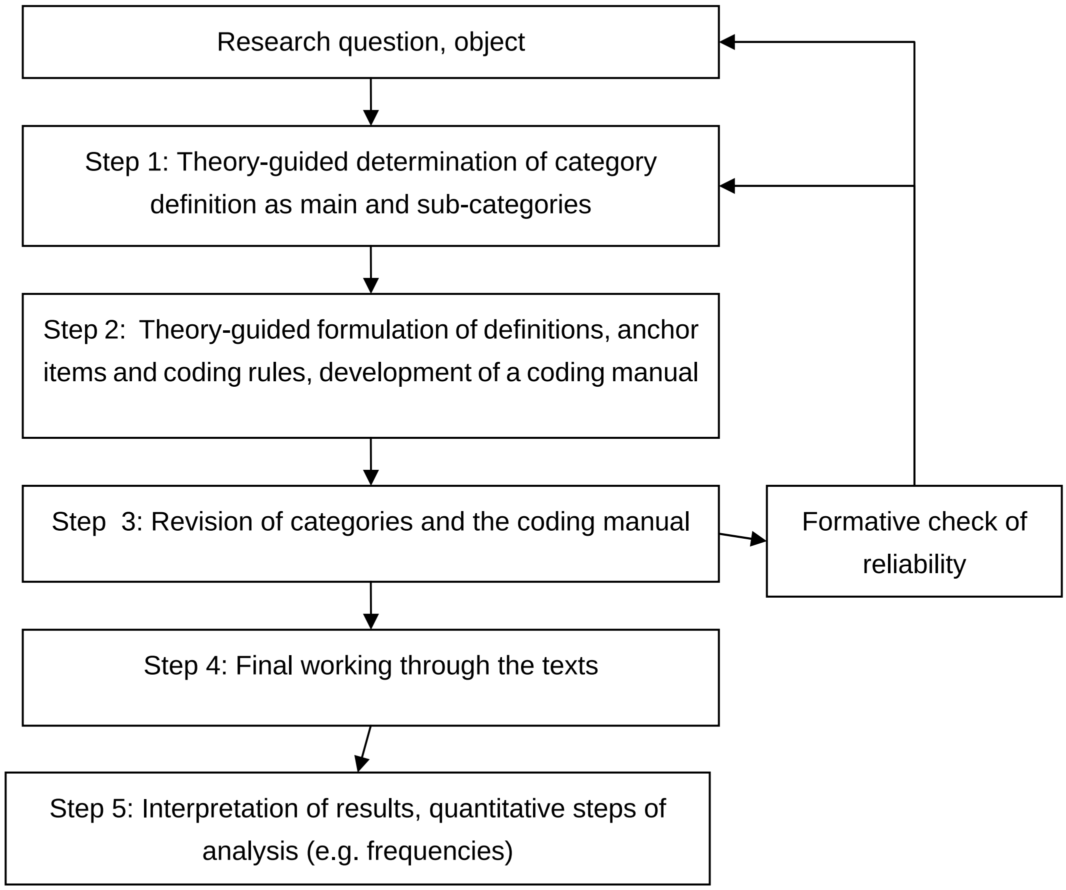


Fig. 1: Model of deductive usage of categories (adapted from Mayring 2000b)

In the following one exemplary item is described, which shows, how the different facets of professional knowledge – pedagogical content knowledge, mathematical knowledge - are linked. A similar item is included in the questionnaire, so that it is possible to connect the evaluation of the data on a rich data base:

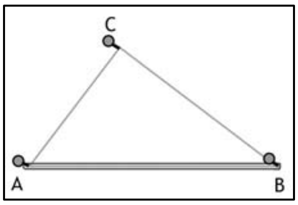
I) The following theorem is valid:

"In any triangle, the sum of the lengths of two sides, a and b, is always longer than, or equal to, the length of the third side, c."

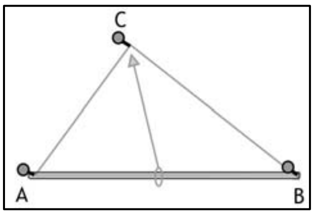
a. Are the following statements equivalent to the above stated theorem?
   i. "In any triangle ABC, the distance from A to B is always shorter than, or equal to, the sum of the distances from A to C and from C to B."
   ii. "There is no shorter route than the direct one."
b. Please formulate a formal equation for this stated theorem.
c. Do you know this theorem? If yes, can you name it?
d. Is this statement valid for only one combination of the sides a, b and c?
e. When exactly is the equality in this statement valid?

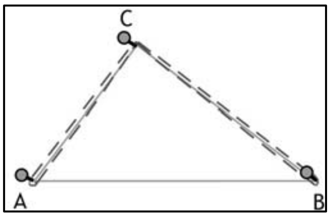
II) The following argument below is given to support the above mentioned theorem:

Below you see a diagram of a board with three nails A, B and C which form a triangle. There is a rubber band tightened around the nails A and B which represents the length of the side c.



If you want to place the rubber band also around the nail C in order to visualise the length of sides a and b, the rubber band has to be stretched.





The sum of the sides a and b is consequently longer than the side c.

a. Which type of proof is it?
b. Judge the proof concerning correctness, generality and being clearly understandable.
c. Please prove the following statement formally.

"The half perimeter of a (constructible) triangle ABC is always longer than each side of the triangle."

d. Which advantages and disadvantages are there of the use of a pre-formal proof in mathematics lessons compared to a formal proof?

OUTLOOK ON RESULTS OF THE STUDY

The study was until now carried out with future teachers from several universities in Germany, Hong Kong and Australia, more are planned. The first results point out, that the majority of future teachers were not able to execute formal proofs, requiring only lower secondary mathematical content, in an adequate and mathematically correct way or to recognise and satisfactorily generalise a given mathematical proof. In contrast, there was evidence of at least average competencies of pedagogical content reflection about formal and pre-formal proving in mathematics teaching. Preferences for pre-formal proving are evident, both with respect to mathematical content knowledge and pedagogical content knowledge. Detailed results are reported in Schwarz et al. (2008), Corleis et al. (2008) and the paper by Brown & Stillman submitted for this ICMI study.

It appears that possessing a mathematical background as required for teaching and having a high affinity with proving in mathematics teaching at the lower secondary level are not sufficient preparation for teaching proof. In the limited time available for initial teacher education courses some time must be devoted to ensuring that future teachers experience proof in such a way that they can in turn, allow lower secondary students opportunities to develop a complete image of proof and proving. Part of this experience should address the plea of Blum and Kirsch (1991) "for doing mathematics on a pre-formal level" and hence providing all students with the opportunity to engage deeply with "pre-formal proofs that are as obvious and natural as possible especially for the mathematically less experienced learner" (p. 186). However, our study also suggests that even students with strong mathematical backgrounds from tertiary studies are not necessarily experiencing proof in such a manner that they can convey a complete image of proving at the lower secondary level.

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UNDERSTANDING THE PROOF CONSTRUCTION PROCESS

John Selden and Annie Selden

New Mexico State University, U.S.A.

Through a design experiment we are investigating how advanced undergraduate and beginning graduate students learn and can be taught to construct proofs. This paper describes the experiment and some results arising from it, including: (1) A description of the formal-rhetorical part of proofs that, if emphasized early, functions as a kind of proving tool and appears to alleviate students' confusion and contributes to an atmosphere of success; (2) a theoretical perspective that allows us to diagnose student difficulties and suggests remedies; and (3) an example that suggests these teaching methods work.

We describe a design experiment, discuss a framework for assessing student progress, and introduce a theoretical perspective for understanding it. Using these, we then describe three examples of student proving difficulties and how they were alleviated. Finally, we provide an example speaking to the effectiveness of our teaching methods.

THE DESIGN EXPERIMENT

We are designing, teaching, and studying a one-semester, three-credit proof course for prospective and beginning mathematics graduate students, although we have sometimes accepted other students such as a high school teacher and several Master of Arts in Teaching students. The course meets twice per week for 75 minutes and its whole purpose is to teach proof construction. There are three teachers, the two authors and a graduate student.

The teaching is a significant modification of the Moore Method (Jones, 1977; Mahavier, 1999). That is, instead of a book, students are given notes, including statements of theorems, definitions, and requests for examples, but no proofs, and only minimal explanations. There are no lectures, and prerequisites such as logic are discussed only as the need arises when considering students' proofs. Students work outside of class and present their proofs at the blackboard in class. Occasionally a proof is started in class by a whole class discussion. We also offer tutoring to any student who needs it. Everything is video recorded, field notes are taken, and this material is analyzed in planning sessions between class meetings in an effort to influence what might be called students' learning trajectories (Simon, 1995). The planning sessions are also video recorded. We are now teaching the third iteration of the course (of a projected eight). The Fall Semester notes include mainly theorems about sets, functions, and some real analysis and abstract algebra. The theorems about sets, functions, and analysis in the Spring Semester notes differ from those in the Fall Semester notes, and abstract algebra is replaced with some topology. Both semesters of the course can be taken for graduate credit, but to date no students have done so.

After a student presents a proof, we validate it, that is, we read it to check for correctness (Selden & Selden, 2003). We "think aloud" so students can see what we are checking. If a proof is, or can be made, correct, we edit it and invite the student author to write it up for addition to the notes. As the need arises during class, we occasionally offer, sometimes extensive, criticism and advice. For example, we have mentioned how to use a statement of the form *P or Q* (by taking cases) or how to prove it (by proving, *if not P then Q*). We have also suggested the usefulness of metaphorical drawings and diagrams, and pointed out that the negation of *P or Q* is not *not P or not Q*. Such comments are always made in the context of student work. About halfway through the course, we start selecting students to present the validation of other students' presented proofs. We then comment on the validation as well as the proof, because we regard validation as a non-trivial ability and an integral part of proof construction.

From our students' perspective, this course has a very practical value. In many advanced undergraduate courses and most beginning graduate courses, mathematics professors assess their students' understanding by asking them to apply that understanding, very often by constructing proofs. Thus students who cannot construct proofs have great difficulty showing they understand their courses.

A FRAMEWORK FOR ASSESSING PROGRESS

Progress in many advanced mathematics courses (say, abstract algebra) can be discussed in terms of students' understanding of various theorems (the isomorphism theorems), concepts (homomorphism or coset), and examples ($\mathbf{Z}_3$). A course on proving needs a similar framework, but one that distinguishes kinds and aspects of proofs, rather than the content of the mathematics. We are collecting and using such distinctions and will mention two, beyond the familiar direct and indirect proofs and proofs by induction.

We see proofs, beyond the simplest ones, as depending on our notes in three increasingly demanding ways. A proof can require a result (1) in the notes, (2) not in the notes, but easily noticed and articulated, or (3) neither in the notes nor easily articulated. We have arranged the notes to provide opportunities to experience each of these, and we have examples of students having difficulties with, and succeeding with, each type.

We also distinguish two aspects or parts of proofs, the *formal-rhetorical* and *problem-oriented* parts (Selden & Selden, in press). The formal-rhetorical part of a proof is the part that one can write based only on logic, definitions, and sometimes theorems, without recourse to conceptual understanding, intuition, or problem solving in the sense of Schoenfeld (1985, p. 74). We call the remainder of the proof the problem-oriented part and it *does* require conceptual understanding and real problem solving. Different skills are needed to construct these two aspects of a proof, and generally writing the formal-rhetorical part of a proof exposes the "real problem" to be solved.

In our proof course, we concentrate first on having students write the formal-rhetorical parts of proofs. We often allow a student, who has not completed a proof, to present whatever he/she has, including just the formal-rhetorical part. This alleviates early student difficulties and contributes to an atmosphere of success. In later student work, starting a proof with the formal-rhetorical part becomes a cognitive tool in the proving process, because doing so exposes what needs to be proved.

THEORETICAL PERSPECTIVE

In designing the proof course we are taking a constructivist perspective, in that we are maximizing students' opportunities to try to construct proofs and to reflect on the results. Our approach is also somewhat Vygotskian, in that some of our criticism and advice is meant to convey to the students what mathematicians regard as an acceptable proof. This assumes considerable agreement about what is acceptable, that is, that there is something one might call a *genre of proof*. Indeed we convey to the students that their task is not just to write convincing arguments, but to write them in a way acceptable to the mathematics community. This seems to alleviate the blockage some students experience when they see a beginning theorem as obvious, because just saying it is obvious is not an acceptable proof.

The above perspective, however, often does not help us understand why a particular student is having a particular difficulty or what to do about it. To that end, we first mention our view of the proving process. While the final written proof is a text, the proving process is a much longer sequence of (mental or physical) actions, some of which directly yield text, such as a bit of the proof or a metaphorical drawing, and some of which do not, such as the act of focusing on some part of what has been done or trying to remember some previous relevant work. Near the end of the process, the text that has been produced may be pruned, reordered, and edited. This is usually needed to produce a proof acceptable to the mathematics community, but also greatly obscures the original proving process. For example, using the definition of convergence to prove *If* $\{a_n\}$ *and* $\{b_n\}$ *are sequences such that* $\{a_n+ b_n\}$ *and* $\{a_n - b_n\}$ *converge, then* $\{a_n\}$ *converges*, one starts by writing the hypothesis. But then one focuses on the conclusion and unpacks its meaning. This requires guessing the limit of $\{a_n\}$, which requires naming the limits of $\{a_n+ b_n\}$ and $\{a_n - b_n\}$. The final written proof does not reflect the order in which these actions, and others, occurred, and those actions not yielding text are not represented at all. The obscuration of the proving process appears to be a major reason proving is so hard to learn. The actions behind a final written proof often need to be reflected upon, influenced by advice, or mimicked, and hence, those actions and their order must be perceived and understood.

In the proving process, an action is a response to an (inner) situation. Such a situation can include anything the prover is conscious of and focusing on, not just the accumulated text. An inner situation cannot be seen by an observer or a teacher, but it can often be inferred approximately. After similar situations occur in several proof constructions with the same resulting action, the common situation may be reified and thenceforth be easily recognized. Also, a persistent link between the situation and a tendency toward the resulting action may be established. For example, in a situation calling for C to be proved from A *or* B, one constructs two independent subproofs arriving at C, one supposing A, and the other supposing B. If one has had repeated experience with such proofs, one does not have to think about doing or justifying this action, one just does it. We call such persistent (small grain-size) linked situation-action pairs, *behavioral schemas*.

We see behavioral schemas as a form of (often tacit) procedural knowledge that yields immediate (mental or physical) actions. Within a broad context such schemas are always available – they do not have to be searched for and recalled before use. The process leading to their enactment occurs outside of consciousness and so is not under conscious control. Perhaps this partly explains why just providing a counterexample to a computational error, such as $(2x+1)/2y = (x+1)/y$, may not prevent its later recurrence. Behavioral schemas depend on conscious input, yield conscious output, but cannot be "chained together" outside of consciousness. Thus a person cannot solve a linear equation normally calling for several solution steps, without being conscious of any of the intermediate steps. Indeed, even in computing (10/5)+7, one is normally conscious of the 2 before arriving at the 9.

Taking a more external, or third person, view and perhaps a larger grain-size, behavioral schemas may also be seen as habits of mind (Margolis, 1993). Just as with physical habits, a person may often be unaware of having a habit of mind, that is, a behavioral schema. Behavioral schemas can play a large role in constructing the formal-rhetorical part of a proof and often play a considerable role in constructing the problem-oriented part. It turns out that some behavioral schemas are beneficial and others are detrimental, and students can be helped to strengthen the former and weaken the latter. Behavioral schemas and habits of mind are more fully discussed in Selden and Selden (2008).

OBSERVATIONS

Below we very briefly describe three student difficulties we have encountered and how they were alleviated.

Moore (1994) described undergraduate transition-to-proof course students who could not prove on their final exam: *If f and g are functions from A to A and $f \circ g$ is one-to-one, then g is one-to-one*. He said that students started in the wrong place, with the hypothesis, instead of supposing $g(x) = g(y)$. Of course it is legitimate to start by *writing* the hypothesis but, like Moore, we have found that a number of our students habitually *focus* on the hypothesis immediately, instead of unpacking the conclusion and trying to prove that. We have found that by patiently guiding students to first write the formal-rhetorical parts of proofs, this detrimental schema can be overcome. The action here is not immediately text producing, but rather psychological – where to focus one's attention.

One normally proves theorems of the form, "For all numbers x, P(x)," by writing in the proof, "Let x be a number," meaning x is fixed but arbitrary (rather than a variable). Some of our students appear to understand the reasoning behind this, but do not do it; they require prompting for a number of proof constructions. We interviewed Mary, a graduate student, and her teacher, Dr. K, about this point in her beginning graduate real analysis course. Mary reported that this action felt somehow inappropriate, but she did it on her homework because she trusted her teacher and wanted a good grade. Dr. K agreed on this point. Mary reported that she had convinced herself that each individual homework proof was correct, but that she had still not felt the action (of letting x be a number) was appropriate until about mid-semester. This is an example of a student first linking a situation to an action based mainly on authority, and only very slowly associating a feeling of rightness with this behavioral schema. The role of feelings and the case of Mary are more fully discussed in Selden, Selden, and McKee (2008).

Finally, we consider Sofia, a diligent first-year graduate student. We began to suspect Sofia might have a persistent difficulty on the fifth day of class when she volunteered to present an argument, but only its first and last lines could reasonably be considered part of a proof. As the course progressed an unfortunate pattern emerged. When Sofia did not have any idea of how to proceed, she fairly quickly produced an "unreflective guess," only loosely related to the context at hand, and the resulting confusion seemed to block further progress. We inferred that Sofia was enacting a detrimental behavioral schema. During tutoring sessions, we tried to prevent the enactment of this schema by suggesting a variety of alternative actions, such as drawing a diagram or looking in the notes for a relevant definition or theorem. As the course ended, this intervention was beginning to show promise.

THE EFFECTIVENESS OF THESE METHODS

While this design experiment is not nearly complete, our tentative methods often seem to be produce quite remarkable results. For example, Sofia, the first-year graduate student mentioned above, could not prove any of the theorems on our take-home pre-test: (1) If A, B, and C are sets satisfying $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then $B \subseteq C$; (2) If f and g are functions from A to A and $f \circ g$ is one-to-one then g is one-to-one; (3) If the number of elements in a set is n, then the number of subsets is 2^n ; (4) For all positive integers n, if $n^2 + 1$ is a multiple of 3, then $n^2 + 2n + 1$ is a multiple of 9; (5) If g is a function (on the real numbers) continuous at a and f is a function (on the real numbers) continues at $g(a)$, then $f \circ g$ is continuous at a.

In contrast, on the in-class final exam Sofia proved that if f, g, and h are functions from a set to itself, f is one-to-one, and $f \circ g = f \circ h$, then $g = h$. Also, on the take-home final exam, except for a small omission, she proved that the set of points on which two continuous functions between Hausdorff spaces agree is closed. We think most mathematicians would be pleased with Sofia's progress.

CONCLUSION

In this paper we have sketched a way of teaching proof construction and described a theoretical perspective allowing us to see, and to tailor responses to, specific student difficulties. While our three examples refer to students with a reasonable grasp of undergraduate mathematics, we have also succeeded with students having very modest mathematical backgrounds. This suggests that our methods could be adapted to courses at the middle undergraduate, beginning undergraduate, or even high school levels. If this could be done widely, it would greatly improve what could be taught and understood in courses like calculus, real analysis, and abstract algebra.

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INFLUENCE OF MRP TASKS ON STUDENTS' WILLINGNESS TO REASONING AND PROVING

Zdenko Takáč

Dept. of Mathematics, Pedagogical Faculty of the Catholic University, Slovakia

The motivation is a necessary part of any educational activity. This takes place also for learning of proofs. Students have to feel the need to prove a statement. We describe tasks which are useful tools for the developing of this feeling. We named these tasks Motivation to Reasoning and Proving tasks (MRP tasks) and we carried out a research concerning with the influence of MRP tasks on students' (grade 5-13) willingness to reasoning and proving. The research confirmed that such tasks can help educators to develop students' mathematical reasoning skills.

THE ROLE OF PROOF IN MATHEMATICS AND SCHOOL MATHEMATICS

Each discipline has its own tools to recognize a true statement. We use proof for this purpose in mathematics. The proof (in some mathematical theory) is a sequence of statements in a language of the theory where each statement is an axiom of logic, an axiom of the theory, or it depends on preceding statements via Modus Ponens or Generalization. In everyday mathematics we use more comfortable concept of proof: a sequence of statements in a language of the theory where each statement is a theorem (already proved statement) of logic, a theorem of the theory, or it depends on preceding statements via any rule of inference.

According to Hanna (1989, 1995) and Knuth (2002), the role of proof in school mathematics is not only to demonstrate the correctness of a result or truth of a statement:

> A proof that proves shows only that a theorem is true; it provides evidential reasons alone.... A proof that explains, on the other hand, also shows why a theorem is true; it provides a set of reasons that derive from the phenomenon itself.

Proof is the holder of mathematical rigour and for many mathematicians also a fundamental notion of mathematics. But proof is the holder of formalism, too and so it is often for students a scarecrow overmuch. Especially, when they are forced to proving in formal degree that is beyond bounds of their possibility. Teacher must very carefully determine what degree of the formal aspects of proof he can demand from students in connection with their grade and mental abilities. If teacher requires more then he can, students acquire a dislike for proving.

Teachers have to see the formal aspect and the substantive aspect of proving. The formal aspect consists of using the logical axioms (or theorems) and the logical rules of inference and the substantive aspect consist of using the theory axioms and the theory rules of inference. Teachers can treat the explanation of proof diversely. They can assign varied importance to the substantive aspects and to the formal aspects of proof. The importance of each aspect depends on pedagogical aims: 1. substantive aspect ("mathematical aims") – developing of students mathematical knowledge: properties of mathematical objects and relations between the objects, 2. formal aspect ("logical aims") – developing of students reasoning abilities: using of the logical connectives, the quantifiers, the formal language and the proof methods.

STUDENTS' MOTIVATION TO REASONING AND PROVING

Since the 1970s the educational psychologists have studied intrinsic and extrinsic motivation. The numerous studies have found intrinsic motivation to be associated with high educational achievement and enjoyment by students (Deci and Ryan, 1985). Intrinsic motivation is when people engage in an activity without obvious external incentives. It is much stronger motivation than the extrinsic motivation in all kind of students' activities.

The most frequent children questions are: "what is it?" and "why?". Especially the second one drives most parents crazy. It is a natural children attribute wishing to know, why things happen as they do. Holding and developing this attribute is one of the fundamental general aims of mathematical education (but not only mathematical). If it is natural attribute, holding it seems be an easy assignment, but the problems with students' motivation to proving indicate that it is not.

In general, any aim achieving demands: 1. to have good conditions, 2. to have relevant abilities, 3. to want to make it. This take place also in mathematical education and of course in proof education. So, if we want a class to prove a statement we need: 1. to have a time and kindly climate in the class (good conditions), 2. students (much more the teacher) to have needful knowledge about mathematical objects that appear in the proof and also needful logical knowledge (relevant abilities), 3. students wanting to convince themselves or convince others (want to make it). The first one is a long-range aim of each teacher. The second one depends on past education and on choosing a suitable statement to prove – appropriate sophistication of proof. The last one is a matter of motivation, which is the aim of this article.

The motivation to proving can be divided: 1. one wants to convince oneself of statement's truth, 2. one wants to convince others of statement's truth, 3. teacher (task assignment) demands the proof of a statement. The first two points are intrinsic motivations, the third one is an extrinsic motivation. The first one is the best precondition for developing the students' abilities to search logically correct arguments, the second one is the best precondition for developing verbal abilities and giving precision to creating arguments. Together, these two points lead to creating correct proofs. It is always the best, if students are solving a problem because of their intrinsic motivation. Then it is an easier job for teachers, they only need to offer a suitable problem and regulate the students' procedure. In this article we shall deal with intrinsic motivation: with fostering the natural one and developing it further.

MRP TASKS

As years pass the children's curiosity weaken and the question: "why?" is not such important for many of them as it was before. Mathematics, more specifically reasoning and proving, is one of the best resources to hold on this curiosity and to develop it to the higher level – the sense of requirement of being able to justify conclusions. Reasoning and proving are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied (NCTM, 2000). According to NCTM teachers would regularly pay attention to holding students' curiosity and developing it. The tasks of following types are supposed to be suitable for this. Type 1: task that looks to have an easy solution, but after dealing with problem exhaustively, it has a different perhaps surprising solution. Type 2: task that can be solved intuitively, but students are not sure of solution's correctness. Type 3: task that has several possible solutions and students have to decide (and verify), which one is correct. We named described kinds of tasks Motivation to Reasoning and Proving tasks (MRP tasks).

We were concerned with influence of MRP tasks on students' willingness to reasoning and proving. We observed pairs of various grade (5-13) classes. Each pair included one experimental class and one comparative class with the same teacher and the same curriculum. Students in experimental classes solved one or two MRP tasks a month for six months (from September of 2007 to February of 2008). Students in comparative classes did not solve these tasks. In April of 2008 we tested both classes and discovered differences between students' willingness to reasoning and proving in these two classes. Next are given a few examples of used MRP tasks.

Task 1: John and Mary raced each other from a place A to a place B and back to A. Mary averaged 25 kmph cycling from A to B and 5 kmph walking back to A. John averaged 9 kmph running from A to B and back to A. Who has won?

Teacher had recapitulated knowledge of the average speed from physics before students (grade 8-10) began deal with task 1. In spite of this most students answered (in a short time): "Mary has won, because she averaged 15 kmph and John only 9 kmph." Only when teacher had encouraged them to convince of correctness of their solution they started to calculate Mary's average speed. They were really surprised that Mary averaged only 25/3 kmph.

Task 2: A dog and a cat raced on the 100 metres straight track there and back. The dog's jump is 3 metres and the cat's jump is 2 metres long, but the cat makes 3 jumps while the dog makes 2 jumps. Who will win?

Most of the students (grade 5-7) answered in a short time: "Nobody will win (or they both will win), the dog and the cat are moving the same speed." They only discovered that the dog would do 6 metres (2 jumps by 3 metres) in the same time as the cat would do 6 metres (3 jumps by 2 metres). Only few of them realized the dog would be behind the cat after turning around at 100 metres. Only when teacher (or successful classmate) had contradicted they started concern with the problem more consistently.

Task 3: Three hens lay three eggs in three days. How much eggs will six hens lay in six days?

This is a typical trick question and students (grade 5-9) very quickly answered six. They were able to correct their solution, but most of them only after teacher's expressing doubts about their result.

Task 4: A knight is at square A1 of chessboard. Is it possible to repeatedly move the knight so that it will be once at each square of chessboard and it will finish at square H8?

Teacher had drawn chessboard and had explained moving a knight before students (grade 10-13) began deal with task 4. Students got enough time to making experiments. They divided into two groups: the members of the first group thought it was not possible, the members of the second group thought it was possible. But only few of the first group members were inwardly convinced that they are true. The others were not sure and so were not sure almost all members of the second group. When teacher or a successful classmate had explained why it is impossible (the knight has to make 63 moves, it starts at white square and after odd number of moves will be at dark square, but the square H8 is white), almost all students very quickly accepted this result.

Task 5: Take a sufficiently big paper and stepwise 50 times bend it. Reckon the thickness of a bended paper, if thickness of original paper is 0.1 mm.

Students (grade 11-13) have got only few seconds and then they had to reckon thickness of a bended paper. Teacher had written assessments of all students on the blackboards and only then they started to calculate the thickness. They were really surprised that it was such a big number.

DIFFERENCES BETWEEN STUDENTS' WILLINGNESS TO REASONING AND PROVING IN EXPERIMENTAL AND COMPARATIVE CLASSES

We tested experimental and comparative classes in April of 2008. Our goal was to discover if students in experimental classes are more willing to reasoning and proving than students in comparative classes. We have done it in three ways:

1. Students solved another MRP tasks. There was an evident difference between experimental and comparative classes solutions of MRP tasks for all grades (5-13). This can be described by the next class episode (grade 12 student of comparative class solved MRP task of type 2 and he did not verified the correctness of his solution):

Teacher: "Are you sure that you are right?" Student: "I think so, but I am not completely convinced." Teacher: "Why did not you make a proof?" Student: "Proof was not required."

Student was not sure about his solution but he did not make a proof, because it was not explicitly required in the assignment.

We do not consider this result to be substantial. One can expect such result, because students in experimental classes were accustomed to tasks of this type and so we consider next two ways of testing to be more substantial.

2. Students solved tasks that explicitly required proof. We paid attention to students' willingness to proving and we came to the conclusions: students in experimental classes were significantly more willing to proving than students in comparative classes; the difference between these classes was not very big for higher grade students (grade 12-13) but it was very big for lower grade students (grade 5-8).

3. Teacher and students discussed the role and importance of proof in mathematics. We came to the similar conclusions as in point 2. Experimental class students considered proof to be a useful tool in mathematics much more than comparative class students. We support connection of this fact to solved MRP tasks by the next class episode:

Teacher: "Why is proof important in mathematics?" Student: "It is like with the race, we all had thought that the runner could not win and then we proved that he won." (She referred to Task 1 in this article.) Teacher: "Do you remember the task?" Student: "Of course I do, I felt sure about my solution and I was wrong. I looked into the problem again at home."

The difference between experimental and comparative classes was again significantly bigger for lower grade students than for higher grade students, similarly as in point 2.

CONCLUDING REMARKS

The conclusion of our experiment is that dealing with MRP tasks helps students to realize importance of proof in mathematics. It develops their intrinsic motivation to proving and it is the first (but necessary) step to creating correct proof in the future. It is interesting that this connection is strong for lower grade students and it is not so clear for higher grade students. We think of this result that higher grade students are accustomed to classical tasks in which they obtain explicit instructions what they have to do. They are not used to confirm their solution, if the task or teacher does not require it explicitly.

We came to the conclusion that dealing with MRP tasks develops students' critical thinking, they feel the need to verify one's or others' statements and not to receive information uncritically. The result is that students' intrinsic motivation to proving raise.

This fact is important not only in mathematics but also in everyday life. Developing students' critical thinking is one of the aims of school mathematics. Students would learn to receive information (also if author is teacher) with some degree of no confidence. It does not mean that they do not respect teacher's professional knowledge. But their respect does not exclude their subconscious desire to verify a new information "in their way" – to formulate "their own" arguments that convince them about truth of the information. Students reach higher degree of understanding, if they are able to find such arguments.

The advantage is that also students' formulating incorrect arguments benefits in developing their critical thinking. In particular, if teacher helps them to understand their mistake.

We were concerned with students' motivation to confirm their statements by creating arguments. At the beginning of this article we asserted: proof is a tool to recognize a true statement. In school mathematics is creating arguments and creating proofs the same thing. Teacher's job is:
- to choose an appropriate degree of formalness of proof – whether he will give preference to substantive aspect or to formal aspect of proof (Takáč, 2007),
- to choose the role of proof in an educational activity. For example De Villiers (1999) presents various roles that proof plays in mathematics: to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, or to systematize statements into an axiomatic system.

The degree of formalness of proof and the role of proof in an educational activity depends on students' grade and their mental abilities.

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IS THIS VERBAL JUSTIFICATION A PROOF? [1]

Pessia Tsamir, Dina Tirosh, Tommy Dreyfus,

Michal Tabach, and Ruthi Barkai

Tel Aviv University, Israel

In light of research results indicating that high school students prefer verbal proofs to other formats, we found it interesting and important to examine the position of high school teachers with regard to verbal proofs. Fifty high school teachers were asked to evaluate given justifications to statements from elementary number theory. Our findings indicate that about half of the teachers rejected verbal justifications. They claimed that these justifications lack the generality property and are mere examples. These claims were made even when the verbal justifications were sufficient for proving the statements.

BACKGROUND

Reform calls throughout the world address the importance of mathematical reasoning. In the USA, for example, the Principles and Standards state that students should be able to "recognize and apply deductive and inductive reasoning" (NCTM, 2000, p. 81). Several studies indicate that high school students tend to use verbal argumentation in order to prove universal elementary number theory (ENT) statements (Edwards, 1999; Healy & Hoyles, 2000), and even point to verbal proofs for universal statements as their preferred type of proof-representation (Healy & Hoyles, 2000). In light of these findings, we have studied to study high school teachers' position with respect to verbal justifications of a wide spectrum of ENT statements, both universal and existential ones as well as true and false ones.

THE STUDY

A group of 50 practicing high school teachers, each with at least two years of experience, participated in the study. The participants were asked to answer a questionnaire that addressed six ENT statements. The statements were chosen to include one of three predicates (always true, sometimes true or never true), and one of two quantifiers (universal or existential). The validity of a statement is determined by the combination of its predicate and its quantifier. In Table 1 we display the six statements with reference to their quantifier, their predicate, their validity and the kind of minimal proof needed for their validation or refutation (written in Italics).

The teachers were first asked to assess whether each of the six statements was true or false and to produce a proof (validation or refutation). All teachers provided correct symbolic or numerical (but not verbal) proofs. Then, they were presented with several correct and incorrect justifications to each statement, and for each justification, they were asked to determine if it proves the statement to be true (or

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false), and to explain their response. The justifications were presented in three representations: numerical, verbal and symbolic. In this paper, we limit the discussion to a sub-set of four, correct verbal justifications, one each for statements S1, S3, S4 and S6 in Table 1.

Predicate / Quantifier	Always true	Sometimes true	Never true
Universal	**S1**: The sum of any 5 consecutive natural numbers is divisible by 5. *True - General proof*	**S2**: The sum of any 3 consecutive natural numbers is divisible by 6. *False - Counter example*	**S3**: The sum of any 4 consecutive natural numbers is divisible by 4. *False - Counter example*
Existential	**S4**: There exists a sum of 5 consecutive natural numbers that is divisible by 5. *True - Supportive Example*	**S5**: There exists a sum of 3 consecutive natural numbers that is divisible by 6. *True - Supportive Example*	**S6**: There exists a sum of 4 consecutive natural numbers that is divisible by 4. *False - General proof*

Table 1: Classification of the statements

The four justifications are presented below; they follow the idea of mathematical induction. That is, the sum of the 'first' sequence of consecutive numbers is shown to be divisible by n (according to each statement), and the generalization to all other related sums in obtained inductively (see Figures 1 - 4).

Statement S1: The sum of any 5 consecutive natural numbers is divisible by 5.
Justification: Moshe claimed: I checked the sum of the first five consecutive numbers, 1+2+3+4+5=15, and found that it is divisible by 5. The sum of the next five consecutive numbers is larger by 5 than this sum (each of the five numbers grows by 1, so the sum grows by 5), and hence this sum is also divisible by 5. And so on, each time we add 5 to a sum that is divisible by 5, and thus we always obtain sums that are divisible by 5. Therefore the statement is true.

Figure 1: the verbal justification for S1

Statement S3: The sum of any 4 consecutive natural numbers is divisible by 4.
Justification: Moty claimed: I checked the sum of the first four consecutive numbers, 1+2+3+4=10, and found that it is not divisible by 4. The sum of the next four consecutive numbers is obtained by adding 4 to this sum (each of the four numbers in the sum grows by 1, so the sum grows by 4). It is known that adding 4 to a sum that is not divisible by 4 will yield a sum that is not divisible by 4 either. And so on, each time we add 4 to a sum that is not divisible by 4, and therefore we always obtain sums that are not divisible by 4. Therefore the statement is not true.

Figure 2: the verbal justification for S3

Statement S4: There exists a sum of 5 consecutive natural numbers that is divisible by 5.
Justification: Mali claimed: I checked the sum of the first five consecutive numbers, 1+2+3+4+5=15, and found that it is divisible by 5. The sum of the next five consecutive numbers is larger by 5 than this sum (each of the five numbers grows by 1, so the sum grows by 5), and hence this sum is also divisible by 5. And so on, each time we add 5 to a sum that is divisible by 5, and thus we always obtain sums that are divisible by 5. Therefore the statement is true.

Figure 3: the verbal justification for S4

Statement S6: There exists a sum of 4 consecutive natural numbers that is divisible by 4.
Justification: Mira claimed: I checked the sum of the first four consecutive numbers: 1+2+3+4=10, and found that it is not divisible by 4. The sum of the next four consecutive numbers is obtained by adding 4 to this sum (each of the four numbers in the sum grows by 1, so the sum grows by 4). It is known that adding 4 to a sum that is not divisible by 4 will yield a sum that is not divisible by 4 either. And so on, each time we add 4 to a sum that is not divisible by 4, and therefore we always obtain sums that are not divisible by 4. Therefore the statement is not true.

Figure 4: the verbal justification for S6

For each of the justifications, we first analyzed whether the teacher participants provided a correct judgment. Then, we analyzed their related explanations.

Note: we use the term *justification* for the verbal proofs which were presented in the questionnaire (see Figures 1-4). We use the term *explanation* to denote participants' explanations when answering our questionnaire. We use the term *proof* to denote correct verification or refutation for a given statement.

FINDINGS

The four verbal justifications presented here are correct. As such, they may serve as proofs to validate statements S1 and S4, and to refute statements S3 and S6. However, not all four justifications are minimal; in S3 and in S4, one example would have been sufficient to refute/validate the claim. The participants' explanations in all the cases exhibit their awareness of the kind of (minimal) proof needed for each type of statement: a general ("covering") proof for validating universal, always true statements (S1) and for refuting existential, never true ones (S6); a (supportive) numerical example for validating existential, always true statements (S4) and a counter example to refute universal, always false statements (S3).

However, the generality of the justifications presented in Figures 1-4 was not noted by all participants, as can be concluded from the analysis of their explanations. We defined five categories of teachers' explanations: The justification (a) is general and sufficient; (b) is general but covers only a sub-set of the cases; (c) is an example; (d) starts with a numerical example, but all the rest is unnecessary and wrong; and (e) no explanation. Category (a) represents a correct reference to the justifications. Table 2 presents the four statements, the categories of explanations provided by the teachers, and the correctness of the judgments (in parenthesis). We now examine each category (each row of Table 2) separately.

(a) *The justification is general and sufficient.* About half of the participants correctly acknowledged the justifications as general and covering all cases. The justification of S3 was an exception, maybe due to the fact that a counter example was obvious as the minimal proof. Participants' explanations to these judgments are illustrated by a typical example of a correct explanation for the justification to S1: "Correct verbal justification. Correct logical inferences. This is like proving using induction verbally" (T2). In this case, the teacher explicitly identifies the underlying logic of the proof – the use of induction.

Predicate	Always true		Never true	
Quantifier	universal S1	existentia lS4	universal S3	existentia l S6
Validity	Valid	Valid	Not valid	Not valid
(a) Identifying the justification as general, relating to all cases	(Correct) 58	(Correct) 48	(Correct) 26	(Correct) 48
(b) Perceiving the justification as general but covering only a sub-set of all cases	(Incorrect) 10	(Correct) 10	(Correct) 10	(Incorrect) 10
(c) Perceiving the justification as an example	(Incorrect) 32	(Correct) 38	(Correct) 36	(Incorrect) 22
(d) Perceiving the justification as starting with a numerical example, but all the rest as not needed and wrong	---	(Incorrect) 4	(Incorrect) 28	---
(e) Did not provide explanation	---	---	---	20
Total Correct judgment	58	96	72	48
correct explanations	58	48	26	48
incorrect explanations	---	48	46	---

Table 2: Teachers' judgments (correct or incorrect) and their explanations for the verbal justifications (percentage) (N=50)

(b) *The justification is general but covers only a sub-set of the cases*. Ten percent of the teachers provided such explanations when rejecting the induction-type, verbal justifications for statements S1, S3, S4, and S6. For statements S1 and S6, such wrong interpretations led these teachers to judge incorrectly that the proofs given by Moshe and Mira were inadequate, while for statements S3 and S4 – they led to correct judgments (S3 required a counter example, and S4 – a supportive example). These participants identified the general aspect of the justification, but not that it covered all cases. This is exemplified in a teacher's explanation for the verbal justification presented to statement S1: "[Moshe] was using a specific example. 1+2+3+4+5=15, 6+7+8+9+10=40, but he did not consider the numbers in between, like 3+4+5+6+7. He had to prove that the statement holds for any given number" (T9). This teacher expressed her understanding of the need to cover all cases, but she did not identify that Moshe's verbal justification does cover all cases. A discussion with the teacher revealed that she understood the words: "The sum of the next five consecutive numbers is larger by 5 than this sum" as denoting that the next sum starts with a number that is larger by five, hence the next relevant sequence will be 6, 7, 8, 9, 10, rather than the sequence 2, 3, 4, 5, 6.

(c) *The justification is an example*. As in category (b), this wrong interpretation of the justification resulted in an incorrect rejection of the verbal justification in the case of statements S1 and S6, but still resulted in correct judgments in the case of statements S3 and S4, Nevertheless, there is a difference to category (b), exemplified by the following explanation for the justification to statement S1: "This is not a general proof. We can ask what will happen for larger numbers, you cannot check all numbers" (T1). T1 clearly stated her view that the justification lacks generality. As opposed to category (b), in which T9 identified generalization for a subset of the cases, this teacher dismissed the justification on the grounds of lacking the critical attribute of generality altogether.

(d) *The justification starts with a numerical example, but all the rest is unnecessary and wrong*. While teachers' explanations in category b and c interpreted the justifications as composed of several numerical instances, explanations in this category perceived the presented justification as an "over doing". Here is an example of a teacher's explanation for statement S3: "[Moty] should have stopped after the first line. The fact that he continued showed his lack of understanding" (T3). In this case the teacher did not state whether, in her view, the part of the justification after the example is valid or not. Her criticism related to the fact that the student was (probably) not aware of the sufficiency of one counter example. Since Moty did not follow this minimal model, she rejected his justification.

This category of participants' explanations was found only for statements S3 and S4, for which a single example is a minimal and sufficient proof. The majority of the teachers were willing to accept a justification which has more than the minimal needed information as legitimate. However, quite a number of teachers (28% for S3, and 4% for S4) did not accept non-minimal proofs.

(e) *No explanation*. Twenty percent of the participants provided no explanations to their judgments. A possible cause for that is the nature of this statement: Existential, Never true statements are rarely addressed in school mathematics.

CONCLUDING REMARKS

In this paper we examined teachers' reactions to verbal justifications of elementary number theory statements, examining their tendencies to accept such justifications and their explanations for their related decisions. The teachers themselves correctly proved the six statements, either symbolically or numerically, but not verbally.

Our findings indicate that about half of the teachers rejected correct verbal justifications, failing to acknowledge the generality of these justifications. About 10% claimed that the justifications include only a subset of the cases in question, and about 35% saw the verbal justifications merely as an example. These findings substantiate findings reported by Dreyfus (2000), that teachers tend to perceive verbal proof as deficient because they lack of symbolic notations. However, Dreyfus (2000) found that teacher rejected in principle verbal justifications. Our findings indicate that teachers had difficulties in understanding what is written in a verbal justification, but not rejecting it in principle. Healy & Hoyles (2000) reported that high school students preferred verbal proofs due to their explanatory power, yet at the same time they expected to get low grades for such proofs.

The everyday practice of teachers involves a constant evaluation of students' justifications for statements. It is likely that verbal justifications of the kinds presented in our study will emerge during interactions with students. Therefore, it is important that teachers will be more familiar with verbal justifications. The findings presented here may serve in the design of mathematics teacher education programs, and of further research, which is needed to better understand and enhance teachers' knowledge of the many aspects of proofs.

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TEACHERS' KNOWLEDGE OF STUDENTS' CORRECT AND INCORRECT PROOF CONSTRUCTIONS[1]

Michal Tabach, Esther Levenson, Ruthi Barkai, Pessia Tsamir, Dina Tirosh and Tommy Dreyfus

Tel-Aviv University, Israel

This study investigates teachers' knowledge regarding their students' construction of correct and incorrect proofs within the context of elementary number theory. Fifty high school teachers were requested to suggest correct and incorrect proofs their students might construct. The suggested proofs were analyzed according to the mode of argument, mode of representation, as well as the types of knowledge evident in these proofs. Results indicate that teachers' suggestions of correct and incorrect proofs students might construct were not always consistent with research regarding students' proof construction.

Construction of proofs is recognized as an essential component of mathematics education (NCTM, 2000). Yet, it is by no means a trivial matter for teachers to introduce and guide their students in proof construction. While past research has focused on teachers' content knowledge of proofs (e.g., Dreyfus, 2000; Knuth, 2002), few studies have focused on teachers' related, pedagogical-content knowledge (PCK). An important aspect of PCK addresses students' ways of thinking (Hill, Ball, & Schiling, 2008). We address two main issues related to proof construction: the appropriateness of the method of proof and the types of representations that are often used when proving.

Different methods of proof may be used for different types of proofs. For a universal statement a general proof, covering all relevant cases, is necessary to validate the statement and a single counter example is sufficient to refute the statement. For an existential statement a single supportive example is sufficient to prove the statement. On the other hand, a general proof, covering all relevant cases, is necessary to refute the statement.

In studying teachers' proof PCK we need to account for what is known about students' thinking related to proofs. Studies have shown that students are not always aware of the necessity for a general, covering proof when proving the validity of a universal statement for an infinite number of cases (e.g., Bell, 1976). Healy and Hoyles (1998, 2000) found that 14-15 year olds have difficulties constructing a complete proof based on deductive reasoning. Balacheff (1991) found that students relate to counter examples as bizarre instances and do not always recognize a counter example as being sufficient to refute a universal statement. Regarding the types of representations used by students when constructing proofs, Bell (1976) found that none of the 36 high school students in his study used an algebraic proof when proving a numerical, universal conjecture.

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Healy and Hoyles (1998, 2000) found that students preferred verbal explanations over other kinds of representations.

This study focuses on high school teachers' knowledge of students' correct and incorrect proof constructions within the context of Elementary Number Theory (ENT).

METHODOLOGY

Participants, Tools and Procedure

Fifty high school teachers with 2 - 15 years of experience participated in this study. A questionnaire consisting of six ENT statements was handed out to all participants. This context was chosen as the related concepts were thought to be familiar to the teachers, enabling the teachers to focus on proving the statements and minimizing difficulties that may have arisen due to misunderstood terminology (see Table 1). The validity of each statement is determined by a combination of the predicate and the quantifier.

Predicate / Quantifier	Always true	Sometimes true	Never true
Universal	**S1**: The sum of any 5 consecutive natural numbers is divisible by 5. *True*	**S2**: The sum of any 3 consecutive natural numbers is divisible by 6. *False*	**S3**: The sum of any 4 consecutive natural numbers is divisible by 4. *False*
Existential	**S4**: There exists a sum of 5 consecutive natural numbers that is divisible by 5. *True*	**S5**: There exists a sum of 3 consecutive natural numbers that is divisible by 6. *True*	**S6**: There exists a sum of 4 consecutive natural numbers that is divisible by 4. *False*

Table 1: Classification of statements

For each of the six statements, teachers were requested to present correct and incorrect proofs that, in their opinion, students would give to these statements. (Note: Henceforth we refer to these six statements by their statement number.)

Analysis of the data

All proofs presented by the teachers were categorized according to their modes of argumentation as well as their modes of representation (Stylianides, 2007). Specifically, we were interested if the teacher would specify a mode of argumentation (such as stating that an example is sufficient to prove an existential statement). We were also interested in the mode of representation used in the proof. Analysis of the proofs resulted in three modes of representation: numeric, symbolic, and verbal.

Finally, the teachers' suggestions of incorrect proofs were categorized according to Fischbein's (1993) theory on the interaction between the formal, algorithmic, and intuitive components of mathematical reasoning. Mistakes related to the nature of proof were associated with a student's formal knowledge. Mistakes within a proof were related to algorithmic and intuitive knowledge. An example of each type of mistake related to S1 is shown in Table 2.

Knowledge Category	Example
Formal	Taking any x, $x + (x - 1) + (x - 2) + (x - 3) + (x - 4) = 5x - 10$ which is divisible by 5. But we may get numbers that are not natural because the student forgot to specify that $x > 5$.
Algorithmic	$x + (x + 1) + (x + 2) + (x + 3) + (x + 4) = 5x + 10$ $5x + 10 = 0$ and $x = -2$. Since x exists, the statement is true.
Intuitive	Because we're talking about 5 numbers, then it must be that the sum is divisible by 5.
Other	They won't pay attention that we are talking about consecutive numbers and therefore they will err.

Table 2: Categorization of mistakes according to knowledge type

RESULTS

Altogether, the 50 teachers presented 763 correct and incorrect proofs. As can be seen in Table 3, teachers presented more correct proofs than incorrect proofs. This was true for each of the individual statements, except for S6 where more incorrect proofs were suggested.

Proofs	S1 (Always true)	S2 (Sometimes true)	S3 (Never true)	S4 (Always true)	S5 (Sometimes true)	S6 (Never true)	Total
Correct	78	80	70	74	72	46	420
Incorrect	70	71	51	34	57	60	343

Table 3: Number of suggested correct and incorrect proofs per statement

In Table 4 we present the percentage of correct and incorrect proofs which explicitly mention the mode of argumentation employed in the proof. For instance, a suggested incorrect proof for S3 was, "Students will present a number of supporting examples in order to validate the statement." After giving a correct general proof for S5 one teacher added, "If the general rule exists then of course such numbers exist. The mode of argumentation was specified more often for existential proofs (S4-S6) than for universal proofs (S1-S3). This observation can perhaps be explained by the fact that universal, valid statements are the most frequent type of statements used in school mathematics, and therefore no need was felt to specify the mode of argumentation (it was probably regarded as obvious).

Proofs	S1 (Always true)	S2 (Sometimes true)	S3 (Never true)	S4 (Always true)	S5 (Sometimes true)	S6 (Never true)
Correct	36	59	63	77	74	56
Incorrect	69	70	65	76	81	72

Table 4: Percentage of proofs explicitly mentioning mode of argumentation

The modes of representation used by the teachers are presented in Table 5 as percentages.

Statement		Mode of Representation			
		Numeric	Symbolic	Verbal	Other
S1 (Always true)	Correct	12	74	14	-
	Incorrect	61	17	17	5
S2 (Sometimes true)	Correct	48	51	1	-
	Incorrect	65	22	10	3
S3 (Never true)	Correct	47	52	1	-
	Incorrect	43	36	19	2
S4 (Always true)	Correct	51	44	1	4
	Incorrect	46	32	19	3
S5 (Sometimes true)	Correct	58	42	-	-
	Incorrect	54	37	5	4
S6 (Never true)	Correct	10	80	10	-
	Incorrect	56	26	18	-
Total	Correct	39	56	5	-
	Incorrect	56	27	14	3

Table 5: Percentage of proofs according to modes of representation

We first note that verbal representations were suggested less than numeric or symbolic ones for both correct and incorrect suggested proofs. In general, teachers used numeric representations for their suggested incorrect proofs and symbolic representations for their suggested correct proofs. Specifically, for statements 2 through 5, statements which require only a supporting example or a refuting counter example, the modes of representation for correct proofs were distributed almost equally between numeric and symbolic representations.

However, for statements 1 and 6, which require a general proof, the mode of representation for correct proofs was overwhelmingly symbolic.

Finally, Table 6 presents the results regarding the types of errors which teachers suggested their students would exhibit when validating or refuting the different statements. Mistakes related to the formal component were by far the most frequent type of error suggested. Perhaps teachers were aware of the formal nature of proofs and believed that students may not have this knowledge. Perhaps, teachers believed that in this context the formal component of proofs is more important than the other, ENT related components.

Statement	Knowledge Types			
	Formal	Algorithmic	Intuitive	Other
S1 (Always true)	60	17	14	9
S2 (Sometimes true)	68	15	14	3
S3 (Never true)	46	35	17	2
S4 (Always true)	53	26	9	12
S5 (Sometimes true)	68	22	5	5
S6 (Never true)	50	25	20	5

Table 6: Percentage of errors relating to knowledge types

CONCLUSIONS

Teachers' suggestions of correct and incorrect proofs students might construct were not always consistent with research regarding students' proof construction. Regarding suggested correct proofs, teachers specifically mentioned that a general proof would be given for the first and last statements, whereas a numerical example would be given to validate or refute the other statements. These suggestions reflect the minimal, mathematical requirement for each of these statements. However, research has shown that students do not always recognize the necessity of constructing a general proof, and even when they do, they may still give an unnecessary example (Fischbein & Kedem, 1982).

Considering the modes of representation, few teachers expected their students to use a verbal representation. Conversely, studies have shown that students appear to prefer this mode over symbolic representations (Healey & Hoyles, 2000). Rather, teachers in this study suggested that their students would predominantly use symbolic representations when constructing correct proofs and numeric representations for incorrect proofs. Yet, studies have also shown that students use numeric examples both correctly and incorrectly. Finally, the suggested proofs were in accordance with the minimal, needed proof for each statement, and thus reflected the teachers' knowledge of the formal nature of proofs.

"It is indispensable for teachers to identify students' current knowledge, regardless of its quality, so as to help them gradually refine it" (Harel & Sowder, 2007, p. 4). The results of this study indicate that teachers ought to be introduced to studies regarding students' proof conceptions in order to increase their pedagogical-content knowledge related to proofs.

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COGNITIVE AND SOCIAL DEVELOPMENT OF PROOF THROUGH EMBODIMENT, SYMBOLISM & FORMALISM

David Tall

University of Warwick, United Kingdom

Proof is a construct of mathematical communities over many generations and is introduced to new generations as they develop cognitively in a social context. Here I present a practical framework for this development in simple terms that nevertheless has deep origins. The framework builds on an analysis of the growth of mathematical ideas based on genetic facilities set-before birth. It unfolds a developmental framework based on perception, action and reflection that leads to distinct ways to construct mathematical concepts through categorization, encapsulation and definition, in three distinct mental worlds of embodiment, symbolism and formalism, which provide the foundation of the historical and cognitive growth of mathematical thinking and proof.

INTRODUCTION

Mathematical proof in today's society uses a formal approach based on axioms and definitions, constructing a formal framework by proving theorems. Mathematics educators research the process, to analyze how mathematicians and students think mathematically and to provide a theoretical framework to improve the teaching and understanding of the subject. In doing so we build on the work of others. However, the human species thinks using a biological brain and the growth of knowledge depends on how this biological entity makes sense of the world. The ideas that we share depend on the concepts developed by our predecessors, our genetic inheritance and our personal experiences in society.

As we reflect on the nature of mathematical proof, we find a process that every mathematician claims to adhere to, yet none can formulate precisely without appealing to implicit meanings shared by the mathematical community. The central question here is to seek the essential foundations of mathematical thinking and proof as it grows within society and within the individual.

Mathematicians speak of 'intuition' and 'rigour', seeing intuition as a helpful personal insight into what might be true, but requiring a rigorous mathematical proof to establish the insight as proven. However, the intuition of a mathematician with a rich knowledge structure (that Fischbein, 1987, calls 'secondary intuition') is clearly more sophisticated than that of a child. It is therefore important to build a framework that takes account of the developing nature of individual mathematical thinking.

THE GROWTH OF MATHEMATICAL THINKING

In this section I outline a framework for mathematical thinking based, on the one hand, on the biological foundation of human thinking and, on the other, on mathematics as developed in our mathematical communities.

I define a 'set-before' as ' a mental structure that we are born with, which make take a little time to mature as our brains make connections in early life,' and a 'met-before' as 'a structure we have in our brains *now* as a result of experiences we have met before'. It is the combination of set-befores that we all share to a greater or lesser extent and the personal met-befores that we use to interpret new experiences that lead to the personal and corporate development of mathematical thinking. In particular, mathematicians come into the world as newborn children, so all of us go through a process of personal cognitive development within society as a whole.

After long periods of reflection, I was surprised to find that just three set-befores form the basis of mathematical thinking. The first is the set-before of *recognition* that enables us to recognize similarities, differences and patterns. The second is *repetition* that enables us to practice a sequence of actions to be able to carry it out automatically. The third is the capacity for *language* that gives *Homo sapiens* the advantage of being able to name phenomena that we recognize and to symbolize the actions that we perform to build increasingly sophisticated ways of thinking.

From these three set-befores, three different forms of concept construction are possible. First, recognition supported by language enables us to *categorize* concepts as formulated by Lakoff and his colleagues (e.g. Lakoff & Nunez, 2001). Repetition allows us to learn to perform operations procedurally by rote, but in mathematics we can also symbolize operations and *encapsulate* these processes as mental objects (as formulated by Piaget (1985), Dubinsky (Cottrill et al, 1996), Sfard (1991) and others), which Gray & Tall (1994) called *procepts*. Language allows us first to *describe* objects for categorization purposes and then to give *verbal definitions*, but a huge shift occurs when we use set-theoretic language to *define* objects to give formal axiomatic structures in advanced mathematical thinking.

Building on these set-befores gives three major ways in which mathematical thinking develops which I term *three mental worlds of mathematics:*

> A world of (conceptual) *embodiment* that begins with interactions with real-world objects and develops in sophistication through verbal description and definition to platonic mathematics typified by euclidean (and also non-euclidean) geometry.
>
> A world of (procedural-proceptual) *symbolism* that develops from embodied human actions into symbolic forms of calculation and manipulation as procedures that may be compressed into procepts operating dually as process or concept
>
> A world of (axiomatic) *formalism* based on axioms for systems, definitions for new concepts based on axioms, and formal proof of theorems to build coherent theories.

In each of these worlds, various phenomena are noted, given a name (which may be any part of speech) and then refined in meaning to give a *thinkable concept* that can be spoken or symbolized with varying levels of rich internal structure, and then connected together in *knowledge structures* (schemas). When thinkable concepts are analyzed in detail, they may be seen as knowledge structures, in a manner that Skemp (1979) described in his ‘varifocal theory’ where concepts may be seen in detail as schemas and schemas may be named and become concepts. This shift between knowledge structure and thinkable concept is, in John Mason’s phrase, achieved simply by ‘a delicate shift of attention’. Further details of the three worlds can be found in published papers available for download from my website: http://www.davidtall.com/papers.

THE COGNITIVE DEVELOPMENT OF MATHEMATICAL PROOF

The development of proof in mathematical thinking matures over a lifetime. The young child experiments with the world, making a grab at something seen, and after practice, developing the action-schema of ‘see-grasp-suck.’ Initially the child develops mentally by experiment. Literally, ‘the proof of the pudding is in the eating.’ As the child grows more sophisticated, proof develops in various ways based on the three set-befores and the individual learner’s met-befores.

Figure 1 shows the hypothesized cognitive development of the child in the lower left hand corner, developing through perception, action and reflection.

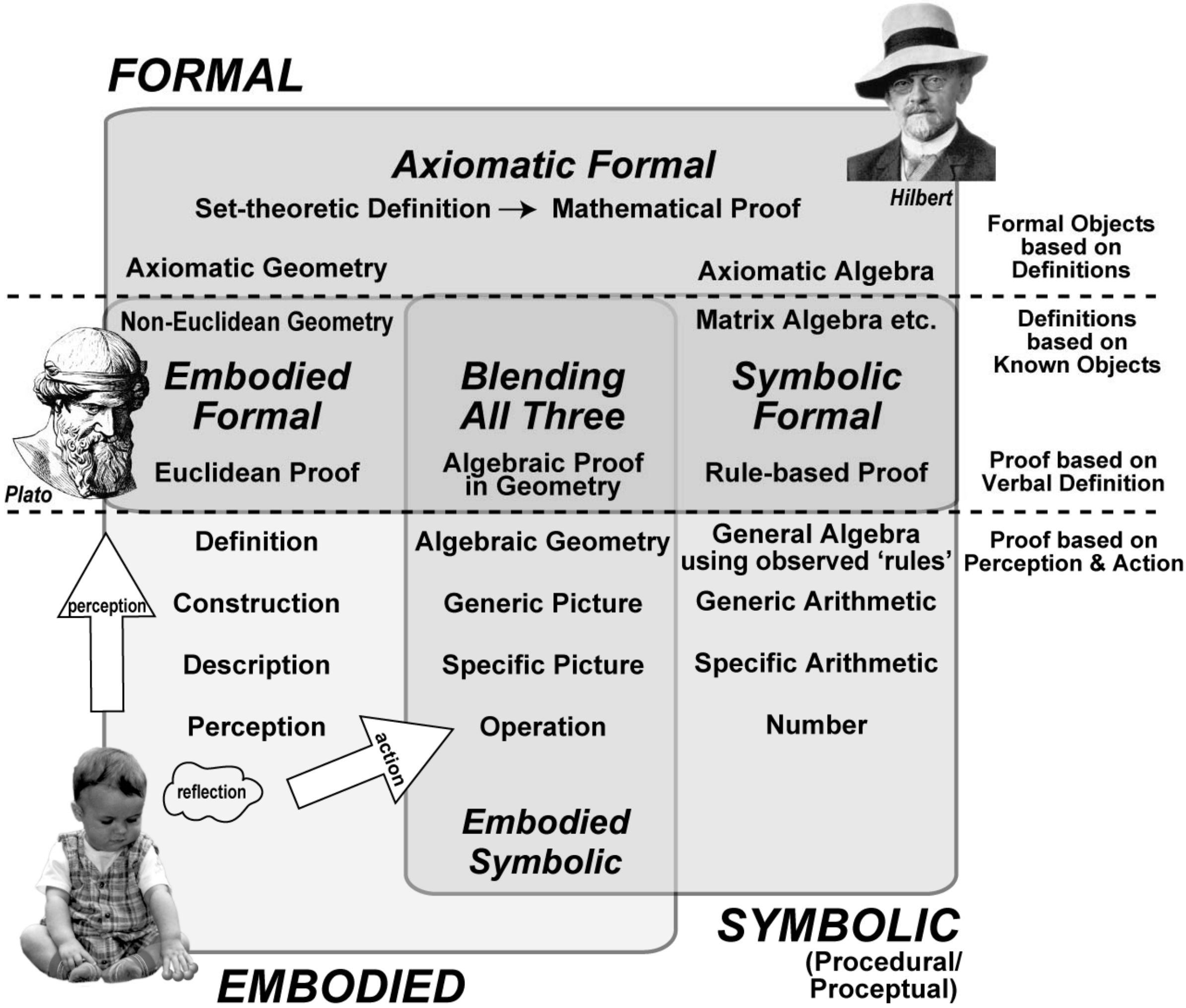

Figure 1: the cognitive development of proof through three mental worlds of mathematics

Perception develops in the embodied world through description, construction, and definition, leading to Euclidean proof represented by the bust of Plato. Even non-euclidean geometry is embodied, being based on mental embodiments of space that have different definitions from Euclidean geometry.

In parallel, the actions performed by the child, in terms of embodied operations such as counting, are encapsulated as symbolic thinkable concepts (procepts) such as number. Arithmetic develops through the compression of counting operations (count-all, count-on, count-on-from-larger) to known facts that may be used flexibly to derive new facts. Symbolic arithmetic benefits from blending with embodied conceptions, through a parallel development in embodiment and symbolism. For instance, the sum of the first 4 whole numbers can be seen as a succession of counters in rows of length 1, 2, 3, 4 and extended in each row successively by 4, 3, 2, 1, to give the sum 1+2+3+4 as a half of 1+4, 2+3, 3+2, 4+1, which is half of 4 lots of 5. This *specific* picture may be seen as a *generic* picture that works for any number of rows whatsoever, so that the sum of the first 100 numbers is $\frac{1}{2} \times 100 \times 101$.

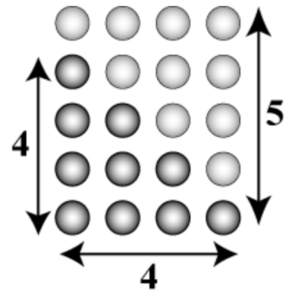

Figure 2

The specific and generic sum of the first few numbers may be generalized as an algebraic proof by writing the sequence 1, 2, …, n above the sequence in reverse as n, …, 2, 1, and adding the corresponding pairs to get n times $n+1$, to obtain a *general* algebraic proof that the sum of the first n whole numbers is $\frac{1}{2}n(n+1)$.

Symbolic operations develop from *specific* calculations to *generic* calculations, to *general* calculations represented algebraically. As this happens, the meanings of the 'rules of arithmetic' also develop in sophistication. Initially it may not be evident to the young child that addition and multiplication are commutative. For instance, calculating 8+2 by counting on 2 starting after 8 is much easier than calculating 2+8 by counting on 8 after 2. However, both calculations can be embodied by specific examples (for instance, seeing that a line of eight black objects and 2 white objects () can be counted in either direction to get the same answer, 8+2 is 2+8 is 10). Such *specific* pictures can again be seen to be *generic* in the sense that the numbers of objects can be changed without affecting the general argument.

A significant shift of meaning occurs when observed regularities such as the shape of a figure in geometry or the property of an operation in arithmetic are formulated in terms of a *definition*. For instance, a figure that has four equal sides with opposite sides parallel and all angles are right angles is called a 'square'. However, when it is *defined* to be a figure with four equal sides and (at least) *one* angle a right angle, a problem occurs. The young child may *see* that such a figure has four equal right angles, but the definition only insists on *one*. An embodied proof that a square, as defined, must have *four* equal angles can be performed by practical experiment in which four equal lengths are hinged to form a four-sided figure that can be placed on a flat table. Changing one angle automatically changes the others and it can be seen that if the one angle moves into a right angle, then the others do so as a physical consequence. Now the need for the implication is established. Such a embodied actions can also be carried out using appropriate dynamic geometry software such as *Cabri*, or *SketchPad*.

Embodied proofs continue to be of value to the learner as they mature, for instance it may be possible to translate them in terms of Euclidean proofs using congruent triangles. Embodied proofs can also be used to prove quite sophisticated statements, such as the fact that there are precisely five Platonic solids with faces given by particular regular polygon. Beginning with equilateral triangles, and considering how many can be placed at a vertex reveals that two are insufficient, three, four or five are possible, to give tetrahedron, octahedron and icosahedron, while six equilateral triangles fit to give a flat surface, so six or more is not possible. A similar argument with squares and pentagons reveals just one possibility in each case (the cube and dodecahedron). Hexagons and above do not fit to give a corner at all. Hence there are precisely five Platonic solids.

A second fundamental transition occurs in the shift from embodied and symbolic mathematics to the axiomatic formal world of set-theoretic definitions and formal deductions. Here, instead of a definition arising as a result of experiences with known objects, a definition is now given in set-theoretic terms, and the formal concept is constructed by proving theorems based on the definition. This leads us to the formal world introduced by Hilbert, as used today by most research mathematicians.

An example of the shift from symbolic formalism to axiomatic formalism can be seen in the nature of a proof by induction. Symbolically, it begins by establishing the truth of a proposition $P(n)$ for $n = 1$, then the general step that ‘if $P(n)$ is true, then $P(n+1)$ is true’ which is then repeated as often as desired for $n = 1$, 2, 3, … to reach any *specific* value of n. For instance, to reach $n = 101$, start the general step with $n = 1$ to get the case $n = 2$, and repeat the general step 100 times to reach $n = 101$. This is a *potentially infinite* proof. However, the formal proof using the Peano postulates is a *finite* proof in just three steps: first establish the proof for $n = 1$, then establish the general step, and then quote the induction axiom, ‘if a subset S of □ contains 1 and, when it contains n it must contain $n+1$, then S is the whole of □ ’. Applying this to the set S of n for which $P(n)$ is true establishes the proof in a single step.

As students pass from elementary mathematics of embodiment and symbolism to the axiomatic formal world of mathematics, they must build on their set-befores and met-befores. Two different paths can be successful. One is to *give meaning* to the definition by using images or diagrams or dynamic change, building on met-befores to construct a *natural* route to formal proof. The other is to *extract meaning* from the definition by focusing on the logic of the proof, to become familiar with the definition and the various deductions that can be made, to build a *formal* knowledge structure that does not depend intrinsically on embodiment (Pinto, 1998).

Formal mathematical proof can then lead to what are termed *structure theorems*, which give rise to new meanings for embodiment and symbolism. For instance, a vector space is defined by formal axioms yet there is a structure theorem that proves that a finite dimensional vector space over a field F is isomorphic to F^n, thus the formal axiomatic system can be embodied by a coordinate system that can also be used for symbolic manipulation. In this way, formal mathematics returns to its origins in embodiment and symbolism.

The individual cognitive development of proof, which relates directly to the long-term social development of proof, builds on the three set-befores of *recognition*, *repetition* and *language*, which give concept construction through *categorization*, *encapsulation* and *definition*, giving rise to three mental worlds of mathematics based on *embodiment*, *symbolism* and *formalism*. Each world develops proof in different ways: embodiment begins with experiment to test predictions, and shifts through description then definition to verbal euclidean proof in geometry. Symbolism develops proof through specific, then generic, then general calculations and manipulations, leading to proof based on rules derived from regularities in symbol manipulation. Formalism is based on set-theoretic definitions and deductive proof.

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CHALLENGES IN DEVELOPING PROOF UNDERSTANDING IN A TECHNOLOGY ENVIRONMENT

Denisse R. Thompson

University of South Florida, USA

Many students in courses that focus on developing understanding of algebra content and properties of functions have access to sophisticated technology, including graphing calculators with computer algebra systems. Such tools have the potential to facilitate students' reasoning abilities by encouraging students to make and investigate conjectures, particularly because many examples can be examined quickly to determine the potential truth or falsity of a statement. This paper explores challenges that may exist in helping students develop proof understanding when such technology is present and raises the question as to whether certain calculator "solutions" should be taken as proofs.

BACKGROUND

The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000) includes reasoning and proof as a standard for all U.S. students throughout all school grades, a position that recognizes the importance of proof and justification to the understanding of mathematics. As NCTM notes, reasoning is a habit of mind and students need opportunities to engage with reasoning and proof in many contexts, including courses in which technology plays a significant role.

Many educators and researchers have investigated students' learning and abilities with proof concepts when using a dynamic geometry drawing tool (e.g., Chazan, 1993; Hadas, Hershkowitz, & Schwarz, 2000; Laborde, 2000). But what about students' proof abilities when using other technologies, specifically graphing calculators that may have computer algebra systems? Dodge, Goto, and Mallinson (1998) offer examples in which students use a diagram, a graph, or computer algebra manipulation to justify some statement. Further, they raise the question as to whether these technology-based solutions should be considered proofs. As noted in Harel and Sowder (2007), proving involves both *ascertaining* to convince oneself of the truth of an assertion and *persuading* to convince others of an assertion's truth. Students who are accustomed to technology in all aspects of their lives (e.g., cell phones, I-pods, MP3 players) may consider proofs done via technology as providing evidence to both ascertain and persuade relative to the truth of an assertion.

The focus of this paper is on issues and/or challenges that arise when students are faced with tasks involving reasoning about algebra or functions when graphing calculator technology, including computer algebra systems, is available. The first section discusses the use of technology to explore conjectures, including the use of graphs, tables, and evaluations on the home screen and the conclusions students may draw. The second discusses challenges that arise when students use logic operators to test the truth of a statement. The third discusses the use of computer algebra systems to simplify manipulations that are typically a part of many proofs. The final section concludes with some issues for further research.

INVESTIGATING CONJECTURES

Many students who struggle to write a proof often struggle because they do not know how to start. Some educators (e.g., Epp, 1998) argue that teachers might ease students into proofs by starting with finding counterexamples, extending to statements where students identify known properties, and then focusing on writing direct proofs. Starting proof work with finding a counterexample can ease students into the proof process because they know how to begin – look for an example that makes the given statement false.

Another potentially effective strategy to engage students with reasoning is with true or false statements with an implied universal quantifier, such as the one in Figure 1 (Thompson & Senk, 1998). The given statement is a proposed conjecture because students must determine if it is true or false for an unstated, but implied, domain. Students do not know if they need an example that disproves the statement or a formal proof. So, a natural way to begin is to try many values of x, and technology is helpful for this purpose. If the student is lucky and quickly finds one instance that does not work, the conjecture is false; if the student finds many examples that work, the student might begin to believe that the conjecture is true and then attempt a formal proof.

Is the following statement true or false:
$\log(x + 3) = \log x + \log 3.$
Explain how you would convince another student that your answer is correct.

Figure 1. A sample conjecture about properties of logarithms

How might the presence of graphing technology help or hinder students' investigations? Many students may graph $y = \log(x + 3)$ and $y = \log x + \log 3$ and notice that the two graphs do not coincide for all values of x. Therefore, the proposed conjecture is false. Students might also generate a table of values for $\log(x + 3)$ and $\log x + \log 3$ and notice that the values are not the same for most values of x. So, again, the conjecture is false.

However, suppose the student inputs $\log(x + 3)$ and $\log x + \log 3$ on the home screen. The student might obtain the same value or might obtain different values, leading to a correct or incorrect conclusion. Figure 2 contains two sample responses.

Solution a Solution b

Figure 2. Two sample solutions to the logarithm statement in Figure 1

It is not clear whether students realize that the calculator evaluates each expression for the value stored in the calculator's memory for x, $x = 1.5$ for Solution a and $x = 10$ for Solution b. Notice that this type of error cannot occur with a scientific calculator because students cannot enter $\log x$ into such technology; rather they must enter the log of a specific value.

Thus, when students use calculator technology to investigate conjectures, they need to use it in a thoughtful manner and not as just a black box. For instance, Thompson and Senk (1993) had students in an algebra class investigate whether $y - 7 = 3(x + 5)^2$ and $y = 3x^2 + 30x + 80$ could represent the same parabola. Depending on the window chosen, the two graphs could appear to be the same, although they are not equivalent. So, students need to realize that graphs can only suggest a statement is true but cannot prove it. There may be values for which two expressions or equations are not equivalent that are not visible given the resolution of the screen.

Such technology has the potential to stretch students' thinking in unexpected ways. Consider the statement in Figure 3 (Thompson & Senk, 1993).

Determine how you could use a grapher to test the truth of the following statement for all values of $x \neq 0$, 1, or -1.

$$\frac{1}{x} = \frac{1}{x+1} - \frac{1}{x-1}$$

Figure 3. A conjecture involving rational functions

Most students rewrote the given statement as two equations, graphing $y = 1/x$ and $y = 1/(x + 1) - 1/(x - 1)$ and then observing that the graphs are not coincident. However, some students rewrote the given statement as a single equation, $y = 1/x - (1/(x + 1) - 1/(x - 1))$, and noted they did not obtain $y = 0$ with holes at 0, 1, and -1 (p. 175). In both cases, students essentially disproved the truth of the given statement.

As these examples illustrate, graphing calculator technology can help or hinder students' ability to reason about important ideas related to algebra and functions.

Yet, to be effective in building students' reasoning, educators need to focus on this technology in the classroom and its role in reasoning.

UNDERSTANDING AND INTERPRETING LOGICAL OPERATORS

Graphing calculators, with and without computer algebra systems, contain logical operators as part of their features. These operators can appear in unexpected ways when students use their technology as an aid to proof. Consider the item in Figure 4.

Prove the following trigonometric identity:
For all real numbers x for which both sides are defined,
$\tan x + \cot x = \sec x \bullet \csc x$.

Figure 4. A sample trigonometry identity proof task

Typically, technology is not needed to prove this identity; rather, students manipulate each side of the proposed identity independently to show that the two sides are equivalent. However, students who have access to graphing calculators, with or without computer algebra systems, may attempt to use this technology in unintended ways.

Figure 5 contains two different technology solutions used by students in an advanced mathematics class that would be taken just prior to calculus.

$\frac{\sin x}{\cos x}+\frac{1}{\tan x}=\frac{1}{\cos x}\bullet\frac{1}{\sin x}$	$\text{solve}\left(\tan x+\frac{1}{\tan x}=\frac{1}{\cos x\sin x}\right)$
True, did it on calculator.	True
Solution a	Solution b

Figure 5. Sample technology solutions to trigonometric identity in Figure 4

Without interviewing students, it is impossible to know exactly what students did. However, given the nature of the response, it appears that the student in Solution a simply typed the given statement into the calculator; the student in Solution b attempted to use the solve feature on a CAS. Note that on a TI84, similar approaches yield 1 instead of True. What should educators make of these responses?

If students were to graph $y = \tan x + \cot x$ and $y = \sec x \bullet \csc x$, the graphs would coincide. But teachers and many students do not consider graphs to constitute a proof, citing the fact that perhaps the graph contains holes not visible given the resolution of the screen.

But what does "True" mean in the two solutions in Figure 5? It appears that the calculator has found the equation works for all values of x which it has checked. But we do not know the number of decimal places that were checked or the possibility of round-off errors that may have caused some values to appear equal that were not. Perhaps there were some values that were not checked that might have disproved the statement. Although the calculator may have checked thousands of cases, students in advanced mathematics classes should know that examples, no matter how many, are not enough to constitute a proof.

So, here is a case where the presence of the technology might hinder students' reasoning ability. Students get "true" and believe that serves as a proof. However, this situation seems to be analogous to the situation with a graph. Just as a graph does not constitute a proof, neither should a calculator logical operator result of "True" constitute a proof. Both a graph and True provide evidence that the statement is *likely* true for all cases, but a formal proof requires other approaches.

USING COMPUTER ALGEBRA SYSTEMS AS A TOOL FOR PROOF

Students might use computer algebra systems with the statement in Figure 4 in other ways as illustrated in Figure 6 (Thompson & Senk, in preparation).

$$\tan x + \cot x = \sec x \bullet \csc x$$

$$\frac{\sin x}{\cos x} + \frac{1}{\tan x} = \frac{1}{\sin x \bullet \cos x}$$

Used CAS typed in $\frac{\sin x}{\cos x} + \frac{1}{\tan x}$ and hit ENTER. It simplified it to $\frac{1}{\sin x \bullet \cos x}$.

Figure 6. A solution using CAS to prove the statement in Figure 4

The solution in Figure 6 uses technology to ease the manipulative facility of the task without removing the general aspect of the proof. In this case, technology does not change the overall expectation of how the proof might proceed.

For many students, technology, such as computer algebra systems, might enable success with proof when students would otherwise be unsuccessful. The student whose response is in Figure 6 appears to understand the process of proof, and uses technology just to make the manipulative work easier.

CONCLUSION

The issues raised in this paper are challenges that educators will continue to face as students use more and more sophisticated technologies in their school curriculum. We need to discuss how these technologies can and should shape our expectations for proof and the standards we are willing to accept. Just as computers have provided an acceptable proof of the four-color problem, perhaps we will eventually need to consider graphing calculators and CAS as acceptable tools to provide proofs of important algebraic and function properties. As Harel and Sowder (2007) suggest, we need more longitudinal studies related to students' proof understanding; I would suggest that some of that research needs to occur for students who have multiple years exploring mathematics with powerful technologies. Educators need to discuss this issue of proof with technology as the number of students taking more advanced mathematics increases. The study group is one means for beginning such a discussion.

NOTE

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A CLASSROOM SITUATION CONFRONTING EXPERIMENTATION AND PROOF IN SOLID GEOMETRY

Denis Tanguay

Université du Québec à Montréal (UQAM), Canada

Denise Grenier

Université Joseph Fourier (UJF), Grenoble, France

We want to bring into the debate a classroom situation submitted to preservice teachers in France and in Quebec, situation in which experimentation should play a central role to solve the problem, without being sufficient to establish the result, a theoretical proof being necessary. Such a situation may lead to a better understanding of how the different phases are linked: exploration, conjectures, argumentation and proof. However, we will witness the difficulties of the students in acknowledging the necessity for a theoretical proof, and how experimentation, in conjunction with some usual classroom contracts about proof, may sometimes contribute to move students away from the relevant reasonings and proving.

1. TWO APPROACHES IN OPPOSITION

As it shows in the Discussion Document of ICMI Study 19, educational research on teaching and learning proof in mathematics is subject to a tension between two approaches in opposition. Very globally, the underlying problem derives from the clash between different forms of communication :

- everyday life communication, through which we use natural language to *convince* our interlocutor, without the quest for truth being necessarily the main concern, but where truth may well be at stake — this form of communication being now referred to as *argumentation* by math-education researchers;
- mathematical communication, where natural language and formal languages are combined, and which relies on validation mechanisms of its own, *formal proof* being the ultimate among them.

1.1. Focusing on the operational feature of proof

Some researchers, such as Duval (1992-93), are convinced that there is a deep gap between argumentation and formal proof (the French 'démonstration') :

> *The development of argumentation, even in its most elaborated forms, does not open up a way towards formal proof. A specific and independent apprenticeship is needed as regards deductive reasoning.* (op. cit., p. 60, our translation)

… and it may go as far as prescribing separation between heuristic tasks and working on proof. Indeed, relatively accurate and sophisticated teaching devices are proposed, where an apprenticeship of the keen deductive structure is targeted :

propositional graphs (Duval & Egret, 1989; Tanguay, 2005, 2007), identification of premises (Noirfoilise, 1997-98; Houdebine & al., 2004), resorting to sheets and files (Gaud & Guichard, 1984), etc. Whether their designers adhere to Duval's dichotomic standpoint or not[1], such devices clearly aim at acquiring competences which are particularly drawn on in proof: not using the thesis as an argument, distinguishing an implication from its converse, correctly using a definition, keeping a minimal control on the logical/formal structure of a mathematical writing, etc.

1.2. Focusing on the meaning of proof

At the other end of the spectrum, influenced by the works of Polya, Mason and Lakatos, stand researchers who estimate that proof should not be the object of an isolated teaching: by laying emphasis on abstract logical mechanisms, independently of the construction of concepts and results to which they are linked, one would reflect a distorted image of the mathematical activity, with proof being at the center, as the goal to achieve, rather than as a tool allowing a better understanding of meanings. The teaching here promulgated is conveyed through activities of exploration, experimentation, search for examples and counterexamples, enunciation of conjectures… It is forecast that the necessity of proving will naturally stem from the process — to validate a conjecture as well as to understand 'why it works' — and that one can expect a relatively smooth transition from argumentation to formal proof (Grenier & Payan, 1998; Grenier, 2001; Mariotti, 2001; Godot & Grenier, 2004...)

1.3. Transposition into the teaching context

The implementation, in the classroom or in the curricula, of the findings of any didactical study almost always goes with alterations, even distortions. The caricatured transposition of the approach described in § 1.1 could take the following form: *its teaching being confined to Euclidean geometry, proof should be produced according to a specific format*[2]*, in isolation from exploring and constructing activities. Statements to be proven are declared to be true ('Show that…') or are quasi-obvious from the figure*. Research has pointed out the pitfalls of this curricular trend. Indeed, according to several studies (Chazan, 1993; Hanna & Jahnke, 1993; Wu, 1996…), from such an approach stems, in students' mind, a strongly ritualistic conception of proving (Harel & Sowder, 1998), proof remaining meaningless and purposeless in students' understanding. Tanguay (2005, §3; 2007) diagnoses this ritualistic scheme as the psychological result of the student being vaguely conscious that he or she remains unable to unravel the

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[1] Designers' motivation may simply derive from the good old pedagogical precept according to which 'difficulties should be tackled one at a time'.

[2] For example, the two columns format in America, the three columns format, or the 'on sait que, … or, … donc' format in France. We reiterate that these are alterations from what has been proposed by research. Duval, for one, insists that there should not be any specific format imposed to the student when he or she is asked to write a proof.

terms of the contract : he or she thinks that proof is about truth of the called in propositions, while it is in fact about validity of the deductive chainings.

But on the other hand, Hoyles (1997) warns against what could be side effects from the second approach, against a too drastic shift from the first (§1.1) to the second (§1.2) in the curricula, with social argumentation leaving no room to reasoning and scaffolding genuinely deductive in nature :

> *Students* […] *are deficient in ways not observed before the* [recent UK] *reforms :* [they] *have little sense of mathematics; they think it is about measuring, estimating, induction from individual cases, rather than rational scientific process.* […] *Given that there are so few definitions in the* [new] *curriculum, it would hardly be surprising if students are unable to distinguish premises and to reason from these to any conclusion.* (op. cit., p. 10)

The aim of the present contribution to ICMI 19 is to report on an experimentation which partly support this warning.

2. THE EXPERIMENT

2.1. Didactical hypothesis

Our starting hypothesis is that understanding the process of proof in its entirety requires that students regularly be placed in the situation of experimenting, defining, modelling, formulating conjectures and proving, with formal proof thus appearing as a **requirement** in establishing the truth of the proposed conjectures. Solid geometry, a field where basic properties are not obvious, strikes us as a source of problems in which the conditions mentioned above may be combined. Indeed, the situation proposed here relates to the activities of defining (Phase 1 ; see § 2.2 below), of exploring via concrete constructions and manipulation (Phase 2), and to the necessity of resorting to proof in order both to validate the constructions done and to ascertain that no others are possible (Phase 2 and Phase 3).

2.2. The situation

The situation was explored in an experiment with students in the third year of a four-year teacher-training programme at UQAM, who are studying to become high school math teachers, and, in a second terrain, with pre-service math teachers in their third year of the *Licence de mathématiques* at UJF. The following three tasks were given to students. *Phase 1 :* Describe and define regular polyhedra. *Phase 2 :* Produce them with specific materials. *Phase 3 :* Prove that the previously established list is valid and complete. The tasks were detailed in a document given to the students. The researchers were both present at the UQAM session, but only one was there at UJF. The students worked in teams of three or four. At UQAM, two teams were filmed. We collected working notes from teams at both universities. More complete reports on the experiment will be available through two articles, still in process of evaluation. We will here focus on findings linked with the particular issue at stake.

3. ANALYSES AND FINDINGS

3.1. The definition phase

The first regularity property which spontaneously came to the fore is the congruence and regularity of the faces of the polyhedron : "The faces are all the same", "It's everywhere the same regular polygon", etc. Depending on teams, other criteria are added : convexity, closure (in the sense that it encloses a finite volume), inscribability in a sphere (one team) or its more fuzzy version : "The more sides it has, the more it looks like a ball". Neither of symmetry, congruence of dihedral angles or equality of degrees[3] is stated. Filmed Team 1 try to find a property about edges, and proposes the formula[4]

Nr of edges = (*Nr of faces* x *Nr of sides per face*)/2,

without realizing that it is true for any polyhedron. It stands out from this phase that students have great difficulty in conceptualizing the dihedral angle[5]. It is more or less surprising, considering that the only 'visibly represented' angles are those between two incident edges : "The angles, there is no need talking about it because, … because it's the polygons that form the angles". It will require the debate episode, when the researcher ask students to decide whether the following polyhedra are regular — the one formed by gluing two tetrahedra, and the star polyhedron (with dihedral angles greater than 180º) formed by gluing square-based pyramids on the faces of a cube — to bring the discussion on and a solution to the issues of equality of degrees and congruence of dihedral angles, dihedral angles being somewhat clarified.

3.2. The construction phase

It should be mentioned from the start that according to the presentation document, students were not to stay confined to construction, but were directed towards arguments relevant to the proving phase, which was to be next :

> *Which geometrical properties (number of edges, of faces, type of faces, angles) should be verified to ensure having a regular polyhedron? For instance, what occurs at a given vertex? Justify your answers. Try to construct as much regular polyhedra as possible…* (Presentation document)

The most striking misconception, shared by several teams (including the two filmed teams), and which explicitly reveals itself in Phase 2, is the following : regular polyhedra constitute an infinite family, with one polyhedron per type of (regular) polygon for the faces, with the number of faces increasing with *n*, the number of sides of these faces, and a resulting polyhedron closer and closer to the

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[3] The *degree* of a vertex is the number of edges (equivalently, the number of faces) incident to this vertex.

[4] We will see in Phase 2 to what extent this focusing on formulae will get.

[5] Yet all the observed students had already taken a course in Linear Algebra, in which the dihedral angle between two secant planes is defined and computed from the plane equations.

sphere (as n goes to infinity). This conception will bring most of the teams into attempts to construct a polyhedron with hexagons[6]. Filmed Team 1, for instance, constructs quickly the tetrahedron and the cube with Plasticine and woodsticks, and then manages to construct the dodecahedron, despite the instability of the construction. Eventually being supplied with jointed plastic hexagons (*Polydron*), the three teammates assemble about ten of these — all lying flatly on the desktop from the start — and then try to raise the ones on the fringe. They blame the rigidity of the material for not being able to do so and to construct effectively the expected polyhedron. They formulate the conjectures according to which the polyhedron with hexagonal faces will have one more 'level' (the French 'étage') than the dodecahedron (1+5+5+1), that it will have 26 faces (1+6+12+6+1) and that more generally, a level should be added each time n is increased by one.

Having constructed the tetrahedron, the cube and the dodecahedron, without its purpose being quite clear, Filmed Team 2 search for a formula which would allow to compute the number of edges knowing the number n of sides for each face. One of the student proposes $n+n(n-2)$, while pointing out on the cube to what corresponds each term of the sum. She notices that the formula does work for the tetrahedron, but not for the dodecahedron. She then proposes $n+n(n-2)+n(n-3)$, and none of the three teammates acknowledges that the formula does not work with the cube! They then assemble some hexagons, trying to understand the decomposition in levels of the hypothesized polyhedron. They formulate the conjecture that it has 20 hexagonal faces (1+6+6+6+1 : three levels plus two 'caps') and that in general, regular polyhedra have two levels when n is odd, and three levels when n is even. Without anymore manipulation done whatsoever, a formula pursuit ensues, bringing the team to propose the following computation : the number of edges is given by $n+n(n-2)+n(n-3)$ and to get the number of faces, one must divide by n (each face has n edges) and multiply by two (each edge touches two faces). The teammates are now giving 16 as the number of faces for the polyhedron with hexagonal faces, without showing annoyance that it does not fit with their previous level decomposition.

3.3. Proving phase and conclusion

None of the teams has been able to produce a satisfactory argumentation that there exists only five regular polyhedra, and that they are indeed the one exhibited in the debate episode of Phase 2. It should be said that the presentation document proposed steps based on the production of graphs (Schlegel's diagrams), and it may have complexified the access to proof. But nevertheless, elementary arguments (the faces around a vertex must not cover 360 degrees of angle or more) were accessible right from Phase 2, but were recognized by only two or three students, who won't be able to use them afterwards.

We still support the hypothesis that the situation contains all relevant elements for

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[6] The same phenomenon has been observed from primary-school teachers: see Dias & Durand-Guerrier (2005).

building a relationship between experimentation and proof, while discerning the role and status of each. Our concern here is about meaning and purpose allocated to the experimental process by a majority among the students. For them, the quest for regularity and formulae has overshadowed any other form of reasoning or judgement; to the point that, for example, members of Team 2 would admit, without batting an eyelid, two distinct and incompatible formulae (cf. § 3.2) to 'justify' the terms of a sequence of three integers! The prevalence of algebra in the curricula and of formulae associated to proving in class, the difficulties linked to dihedral angles (cf. § 3.1), the notion of cognitive unity proposed by Mariotti (2001), none of these are sufficient in our view to explain such a downswing, to understand why the potentiality offered by the experimentation have contributed so little in the commitment of the students to the proving process. We argue that there is here a matter of investigation for educational research, and it may start by being debated at ICMI 19.

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WHAT HAPPENS IN STUDENTS' MINDS WHEN CONSTRUCTING A GEOMETRIC PROOF? A COGNITIVE MODEL BASED ON MENTAL MODELS

Stefan Ufer[1], Aiso Heinze[2], Kristina Reiss[1]

[1] *University of Munich, Germany*

[2] *Leibniz Institute for Mathematics and Science Education, Kiel, Germany*

Based on psychological models of knowledge and deductive reasoning we provide a cognitive model of the proving process in geometry. We derive central prerequisites of the ability to construct geometry proofs and show how they interact within the model. In the sequel, we review empirical evidence in support of the model and discuss implications for research in mathematics education.

1 INTRODUCTION

One important basis for research and development in mathematics education are models about cognitive processes during individuals work on mathematical problems. In this contribution we will adopt some ideas from cognitive psychology to describe a model of cognitive processes related to geometric proof processes. Since we discuss the issue of proving from this perspective our approach emphasize different features than for example Boero's model (Boero, 1999) which take a mathematical perspective or Duval's ideas (Duval, 2002) which describes cognitive obstacles of valid reasoning in terms of the content, epistemic value and status of propositions.

Like for any model, we do not claim that our model is identical to reality, but we want to provide a framework to derive predictions about the ability to prove and possible related problems. In this short contribution we restrict ourselves to cognitive aspects, having in mind that there are also beliefs, affects, and interests influencing the understanding of geometric proof. Moreover, here we consider only direct proofs though the model can be extended to other proof types.

2 A COGNITIVE MODEL OF THE GEOMETRIC PROVING PROCESS

In this section we will describe a model of individual cognitive processes during the proving process. This model assumes that basic features of the concept of mathematical proof are known to the students, e.g. that they are to some extent aware of the epistemological values and the functions of proof (e.g., Hanna & Jahnke, 1996).

The ability to construct proofs requires certain knowledge and certain skills to be mastered by an individual. In the sequel we will address particularly the content knowledge, the ability to reason deductively and metacognitive competencies which are essential for our cognitive model on geometric proving.

Structure of content knowledge

To be able to construct proofs in any mathematical field, an individual needs knowledge about the content of the field. On the one hand, this content knowledge includes conceptual knowledge connected to central notions of the field and mathematical connections between them (logical relations between properties of geometric figures, hierarchical classification of figures, connection with figural representations of the concepts). On the other hand, knowledge on a mathematical content can be available as procedural knowledge, i.e. automatized procedures to achieve certain goals (e.g. Anderson, 2004). These procedures are usually bound to a specific context, e.g. calculation problems: "If the goal is to calculate an angle in a triangle, and the other two angles are given, then calculate it as 180° minus the two given angles". In the context of proof it is possible that well known proof steps or even a combination of proof steps are already coded as procedural knowledge which are activated by the recognition of certain features of a proof problem (e.g. subfigures in a geometric figure). In this case, a deductive reasoning process – based conceptual knowledge – is not necessary for this part of the proof.

Deductive reasoning by mental models

The ability to perform a deductive reasoning step must be regarded as a potentially important predictor of proof competency. Johnson-Laird, Byrne, and Schaeken (1992) describe the process of deductive reasoning not in terms of formal or symbolic manipulation of logical statements, but rather as a cognitive activity involving semantically rich mental models of the reasoning context. In a geometric context, giving a figure of the problem situation supports the construction of this mental model. Nevertheless, a mental model may as well include non-figural information such as the congruency of certain line segments or angles.

The first step to solve a proof problem is thus to construct an adequate mental model, which is usually provided – at least partially – as a figure within the problem formulation. Second, a first conclusion is derived from the model (on the basis of what would be described as an inductive reasoning process in mathematics education). In the model of Johnson-Laird et al. the validity of the conclusion is finally checked by ensuring that no alternative model of the premises exists that contradicts the conclusion. For a mathematical proof step this kind of deductive reasoning is incomplete and not sufficient. An argument has to be provided that meets the requirements of the mathematical community and ensures that such a model does indeed not exist. Finally, the conclusion is incorporated into the mental model of the situation as additional information and – if the proof is not finished - the process starts again.

Deductive reasoning and content knowledge

Several aspects of content knowledge in geometry seem to be necessary to perform deductive reasoning in a geometric proving process. A mental model as described in the previous section is not equal to a simple mental graphical image as a copy of a figure given in the geometric problem. Usually, a given figure is restructured on the basis of the individuals' knowledge already in the process of perception. Based on prototypical pictorial concept representations that occur as subfigures, possible conclusions might be drawn automatically if the corresponding arguments are mentally connected to the concept representations. This may occur in two different ways: (1) The recognition of the concept representation activates the proper argument (e.g. the name of the corresponding theorem) and thus leads to a kind of automatic deductive inference. (2) For a pictorial concept representation (e.g. two vertical angles between two crossing lines), only the conclusion as property (the two angles are congruent) is activated as knowledge. This helps in particular for calculation problems. If the argument is not available automatically, it has to be found within the base of conceptual knowledge, which is a more complex cognitive task.

Moreover, if several conclusions are drawn automatically, triggered by recognition of familiar subfigures, then a multi-step proof (from a mathematical perspective) can be reduced in its cognitive complexity. In special cases a multi-step proof problem may be represented and processed mentally even as a single-step problem.

The case of proofs with more than one step

Empirical studies have shown that proof problems with more than one step are usually much more difficult for students than one-step proofs (e.g. Heinze, Reiss, & Rudolph, 2005). Some differences between these two classes of proofs have been described by Duval (2002). In order to construct proofs consisting of several proof steps (i.e. the use of several theorems that are not likely to be applied within one step of deductive reasoning), a more complex planning and coordination of the proving process is necessary. Usually, the reasoning process starts by establishing at least one potentially valid intermediate hypothesis from the mental model and then filling the gap between these intermediate hypotheses and the assertion by constructing a proof step. Nevertheless, these intermediate hypotheses and the conclusion for possible proof steps cannot be constructed at random. There are a lot of possible inferences that may be drawn from one model, but usually an individual chooses an inference that is likely to bring it closer to the assertion of the proof. To judge which arguments are more useful in this sense, the individual needs knowledge to judge the “distance” between the actual state of the mental model and the assertion (in the sense of reducing the problem space).

Spatial and structural aspects of the mental model may influence this judgment. A very rough, but sometimes helpful, measure may be the spatial distance of two elements in the mental model – close elements may be related in some sense. Apart from this spatial distance, two elements may also be connected on the basis of a geometric structure that was identified when perceiving the original figure. For example, two angles may be regarded as being connected very closely, if they are angles in a rectangular triangle, even if they are very far apart within the original figure. As in this example, this connection is mediated by relations between several mathematical concepts (angles, triangles, rectangular triangles, common sides etc.). To detect the connection between the two elements, these conceptual relations must be available to the individual in the sense of mental associations, which are modeled by chunks (cf. Anderson, 2004). Thus it can be expected that a better quality of conceptual knowledge, in the sense of more elaborated chunks, may lead to a better ability to plan and supervise the proving process. In some cases such conceptual connections may be helpful even if they are not supported by the structure of the figure in the mental model. Apart from the amount of chunking within conceptual knowledge a further aspect may influence this ability to judge the distance of two mental models: The possibility to find structural connections increases with the amount of structure detected within the model. So individuals with a better ability to structure their mental models can be expected to perform better when planning their proving process.

Proving and metacognition

Apart from these content-specific aspects, more general skills and strategies influence the proving process. In research on problem solving, several prerequisites were identified (see Schoenfeld, 1992 for a summary). Among these are the availability of problem-solving strategies like working backwards and forwards, identification of invariants, use of symmetries and patterns etc. Moreover, skills for monitoring and eventually adapting the plan for the problem solving process are necessary to avoid an effect that could be identified in novice problem solvers: Some of them fail because they start with applying a specific strategy after a short process of planning and follow this strategy regardless of success or frequent failure of their attempts. Reflective processes and an adaption of strategies were rare (cf. Schoenfeld, 1992). Of course these processes monitoring depend on knowledge on the problem situation (mental model and "distance" between models), but the use of this information requires further complex metacognitive abilities.

According to this, the influence of problem-solving skills is of course mostly expected for multi-step items. Nevertheless, since a mental model cannot be arbitrarily detailed, it should not be neglected that also structuring of the mental model for single-step proofs requires some coordination skills.

3 SUPPORT BY EMPIRICAL DATA

The model described above is derived mainly from psychological theories and observations from research in mathematics education. In the sequel, we will shortly review some empirical data supporting the described model.

In an explorative study (N=341), Ufer, Heinze and Reiss (2008) compared the effect of conceptual knowledge, procedural knowledge in calculation contexts, and problem solving skills on the ability to construct geometric proofs. All three predictors showed significant influence, but the influence of conceptual knowledge was much stronger than the effect of the two other predictors. This is in line with the model described, since quite general problem solving skills can only be applied, if sufficient knowledge is available. In a longitudinal study (N=196), Ufer and Heinze (2008) found that the difficulty of multi-step proof problems that admit automatized reasoning decreased within one school year.

Further empirical evidence for the reasoning by mental model theory can be derived from a small intervention study (N = 64) by Cheng and Lin (2006). They succeeded in fostering students' ability to construct multi-step proofs by showing strategies to structure the figure given with the proof problem - and thus to help students enriching their mental models - using different colors while reading the problem (reading and coloring strategy, Cheng & Lin, 2006).

Moreover, the study of Weber (2001) can be interpreted in the sense of our approach. He investigated a difference between undergraduate students and doctoral students in mathematics when solving mathematical proof problems. Both groups had the necessary mathematical knowledge for the proof tasks but the undergraduates showed problems in activating it in the proving process. Weber (2001) explained this by lack of strategic knowledge. In the context of our approach applying this strategic knowledge can be described as the ability to distinguish helpful proof steps from irrelevant ones by judging the "distance" between the actual state of the mental model and the assertion of the proof. From this perspective, the undergraduates were not able to select the right theorems and concepts from their knowledge, because they had no idea, which direction would bring them closer to the solution. The number of irrelevant inferences reported by Weber (2001) supports this interpretation.

4 IMPLICATIONS FOR RESEARCH ON (GEOMETRIC) PROOF

If our model proves to be useful for describing cognitive processes relevant to the construction of proofs in geometry, then several questions for further research arise. First, the exact distinction and the interdependency between conceptual and procedural knowledge is still an open problem, not only from a psychometric and psychological perspective, but also from a didactical point of view (e.g. in the context of arithmetic, Gilmore & Inglis, 2008).

Apart from lack of content knowledge, the problems of students with geometry proof may arise from their ability to construct and to use adequate mental models of the problem situation. A model may merely consist of a mental image of the figure given in the task (e.g. a special rectangular triangle), or it may contain further information encoding possible alternative models (e.g. a class of triangles with a common hypotenuses and one point on the Thales' circle). If a student can construct generic mental models, invalid conclusions become less probable and thus the complexity of the proving process reduces. Possibilities to foster this ability to construct more generic mental models are a challenge for mathematics instruction, for which in particular dynamic geometry tools may be useful.

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MATHEMATICAL GAMES AS PROVING SEMINAL ACTIVITIES FOR ELEMENTARY SCHOOL TEACHERS

Greisy Winicki - Landman

Department of Mathematics and Statistics

California State Polytechnic University, Pomona – USA

This presentation draws on an initiative with pre-service elementary school participants. The description of the design and the implementation of a course based on mathematical games and the analysis of a specific game with some of its variations are presented and discussed. Different roles of mathematical proofs that emerged in the analysis of that game will be analyzed.

INTRODUCTION

Although proof is a key component of mathematics, there is no agreement about how this central role should be echoed in mathematics education. Sometimes, mathematics teachers expose their students to the ritual of proving but not enough to the need and the essence of a mathematical proof. Mathematical proofs must be an integral part of mathematics teaching at all grades, including elementary school. Hence, elementary school teachers need specific opportunities to develop their own experiences proving, to broaden their own conceptions about proofs, to explore different ways to present a mathematical proof, and to be aware of the different roles proofs play in a mathematical endeavor.

In the framework of a course for pre-service elementary school teachers, I decided to expose the participants to mathematical games. A mathematical game is a special kind of mathematical problem in which two players take turns making moves and a player cannot decline to move. The problem is always to find out which player – the first or the second – has a winning strategy. A *winning strategy* is a complete description of the moves a player can choose in order to guarantee a win under every possible circumstance. A winning strategy is said to be so when it is well formulated and it is proved to lead always to a win. The 'flavor' of the game vanishes when the two player know a winning strategy. This is one of the main differences between a mathematical game and other games.

From a *pedagogical perspective*, mathematical games constitute a unique opportunity to engage the participants in meaningful mathematical activities because they facilitate mathematical communication as well as reinforcement of key mathematical concepts using a non-intimidating approach. From the *content perspective*, mathematical games foster a suitable environment for the discussion of proof because they enable: to collect cases and to look for patterns; to work systematically; to formulate conjectures; to test conjectures and refine them if needed; to decide whether to look for a proof or a refutation, and when to do so; to set up standards of an argument to constitute a proof; to identify the certainty factor embedded in a proof; to analyze a proof and look for possible generalizations.

THE SETTING

A fourteen-weeks course was developed and taught as part of the Teaching Certificate Program for elementary school teachers. The first step of the design of the course consisted of the selection of almost fifteen mathematical games. Some of them are described by Fomin et al (1991) and Winicki-Landman (2004, 2005). This seleection followed four criteria: connection to contents taught in elementary school; possible implementation of different types of problem-solving heuristics; simplicity of needed equipment; and possibilities to formulate variations and generalizations.

After the games were selected, a sequence of games was build and each meeting was planned. The format of each ninety-minutes meeting was essentially invariant. During the first ten minutes, the rules of a game were presented and demonstrated orally. The participants played against the instructor and some of the examples remained on the board. During the following twenty minutes, the participants played the game in small groups and collected data spontaneously. After that time, they were encouraged to formulate conjectures of a winning strategy and to test them by playing against the instructor or among themselves. They got five more minutes to refine them and to test them within their small groups. For homework, the participants had to write a) the rules of the game, b) the formulation of a winning strategy and, c) a proof of the winning strategy.

A SNAPSHOT OF THE OUTCOMES

One of the goals of this course was to provide pre-service elementary school teachers with specific opportunities to develop their own experiences doing genuine mathematics in general and proving in particular. Other goals of this course were: to broaden their participants' conceptions of proofs; to develop their awareness of the different roles played by proofs in the process of doing mathematics; to explore different ways to present a mathematical proof and; to evaluate whether a justification constitutes a proof.

To illustrate the learning potential embedded in these games, a specific mathematical game will be analyzed following the steps suggested by the participants. This game was presented almost at the end of the course and the participants were reasonably familiar with the complete process of analysis a mathematical game as it was modeled during the course. This process includes playing the game several times, formulation of conjectures connected with the winning strategy, testing the conjectures by playing the game following the conjectured strategy, and finally, looking for a justification of whether the conjectured strategy is indeed a winning strategy.

The observations and the results are presented following the original order of appearance during the implementation of the game in class and the subsequent discussion. The current proofs, although inspired by the ideas suggested and formulated by the participants, were polished and written later by me for communication purposes with the reader.

Rules of the game

A natural number is written on a blackboard. Players take turns subtracting from the number on the blackboard one of the natural powers of 2 that is smaller than the number, and replacing the original number with the result of the subtraction. That result is to be a positive number. The player who writes the number 1 wins the game. After the explanation of the rules, a couple of examples were presented in order to illustrate the game and verify the understanding of the rules.

The participants were invited to play and to keep in mind the following questions: a) Is there a sequence of moves that leads one of the players to a certain win? b) Which sequence is it? c) Which one of the players may be able to use it?

After fifteen minutes of playing the game, the following observations were made by the participants. I was the moderator of the interchange of ideas and also responsible mainly for maintaining the consistency of the mathematical language.

Observation 1: If player A leaves the number 2, player B has exactly one legal move (2-1=1) and by doing so B will win by leaving 1. Thus, 2 is a losing number, because the player that leaves it, loses the game.

For communication purposes, a working definition was introduced. A number is a *winning number* for a player if whenever it is left by this player, he/she has at least one way to win the game.

Observation 2: If player A leaves the number 4, player B may subtract 1 or 2. On his turn, player A can subtract 2 or 1 correspondingly and by doing so A wins the game. Thus, 4 is a winning number because for every move of player B, player A has a way to reach 1 and by doing so, player A wins the game.

Observation 3: If player A leaves 7, player B may subtract 1, 2 or 4. On his turn, player A can subtract 2, 1, or 2 correspondingly and by doing so player A leaves 4 or 1, which are winning numbers. Thus, 7 is a winning number.

Representing this justification by means of a tree was suggested and the participants appreciated the visual representation of the verbal statement based on an exhaustive type of reasoning. Some students saw in the construction of such a diagram a possible method to be used in other cases.

Recognizing a pattern among the winning numbers - 1, 4, 7 -, a participant asked whether the number 10 is a winning number. Using another tree he managed to prove that in this game the number 10 is indeed a winning number. A conjecture was formulated: in this game, all the natural numbers of the form 3n+1 are winning numbers. At this moment I thought "… and only them", but I decided not to interfere with the flow of the events. After playing some more rounds, the participants, had no doubt that the conjecture was true but they wanted to know "*where does the coefficient 3 comes from*." They were already convinced that the result was true but they were looking for a proof that explained *why* it is so. This was a delicate decision for me: my first thought was to suggest a proof by Mathematical Induction but I immediately abandoned the idea because: a) these proofs tend to be "proofs that prove" but rarely "proofs that explain" (Hanna, 1989); b) the participants were not very familiar with this type of proofs. The challenge to find an *explanation* but, definitely not to the need for more certainty, lead me to suggest the following exercise. Choose any two consecutive powers of 2 and add them. What do these sums have in common? The sum of two consecutive powers of two is always a multiple of three because:

$$2^k + 2^{k+1} = 2^k(1+2) = 2^k \cdot 3 \quad k \in N$$

The participants were asked to consider some of natural powers of two and to divide each one of them by 3. The data was organized in a table that led the students to formulate powerful observations: the remainders of these divisions cannot be zero because 2 and 3 are relative prime numbers; the remainders constitute an alternating sequence of 1's and 2's; the powers of 2 can be classified into two categories: those that leave remainder 1 when divided by 3 and those that leave a remainder 2 when divided by 3; if the exponent is even, the remainder is 1 and if the exponent is odd, the remainder is 2. These discoveries explained the connection between the powers of 2 and the number 3 as a coefficient in the pattern found for the winning numbers.

It was really surprising to all the participants to discover that while the numbers to be subtracted in this game constitute a geometric sequence, the winning numbers constitute an arithmetic sequence. Using the categorization presented by Movshovitz-Hadar (1988), this was a surprise of the type a common property in a random collection of objects because at the first steps of the game, the participants could not suspected that there exists a simple pattern that describes all the winning numbers.

In order to check the participants' understanding of the results proved I formulated the following questions: Assume the chosen number is 1000. What player has a winning strategy? What if the chosen number is 1001? And what is it is 1002? The participants worked on them for a couple of minutes. Through the application problems, they observed that the proof not only proved the existence of winning numbers and showed their form, but also helped them make decisions on how to proceed in order to stay on the 'good track' of the winning numbers. In that sense, that proof played a role of *illumination* or *explanation*, as described by de Villiers (1999).

After that comment, a participant suggested some variations of the game: what if we play with powers of 3? Knowing the potential of her proposal, I assigned the analysis of this variation as a homework problem.

CONCLUSION

Although the game presented is easy to play, its mathematical richness is immense: it involves basic concepts like power of a number, base, exponents and more advanced ones like arithmetic sequences, modular arithmetic, and complete induction. This was, maybe, one of the factors that made the participants willing to be involved in the enterprise of analyzing the game and formulating a strategy to win.

During the activity described, proof appears as a natural step in the mathematical discourse and not as an imposed ritual: the students asked for the proof in order to be sure that by following the strategy they will win "no matter what". Using the terminology presented by de Villiers (1999), for these participants the proof served as a verification or confirmation tool. Some others wanted to know where these patterns came from or why are they arithmetic sequences, and the answer to each one these questions is also within the proof of the theorem. For them, the proof served as an explanation tool.

It is important to note that there was a conjecture that the participants formulated after looking at the pattern of numbers in a table and it was accepted as true without proving it. Sometimes this can be dangerous, but in this case the participants were convinced of the truth of the conjecture and they didn't need its proof. It is possible that if the instructor would had tried to lead towards its proof, it would have being done but the rhythm of the discussion would have been broken, as well as the participants' feeling of ownership of the product that was been created. This feeling of ownership and genuine involvement in the development of the discussion is illustrated by the number and the quality of the suggestions made by the participants.

For all the participants, even if they were not aware of it, the approach chosen for the proof of the strategy of the first game, allowed its generalization because the same 'pattern-of-proof' appeared during all the activity. It also enabled the discovery of the winning sequences without even playing! So, it can be said that proof also served as means of discovery.

This type of seminal situation created by a mathematical game and the chain of its different variations constitute a concrete opportunity for participants to be involved in doing new mathematics. But, in order to be successful, this situation needs to be seasoned with a bit of intellectual surprises of the type described by Movshovitz-Hadar (1988) and loads of tolerance and respect among the participants.

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OPERATIVE PROOF IN ELEMENTARY MATHEMATICS

Erich Ch. Wittmann

Technische Universität Dortmund, Germany

The paper is based upon an understanding of mathematics education as a "design science". By means of typical learning environments the notion of "operative proof" and some aspects of its theoretical background are explained. Particular emphasis is put on embedding operative proofs into the practice of skills.

The basic intention of the "elementary mathematics research program" as delineated in Wittmann (1988) is to make mathematics accessible in an authentic way also at the elementary level to both student teachers and students. In this program the grammar of non-symbolic representations and the notion of "operative proof" are playing crucial roles. The present paper gives an account of how progress in these fields has been made practical in the developmental research carried out in the project "mathe 2000".

In the first section some learning environments are described which include operative proofs. These examples serve as illustrations for the second section in which the notion of operative proof is defined, and for the last section in which some aspects of the theoretical background will be discussed.

1. LEARNING ENVIRONMENTS INCLUDING OPERATIVE PROOFS

The following learning environments from the "mathe 2000" curriculum cover the whole spectrum from grade 1 to 4. It is at this level that the specific features of operative proofs become particularly clear.

1.1 Even and odd numbers

In primary mathematics counters are a fundamental means of representing numbers in primary mathematics. Usually they are understood as "teaching aids" which have been specially invented for this purpose. However, their status is primarily not a didactic, but an epistemological one: Greek arithmetic at the times of Pythagoras underwent a period which is called "ψηφοι-arithmetic" and can be considered as the cradle of arithmetic (Damerow/Lefèbre 1981).

In the "mathe 2000" curriculum odd and even numbers are introduced in grade 1 in the Greek fashion by means of double rows of counters and double rows with a singleton (Fig. 1).

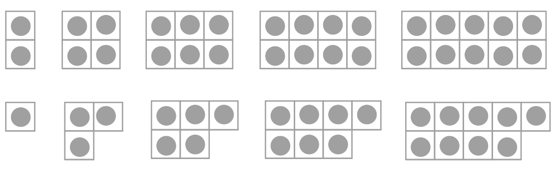

Fig. 1

These patterns are painted on cardboard and cut off so that children can operate with the pieces and form sums of numbers. The first exercises make children familiar with the material. The next exercise asks for finding sums with an even result. This is a first suggestion to look at the structure more carefully.

The subsequent task is more direct: Children are asked to reflect on the results of the four packages of sums in Fig. 2: “*What do you notice? Can you explain it?*”

$4 + 6 =$	$5 + 1 =$	$2 + 1 =$	$1 + 8 =$
$6 + 8 =$	$7 + 3 =$	$4 + 3 =$	$3 + 6 =$
$8 + 4 =$	$9 + 5 =$	$6 + 5 =$	$5 + 4 =$
$10 + 2 =$	$5 + 7 =$	$8 + 7 =$	$7 + 2 =$

Fig. 2

At this early level teachers are expected to refrain from pushing the children. All they should do is to listen to children’s spontaneous attempts in coming to grips with the underlying patterns.

In grades 2 and 3 even and odd numbers are revisited in wider number spaces. This is necessary as some children have to realize that 30, for example, is an even number although 3 is an odd number. Children are again given little packages of tasks similar to those in Fig. 2 with bigger numbers and asked the same questions. At this level the even/odd patterns are recognized more clearly and expressed in the children’s own words more precisely. In the manual teachers are “admonished” to be content with children’s spontaneous explanations and not to enforce a “proof”.

In grade 4, however, children are expected to have enough experience with even and odd numbers and to be ready to prove by means of the above representations that the sum of two even numbers and of two odd numbers is always even, and the sum of an even and an odd number always odd. Children realize that no singletons occur when even patterns are combined and that in the case of two odd patterns the two singletons form a pair and yield again an even result. Furthermore children see that the singleton is preserved if an even and an odd pattern are combined and that in this case the result is odd. The teacher’s task is to take up children’s attempts and to assist them in formulating coherent lines of argument.

The formal proof will be addressed in higher grades by using the language of algebra. As a matter of fact it expresses exactly the same relationships. As experience shows operations with patterns of counters are an excellent preparation for algebraic calculations.

1.2 A learning environment on long addition

This unit for grade 3 starts as an exercise in long addition and uses cards for the digits 1 to 9. The students are given the following task: *Use the digit cards from 1* *to 9 to make three three-digit numbers and add these numbers so that you get a sum below 1 000*. From the mathematical point of view the restriction "below 1000" is not necessary. However, below 1000 there are 26 different results, a set that can be well managed by third graders.

In the first round the students calculate according to the given rule, and get many results, including of course wrong ones. In the next round the results are collected and compared, then ordered and with some hints from the teacher the students discover patterns, identify wrong results and correct them, fill gaps, and, perhaps with the teacher's assistance, they eventually arrive at the complete list of possible results: 774, 783, 792, 801, 810, 819, 828, 837, 846, 855, 864, 873, 882, 891, 900, 909, 918, 927, 936, 945, 954, 963, 972, 981, 990, 999.

These numbers are exactly the multiples of 9 in the interval [774, 999] or the numbers whose digits add up to 9, 18 or 27 (divisibility rule for 9).

The standard proof of this pattern uses modular arithmetic and is not accessible to students at this age.

The following operative proof uses a representation that the students are familiar with: the place value table. The basic idea is due to Heinrich Winter (1985). In this representation the total of the digits of a number has a very concrete meaning: it is just the number of counters needed for representing the number on the place value table.

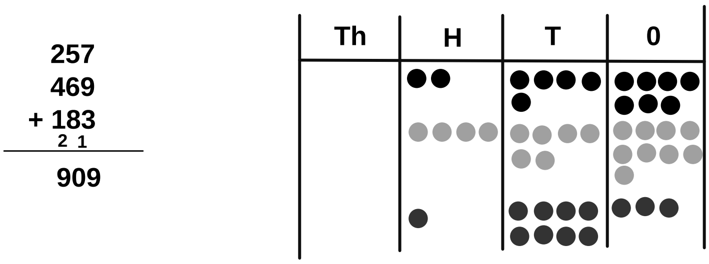

Fig. 3

Fig. 3 shows a calculation and its representation on the place value table: Altogether $1 + 2 + \ldots + 9 = 45$ counters are needed for representing the three numbers to be added. In order to determine the result on the place value table we have to push the counters in each column together and to replace groups of ten counters in one column by one counter in the next column. Each carry reduces the number of counters by 9. As we started from 45 counters and as 45 is a multiple of 9 the total of the digits of the result can only be a multiple of 9. In the example of Fig. 6 there are three carries ($1 + 2 = 3$), so the sum of the digits of the result is necessarily $45 - 3 \cdot 9 = 18$.

This explanation is *independent* from the special setting of this exercise. Therefore for *any* addition the sum of the total of the digits of the given numbers differs from the total of the digits of the sum by a multiple of 9. This multiple is determined by the carries.

Winter (1985) has shown how the standard divisibility rules can be derived from operating on the place value table. As far as the rule for 9 is concerned the crucial point is that a move of one counter from one column to an adjacent column changes the number by a multiple of 9. Therefore the total of the digits of a number differs from the number itself by a multiple of 9.

2. THE NOTION OF OPERATIVE PROOF

Proofs as considered in the previous sections are called *operative proofs* as they have the following properties:

– they arise from the exploration of a mathematical problem,

– are based on operations with "quasi-real" mathematical objects, and

– are communicable in a problem-oriented language with little symbolism.

Strictly speaking, the term "operative proof" is not quite correct as it is not the proof which is "operative" but the whole setting. However, for the sake of brevity the term seems acceptable. Operative proofs have received growing attention since Zbigniew Semadeni's seminal paper on "pre-mathematics" (Semadeni 1974). His ideas were elaborated on in Germany by Arnold Kirsch (1979), Heinrich Winter (1985) and others and in Japan by Mikio Miyazaki (1995). These authors called proofs of this kind "pre-formal" or "pre-mathematical proofs" or "explanations by actions on manipulable things". These descriptions indicate some concerns about the status of such proofs and at the same time an unquestioned respect for formal proofs. However, research in the philosophy of mathematics and a re-thinking of the role of proof in the community of mathematicians have changed the situation considerably. An excellent overview is given in Hanna (2000).

As the first example in section 1.1 shows operative proofs represent the most elementary form of proof that came along with the first attempts to shape a discipline called "mathesis". Operative proofs refer not to symbolic descriptions of mathematical objects within a systematic-deductive theory but directly to these objects via representations that allow for "concrete" operations. These operations are generally applicable independently of the special objects to which they are applied. So it is not from special cases that the generality of a pattern is derived but from the *operations* on objects. There is a close link here to Jean Piaget's epistemological analysis of mathematics (Beth/Piaget 1961, chapter IX) that has led to the formulation of the "operative principle", a fundamental didactic tool in German mathematics education (see Wittmann 1996, 154 ff.).

3. THEORETICAL BACKGROUND OF OPERATIVE PROOFS

The notion of operative proof is based upon some theoretical positions from various disciplines. In this section two aspects will be described.

3.1 The quasi-empirical nature of mathematics

Operative proofs depend on appropriate representations of mathematical objects. It was Imre Lakatos who first pointed to the fact that mathematical theories are always developed in close relationship with the construction of the objects to which they refer (Lakatos 1976). In each theory the mathematical objects form a kind of "quasi-reality" which permits the researcher to conduct experiments similar to experiments in science. In the last decades the importance of the "quasi-empirical" view for mathematics education has been more and more recognized.

At school level *informal* representations of mathematical objects are indispensable as they provide a "quasi-reality" which is easily accessible. Patterns become in a sense "visible" when informal representations like counters, the number line, the place value table, calculations with numbers and constructions of geometric figures are used.

The "quasi-reality" of mathematical objects forms a world of its own which Yuri Manin in a letter to ICME 7 aptly called a "mathscape". As the theoretical nature of mathematical objects is imposed on these representations this mathscape is well suited to support the building of theories at whatever level by conveying meaning, stimulating ideas and providing data for checking mathematical arguments. Unlike Hilbert's fictitious mathematician who has cut the ontological links the working mathematician and the learner act in a "visible" mathscape. The following statement by D. Gale summarizes this position very neatly (Gale 1990, 4):

> *The main goal of all science is first to observe and then to explain phenomena. In mathematics the explanation is the proof."*

3.2 Practicing skills in a productive way

When "mathe 2000" was founded 20 years ago it was a conscious decision to pay particular attention to basic skills in order to escape the fate of many curriculum projects in the sixties and seventies which failed because they neglected basic skills. Traditionally, "practice" is linked to the proverbial "drill and practice" which of course is not compatible with the objectives of mathematics teaching as we see them today. So a new approach to practice had to be developed which deliberately combines the practice of skills with higher objectives ("competences") like mathematizing, exploring, reasoning and communicating. This type of practice bas been called "productive practice" (Wittmann & Müller 1990/1992). The basic idea is quite simple: For practicing skills appropriate mathematical patterns are used as contexts.

Learning environments designed accordingly always start with extended calculations, constructions or experiments. In this way a "quasi-reality" is created which allows for observing phenomena, discovering patterns, formulating conjectures, and last not least for explaining, that is proving, patterns. The operations on which these operative proofs rest are introduced in this first phase in a natural way. While the environment is being explored more and more deeply reference to this quasi-reality is made continuously. In particular the practice of skills is called upon again and again for checking and verifying arguments.

Developmental research in the project "mathe 2000" has shown that the addition table, the multiplication table, and the standard algorithms for addition, subtraction, multiplication and division are so rich in patterns that there is no need to introduce additional contents for developing higher objectives of mathematics teaching. It is crucial, however, to select standard representations of numbers that incorporate fundamental mathematical relationships and so allow for operations that can carry operative proofs (Wittmann 1998). In arithmetic counters provide the representations of choice. For example, rectangular patterns of counters allow for representing the multiplication of natural numbers with the full information of the arithmetical laws. It is important, however, to secure the coherence of the representations employed. Instead of a wild variety of different ad hoc representations parsimony is indicated. In this line a comprehensive presentation of arithmetical theories for teachers has been developed in Müller, Steinbring& Wittmann (2004).

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REFUTATIONS: THE ROLE OF COUNTER-EXAMPLES IN DEVELOPING PROOF

Walter Whiteley

York University, Toronto, Canada

Abstract: As a research mathematician, I generate and read a 'proof' through the lens of a space of samples/possible counter-examples to 'make sense' out of the steps and each statement. As a teacher of mathematics, I find that students do not have a suite of examples/counterexamples at hand, and too often cannot tell whether a provided sample makes a key statement true or false! Faced with this, work with formal proofs becomes a strange exercise in 'meaningless' formalism and abstraction, rather than a support for making sense of the connections of mathematics. I urge that care be taken to help students build up appropriate sample spaces of examples and counter-examples for the concepts being explored through proofs and reasoning.

MY BACKROUND

I trained in Mathematical Logic and work as a research geometer, problem solving and communicating mathematics with other researchers in a variety of pure and applied areas in mathematics, computer science, engineering, and biochemistry. I have been working with undergraduate and graduate students learning to research, read and write mathematics. I have taught mathematics courses for over 35 years, including 15 years of teaching pre-service and in-service teachers, often in courses in geometry.

Throughout the last decades, I have had a number of occasions to reflect on my own practices as I 'do mathematics'. What are the processes occurring in my problem solving? Reasoning through examples and counter-examples, making conjectures, evaluating plausible ideas, working with sketched proofs, and writing. In assessment of students, and refereeing research papers, I have also had occasions to critically analyze presentations of reasoning, and my processes in making sense of others' reasoning.

REFLECTIONS ON EXAMPLES AND COUNTER-EXAMPLES

In observing my students, in courses like Linear Algebra and Mathematical Logic, I observed that the most challenging task for students was not reproducing a 'memorized proof', but in assessing whether an assigned 'If … then …' statement is true or false. They were very challenged to provide a counter-example to a false claim, or to provide an illustrative example for a true claim. In Logic, only the strongest students could generate a finite model to show an argument was not valid. In Linear Algebra, students were challenged to investigate 'reasoning' with appropriate examples and counter-examples. This clearly was a skill that needed to be learned.

I recall teaching a graduate course in Graph Theory. The most common way of developing a proof was to show there was no counter-example. The struggle to imagine properties such a counter-example would have to have, builds on the capacity to develop counter-examples and to know what features matter and do not matter in this search. Unless the students are at ease with examples that make the conclusion false, they have little chance to develop this reasoning.

In contrast, I constantly read mathematical statements with a suite of examples and non-examples to explore the meaning of the claims in contexts. Being able to set aside false statements (through counter-examples) and to select a possible flow of plausibly true statements, through examples, is an essential tool of my work, and essential part of communicating the meaning of mathematical claims to non-mathematicians – and an essential skill that I model with my students.

This reality is not what is commonly presented as 'mathematical logic'. It is common to wonder: why would someone who has just read a proof, then work through an example? The 'abstract proof' is supposed to be superior reasoning – and nothing should be added by exploring examples, and partial counter-examples. A proof by contradiction shows there are no counter-examples. What could an example, and non-example add?

Yet this is what I do, as both a pure and an applied mathematician. The description of Brown (1997 p.168) confirms that this is what mathematicians often do. It is a key process described in the classic book of Lakatos (1976). These practices also carry the 'sense' of mathematical claims and processes, and the mathematical connections (reasoning) to engineers, physicists, and others who use mathematics. It is also what students can use to do meaningful mathematics – and a skill they should practice.

Reflections on Mathematicians' Ways of Knowing.

In an interdisciplinary comparison of 'ways of knowing' I observed that a 'counterexample' has a clarity and status in math that is qualitatively different than in other disciplines. Whereas an event with alternate outcomes can generate a debate in, say, economics, a counter-example causes a complete shift in the mathematical discussion. As I observed at a recent research workshop, even a sketched proof did not have that immediate impact, while collections of examples and counter-examples provoked the refinement of conjectures and furthered our reasoning.

This is not the reality presented in most texts. It is also not the reality presented in many classrooms. It is the reality experienced daily by workers in applied mathematics, statistics, and the practice of some portions of pure mathematics. This is my world of making sense of mathematics, and the applications of mathematics.

Over the last decade I have noticed that a great deal of writing in Mathematics Education responds to a mythical vision of mathematics, one restricted to Pure mathematics. This is how math is typically represented in classrooms and in carefully edited and constrained publications. The world of `proofs' and reasoning without a significant role for examples and counter-examples is part of that myth. Here are a couple of key quotes that capture this disconnection:

> An axiomatic presentation of a mathematical fact differs from the fact that is being presented as medicine differs from food. It is true that this particular medicine is necessary to keep mathematicians from self-delusions of the mind. Nonetheless, understanding mathematics means being able to forget the medicine and enjoy the food. Gian-Carlo Rota 1997

> There's another unrecognized cause of the failure [of mathematics education]: a misconception of the nature of mathematics. A philosophy of mathematics that obscures the teachability of mathematics is unacceptable. Reuben Hersh, 1997

> Only professional mathematicians learn anything from proofs. Other people learn from explanations. A great deal can be accomplished with arguments that fall short of proofs. Raol Boas, 1980

In these terms, the suite of examples and counter-examples are the meaningful food of mathematics, the learnable grounding for explanations. A philosophy of mathematics that reflects the intimate meshing of modeling and interdisciplinary collaborations – which is how most students will use mathematics – will find a larger role for these foundations of reasoning in mathematics.

One other aspect of developing a range of examples and non-examples is in learning and creating definitions. John Mason (2003) writes about the importance of students' developing an appropriate range of examples and non-examples as they learn new concepts. That is, they should recognize what can be varied in an example, and what is essential and cannot be varied. This is a fundamental part of learning mathematics – and should be explicit.

This is an issue that comes up as early as when the young child first learns about triangles: iconic shapes and standard orientation vs. a range of shapes and orientations, as well as non-examples with non-straight edges, with extra (concave) corners, with open gaps (Clements 1999). Failure to recognize these shapes is a problem throughout elementary school. The problem is still present for senior high school students struggling to recognize when theorems about triangles in circle geometry apply to a figure (Bleck 2008).

Sadly, students do not get much experience in making definitions and developing their skills with stretching examples to extremes – a common technique in mathematics – and pulling non-examples close to the 'boundary' of the concept. As a first-year university student, I saw that this skill with the examples to the extremes was critically effective – and was immediately identified by my instructors as showing mathematical talent. However, many of my peers did not have this skill – and struggled.

Reflections on Visual and Kinesthetic 'Proofs'.

For many centuries, there have been debates on the validity of 'visual reasoning' in mathematics. In the 19th century, there was an active campaign to reduce or eliminate ‘motion’, ‘time’ and ‘visual/kinesthetic reasoning’ from formal mathematics. Rafael Nunez’ videos of upper year analysis classes vividly illustrate the critical disconnect between the formal, set theory presentation on the board (and the associated ‘proofs’) and the motion-based gestures and actions of the instructor. It is likely that the gestures are closer to the cognitive processes of the instructor than the blackboard material. As such, these gestures are closer to the cognition that the students need in order to develop a meaningful mastery of even this ‘pure’ mathematics.

One of the critiques of visual reasoning is that visuals are too specific to be used in general proofs – they are ‘merely’ examples. There are several observations on this issue. One is that visuals are strong particularly because they are examples. However they can carry general reasoning as symbols for the general case, provided the readers bring a range of variation to their cognition of the figure. A second observation is that we can develop conventions and expressions that are ‘partial’ in the sense that certain portions are missing (I often cover them with a blank patch, in a program like GSP).

It might surprise mathematics educators that even as a senior mathematics researcher, I make regular use of physical, kinesthetic, dynamic models – embodied examples and non-examples. Like my students, I find I understand the mathematics better when I have both a proof and some key models, even if I produced the first ‘proof’! When discussing with graduate students their current work, I may pose a verbal question, but I typically turn to a visual example, and when feasible, I pick up a physical example from the collection of models in my office, to center our discussion. I am always reasoning with images and a physical ‘sense’ – and these external visuals and examples are a way to model this for students who have experienced the presentation of mathematics as an abstract, often purposely ‘meaningless’ exercise. A possibly apocryphal saying is that ‘a mathematician is someone who does know what he (sic) is talking about!’ The ‘game’ of logic is presumed to be syntax: form and formalism; not semantics: meaning and a sample space of examples and non-examples, often in visual form.

Making Sense of Mathematics.

> If mathematics were formally true but in no way enlightening this mathematics would be a curious game played by weird people. Gian-Carlo Rota, 1997

Doing mathematics is an intensely human experience, and for me this is embedded in making sense out of the world. I enjoy working with interdisciplinary teams that are making sense of problems in engineering, or in predicting how proteins move, or don’t move, in order to develop understanding of diseases. I enjoy looking at footprints in the sand at a beach, and understanding the mathematics of why they fill with water as I step forward. I look at buildings and enjoy contemplating the choices the designer made in bracing (and hopefully overbracing) the structure.

I think in mathematics with all my senses and all my cognition, awake and asleep. I bring meaning to this process, and as I take results from new papers, I try to extract meaning from them. This is not always easy if the writers do not believe their readers need 'meaning', or if conventions dictate that the 'sense of the argument' does not belong in the publications. As if mathematics is indeed presented as a curious game played by weird people.

I was struck a few years ago when, during a presentation to a large group of high school students on 'learning to see like a mathematician', a student asked whether what they saw as polished proofs in their books was actually the way mathematicians think. Explicitly, they pointed out that what they saw in books was not how they thought, and they were concerned that they did not 'belong' in mathematics! This is an example of making mathematics unlearnable, alien and intimidating – by hiding the reality of how mathematics is done, and hiding the diversity of practices, reasoning, and problem solving represented in the current term: the mathematical sciences.

As soon as a classroom becomes the scene of the 'sense making game' (Flewelling), the role of examples and counter-examples is expanded. The search for a counter-example is, in my experience, essentially 'sensible' – searching for a cognitive 'fit' and discarding what does 'not fit'. This is learned aesthetic developed through years of 'making sense' of our mathematics, and laying a solid foundation for further sensible reasoning.

As an instructor, I was struck by the absurdity of students' replicating strings of statements as a 'proof', when they were, in fact not able to take a given example and a given line of a proof and say whether this statement was true, or false, for the example. It was a truly meaningless exercise. As a result, I modified my instruction to give a much higher priority to developing this basic ability.

Since that time I have further expanded my awareness that the ability to probe statements with examples and non-examples is an essential skill. It is learnable, and teachable – but it takes time, planning and assessment. Without developing this ability, it is highly questionable that formal abstract proofs serve even those of our students who are planning to become pure mathematicians. It is often through the examples and counter-examples that we recall the results, and reconnect to the reasoning. If this is true for expert mathematicians, how much more true it is for students who are learning to use the mathematics.

My experience is that without developing an appropriate range of examples, and non-examples, as well as diverse ways to vary the examples, our students are also poorly prepared for communication with collaborators within interdisciplinary contexts, or within school classrooms.

Connections back to proofs and proving in mathematics education.

My observation is that the roles of 'refutation' and of 'sample' are seriously underplayed in developing reasoning within mathematics education – and that the lack of this capacity radically handicaps student learning of proof and proving.

Some students develop this capacity, along the way, and employ it naturally in their learning.

It is unusual, but sophisticated, to also use examples and non-examples to probe the assumptions of a theorem or conjecture to see whether the assumptions are necessary. This too is a contribution that counter-examples and examples can make to the proofs and proving in mathematics education.

This ability is too important to leave to chance, or to use as an implicit screening criterion for success or failure. This is what happens unless we cultivate the ability with care and assist the students to reflect on its role and contributions. Sadly, my reflections back through my decades of learning, and teaching, confirm that too often my classmates, and my students lacked this ability, particularly the ability to appropriately vary examples and counter-examples. We can do more, if we attend to this, through all stages and all ages of learning mathematics.

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TALKING POINTS: EXPERIENCING DEDUCTIVE REASONING THROUGH PUZZLE DISCUSSIONS

Jeffrey J. Wanko

Miami University, United States

The proliferation of Language-Independent Logic Puzzles (LILPs), popularized by Sudoku puzzles, has provided us with an opportunity to extend the discussion and discovery of deductive reasoning beyond the classroom. When viewed as more than a pastime or recreation, LILPs can provide students with some foundational experiences that they may be able to apply to other logic problems. In a pilot study, fifth grade gifted students scored significantly higher on a test of general logical reasoning after engaging in a ten-week curriculum around LILPs. This researcher found that students made substantial gains in their deductive skills which may lay the foundation for later mathematical reasoning tasks and mathematical proofs.

BACKGROUND

Language-Independent Logic Puzzles (LILPs)—such as Kakuro, Nurikabe, Masyu, Shikaku, and especially Sudoku—have become more prevalent in puzzle magazines and in more common cultural settings. While they still trail in popularity to crossword puzzles (and other language-dependent puzzles), LILPs have been receiving more attention in recent years as puzzle-solvers become familiar with them. Sudoku has clearly led the way—taking off in 2005 through their appearance in newspapers, puzzle magazines and books, and in board games and computer programs (Dear, 2005). While other LILPs have not become as ubiquitous as Sudoku there is no shortage of types of logic puzzles which challenge the solver to apply deductive reasoning in finding a solution (Fackler, 2007).

Some LILPs are simple to understand, have few rules, and the deductive reasoning required for finding solutions is relatively easy to develop. For example, Shikaku puzzles require the solver to divide the board along the grid lines into non-overlapping rectangles so that exactly one number appears in each rectangle and that number is equal to the area of the rectangle (Figure 1). Novice Shikaku solvers can quickly devise typical solution strategies—including identifying numbers for which only one possible rectangle can be drawn and finding squares which can be covered by a rectangle associated with only one number in the grid. However, even with simple solution strategies, Shikaku puzzles can be quite difficult—especially when the grid is quite large and the numbers in the grid have many factors and possible corresponding rectangle placements.

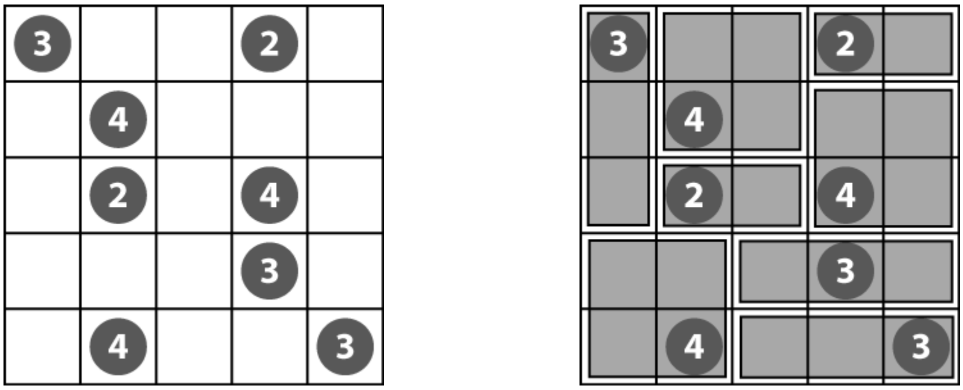


Figure 1: Example Shikaku puzzle and solution

Other LILPs have many more rules and varying degrees of complexity in the discovery of solution strategies. Like Shikaku, Nurikabe puzzles (Figure 2) also appear as a lattice of squares with some numbers placed in them, but have the following rules:

1. Your goal is to create white regions surrounded by black walls.
2. Each white region contains only one number and that number corresponds to the area of that region.
3. The white regions must be separated from each other (but can touch at the corners).
4. Numbered squares cannot be filled in.
5. The black squares must be linked in a continuous wall.
6. Black squares cannot form a square 2 x 2 or larger.

Elementary solution strategies for solving Nurikabe puzzles are similar to those for Shikaku puzzles, but novice solvers are often frustrated by the number of rules that they must follow. However, larger and more complex Nurikabe puzzles can involve solution strategies that require a novel approach to the puzzle and variations on other deductive approaches.

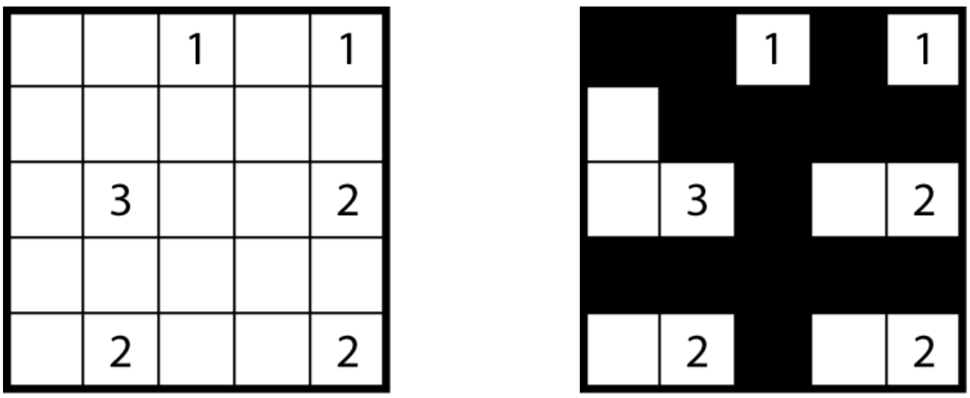


Figure 2: Example Nurikabe puzzle and solution

Puzzle types that are unfamiliar to the solver require the development of new solution strategies—an exercise not only in problem solving, but in applying principles of deductive reasoning. This activity—of solving a puzzle type that is new to the solver—often puts groups of people with varying levels of puzzle solving experience on more equal footing, essentially making novices out of everyone. At this point, the conversations about solution strategies become centered around the underlying logic structure of the puzzle.

FUNCTIONS OF PROOF IN THE CLASSROOM

When students are given a new type of LILP and are asked how they might start looking for a solution, their responses can vary greatly. But by discussing their ideas collectively, the class typically devises some of the elementary solution strategies and individuals begin to explore more complex approaches. This was the basis of a supplemental puzzle curriculum that was used in a fifth grade gifted class of twenty students.

One day a week, for ten weeks, this researcher taught a lesson built around one type of LILP. Lessons typically comprised a description of a puzzle type and its history and/or a translation of its name, a picture of a sample solution, and a discussion of the puzzle's rules. Students often worked through one sample puzzle together as a whole class while discussing various strategies that individuals posed for working toward a final solution. Students worked individually or with a small group to explore their ideas—making conjectures and testing their validity.

The process that students developed closely models what de Villiers (1999) describes as the roles of mathematical proof: verification, explanation, discovery, systematization, intellectual challenge, and communication. Each of these is described below.

Students modeled *verification* as they analyzed a possible move in a puzzle against the predetermined rules that were set for each LILP. Students practiced *explanation* as they constructed and made arguments with their classmates for whether a move would be possible or whether a strategy might lead to a solution. *Discovery* involves more than simply finding a solution—for example, students discovered patterns in the placement of numbers in Nurikabe puzzles that resulted in classifying sections of a puzzle as similar to ones they discovered in other problems.

Students also practiced some early forms of *systemization* as they refined their solution strategies and moved from an informal series of correct steps, to an almost algorithmic approach to starting each puzzle before finding a new wrinkle. *Communication* extends the explanations that students engaged in and describes the process of social interactions that were made as students worked to persuade others of their strategies. Students also experienced an appreciation for *intellectual challenge* as they worked toward finding a final solution and the intrinsic reward of solving a puzzle.

In this setting, students modeled these functions of proof as they developed solution strategies to LILPs, thus laying the foundation for a better understanding of proof when they encounter it in their mathematics class.

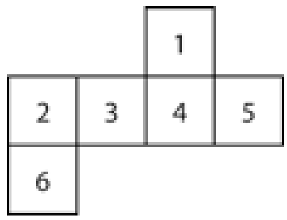
GROWTH IN REASONING SKILLS

Prior to instruction, students were given the Logical Thinking Inventory (LTI), a series of twenty multiple choice questions that require deductive and/or spatial reasoning for solving (see examples in Figure 3).

Four students are standing in a line. Pablo is to the left of Nicole. Suzanne is to the left of Wally. Wally is to the right of Nicole. Suzanne is between Nicole and Pablo. Who is the first person on the left?

A. Nicole
B. Pablo
C. Suzanne
D. Wally
E. I don't know—no response

If the shape at the right is cut out and folded to make a cube, which of the following statements is true?



A. 2 is opposite 5
B. 3 is opposite 6
C. 1 is opposite 6
D. 4 is opposite 6
E. I don't know—no response

Figure 3: Sample questions from the Logical Thinking Inventory (LTI)

After the ten weeks of instruction using the LILP curriculum, students were again given the LTI. Mean scores on the LTI increased from 68.3% correct to 77.0% correct, a statistically significant increase (Wanko, under review).

This pilot study is being expanded in 2009 to include students from various grade levels and academic backgrounds across a number of sites.

CONCLUSION AND FUTURE WORK

While it is premature to conclude that a curriculum of collaborative work on developing solution strategies for LILPs can increase deductive reasoning skills that are applicable to mathematical proofs and other logical reasoning situations, there is some evidence that this approach may have merit. Students in the pilot study displayed a significant amount of transference of the deductive skills they developed in solving puzzles to the logical thinking and spatial reasoning skills needed on tasks that were not at all similar to the puzzles. The informal language that students developed and used in the classroom while collaborating on solution strategies was quite telling, as the students employed phrases like, "This rectangle has to go here because…" and "I know that this square is shaded in because…" These phrases were quite common and are precursors to the basic arguments needed in deductive proofs. Finally, the stages of development that students traversed while engaging with the puzzles were parallel to those recognized as roles of mathematical proof.

In addition to a larger scale version of this pilot study, it would be helpful to investigate the impact of a puzzle-based curriculum over several years prior to students' introduction to formal proof in a high school mathematics course, comparing the results of students exposed to logic puzzles to those who have a more traditional curriculum. Studies such as these would be helpful in understanding the role that recreational puzzles—such as Sudoku and other LILPs—may play in developing deductive reasoning skills.

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UNDERSTANDING PROOF: TRACKING EXPERTS' DEVELOPING UNDERSTANDING OF AN UNFAMILIAR PROOF

Michelle H. Wilkerson-Jerde, Uri Wilensky

Northwestern University, USA

In this study, we adopt the notion of dense connection *in the understanding of mathematics, and trace the development of these connections over time as participants make sense of an unfamiliar proof. By representing participants' verbalized sensemaking with a network of ideas and resources that changes over time, we can investigate what features of a mathematical proof play more or less central roles in one's developing understanding of that proof. Preliminary results indicate that though all participants in the study were at a graduate level of study or above, different participants revealed different aspects of the proof (a formal definition, a specific example, or a specific property or component of the focal mathematical idea) to be central to their developing understanding.*

INTRODUCTION

One of the most important aspects of mathematical proof is the relationship between a reader and a proof as a disciplinary tool – that is, how does one use a proof to learn and make sense of the mathematical ideas contained within? In this project, we provide expert mathematicians (graduate students and university professors) with an unfamiliar mathematical proof, and ask them to think aloud as they make sense of it. We use these interviews to trace how experts construct their own understandings of the mathematical ideas contained within the proof, and identify which aspects of the proof serve as hubs or remain on the periphery of this developing understanding.

Unlike several studies of expert mathematical knowledge and expert mathematicians' proof practices, this study concentrates specifically on experts as they interact with an *unfamiliar mathematical idea*. We believe that such an approach may begin to address the discrepancies often cited between novice and expert practitioners of mathematics – namely, that novices rely on empirical and informal knowledge, whereas experts rely on coherent, formal definitions when thinking about mathematics (Vinner, 1991; Schoenfeld, 1985; Tall, 1991; Sfard, 1992; Dubinsky 1992). While certainly experts are able to describe their well-established mathematical understandings in such a way, this does not necessarily suggest that experts *learn* about new mathematics this way. As such, we believe that a deeper look into how individuals with a deep mathematical knowledge base *construct* such knowledge may yield different implications for secondary and tertiary mathematics education than expert/novice studies that focus on mathematical ideas that experts already understand well.

ANALYTIC FRAMEWORK

In both the formulation and the analysis of this study we relied heavily on the notion of *knowledge as dense connection* (Skemp, 1976; Papert, 1993), and were interested to investigate the extent to which expert knowledge, and particularly the development of this knowledge, can be described in the context of new and unknown mathematical content. The notion of mathematical knowledge as connected elements accounts for a number of aspects of expertise – for example, one of the identifying aspects of expertise is the ability one has to deconstruct and reconstruct mathematical knowledge in new and different ways (Tall, 2001); and it is certainly expected that different participants, with their varied experiences and backgrounds, may have different ways of "slicing up" the elements of the proof in order to construct their own understanding (Wilensky, 1991). As such, the coding system described below was developed using a bottom-up iterative process (Clement, 2000), though connections to existing literature were made when these relationships became apparent during development of the codes.

Our coding scheme consists of two levels – *ways of understanding* and *resources for understanding* – that closely mirror Sierpinska's (1994) distinction between acts of understanding and resources for understanding.

Ways of understanding include *questions*, *solutions*, and *explanations*, and align well with Duffin and Simpson's (2000) descriptions of *building*, *enacting*, and *having* understanding.

Resources for understanding include *parents*, *definitions*, *fragments*, and *instantiations* (examples provided by the proof itself, introduced by the reader, and so forth). Several resources for understanding can be identified within a question, solution, or explanation: for example, if a participant questions how two definitions presented within a proof are related to one another the statement would be coded as a question involving two definitions; if a participant makes sense of a definition by enacting it on an example provided within the proof, this would be coded as a solution involving a definition and an instantiation. The coding system is described in much more depth in Wilkerson and Wilensky (2008).

Research Questions

In keeping with the themes of the ICMI Study as outlined in the Discussion Document, we believe that this study (a) begins to address questions of individual differences in how one understands and makes sense of proofs, (b) identifies what aspects of proof (definition statements, examples, detailed description of processes and machinery) serve as central components of one's understanding, and (c) provides a language with which to investigate how learners interact with disciplinary materials in order to make sense of new and unfamiliar mathematical ideas. For this paper, our research questions include:

1) What aspects of a proof play a more central role in one's developing understanding of the mathematical ideas contained within?

2) What are the similarities and differences between different individuals and the proof elements that find more or less central to their understanding?

METHODS AND DATA

Participants

10 participants, including 8 professors (assistant, associate, and full) and 2 advanced graduate students from a variety of 4-year universities in the Midwest participated. Participants were identified primarily through university directory listings, and contacted via email to see if they would agree to be interviewed.

Protocol

Students and professors who wished to participate were given semi-structured clinical interviews using a think-aloud protocol (Ericsson & Simon 1993). Each was provided with the same mathematics research paper (Stanford, 1998; see below), selected for its accessibility in terms of topic and vocabulary. They were asked to read the paper aloud and try to understand it such that they would be able to teach it to a colleague. Interview data was videotaped, transcribed, and coded using the TAMSAnalyzer software (2008).

Proof

The research paper provided to participants (Stanford, 1998) concerns *links,* which can be thought of informally as arrangements of circles of rope that are entwined with one another, and the conditions under which those circles can be pulled apart. If a link has the property that when any single circle is removed from the arrangement, the rest can be pulled apart, that link is said to be *Brunnian*. If in a given two-dimensional representation of a given link, there are n distinct collections of over- and underpasses that, when switched, make the loops fall apart, the link is said to be *n-trivial.* The proof establishes a systematic relationship between the properties that make a link *Brunnian* and *n-trivial*, such that any Brunnian link can be described as *(n-1)-trivial*.

Analysis

For the construction of each experts' network, each participant's *resources for understanding* were converted into network nodes and *ways of understanding* into links between those nodes. For example, when a participant asks how two definitions (say, the definitions of trivial and of Brunnian) are related, this is reflected in the network by establishing a question link between trivial and Brunnian. If later the participant tries to find out how the two aforementioned definitions are related by manipulating the Borromean Rings as a specific example of a Brunnian link, this is reflected in the network by establishing a solution link between Brunnian, trivial, and the Borromean Rings.

After the network is built, it can be analyzed to determine which nodes, or mathematical resources presented in the proof, served a more central role in each experts' sensemaking. Network measures were computed using the statnet package (Handcock, et al, 2003) in the R statistical computing environment. The measure used in this report, betweenness[1], is a measure of the extent to which a given network node (or element of the proof) serves as a "bridge" between other nodes. In other words, a proof element with high betweenness is one that enables the participant to form many connections between other elements of the proof.

RESULTS

At the time of this proposal, data is in the process of being analyzed. Preliminary findings, however, suggest that while there is relative consistency in experts' *ways of understanding*, they use very different *resources for understanding* introduced by the proof while making sense of the ideas presented within. In other words, while experts seem to be relatively consistent in the number of questions, solutions, and explanations they discuss as they read through the proof, the specific aspects of the proof discussed within each of these questions, solutions, and explanations differ greatly. For some experts, specific instantiations of the mathematical object being explored serve a central role in building a densely-connected description of the proof; while for others a formal definition or several small components of the mathematical object serve this purpose.

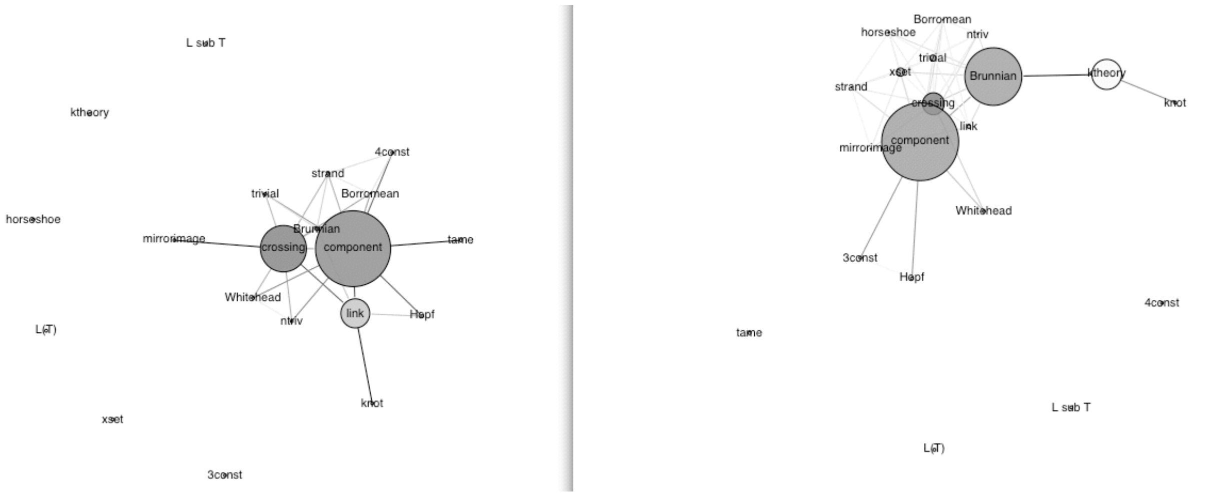

Figure 1. Network representations of two participants' coded interviews

In the graphic above, the network produced from each participant's entire coded interview is featured. The darkness of links between any two elements represents the frequency with which those elements were mentioned together. The color of elements indicates which code each element belongs to; the most visible here are *fragments*, that is, smaller pieces of the main idea to be proved (green), formal *definitions* (red), *parents* or background knowledge introduced by the participant (white). The graphs indicate that Joe's network is more dense, but that Ana more frequently linked the same elements together. Furthermore, definitions,

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[1] Betweenness $C_B = \sum_{i \neq j \neq v} \frac{g_{ivj}}{g_{ij}}$, where g_{ivj} is the shortest path between i and j through v, and g_{ij} is the shortest path between i and j.

background knowledge, and pieces of the larger proof all played a much more important role for Joe's developing understanding, while Ana made sense of the proof mostly in terms of its smaller pieces only.

(average betweenness[2])	Joe	Ana	Mark
Fragments	.28	.05	~0
Parents	~0	.09	~0
Definitions	.02	.09	~0
Examples	.43	.05	~0
Constructions	.25	.71	1

Table 1: Betweenness of different types of proof elements for Joe, Ana, and Mark

In addition to exploring how different individuals might utilize different components of a proof in order to make sense of it and the mathematical ideas contained within, it is interesting to consider what elements are important for all participants, regardless of their "proof style". The table above shows that while Joe, Ana, and Mark relied on different types of proof elements to very different degrees (Joe heavily relied on fragments and examples; Ana had a more distributed focus and used her background knowledge more), constructions – that is, examples that were constructed by the participant on-the-fly to illustrate, test, or otherwise investigate the claims laid forth in the proof – served as an important bridging element for all three participants.

CONCLUSION

In order to access the aspects of expertise that might best inform educational practice; it is important to recognize that the *mechanism* by which experts come to know mathematics should be investigated in addition to the structure of that knowledge they already have. In this paper, we outline a method for representing experts' active sensemaking while reading a proof, and some analytical tools for evaluating what parts of a proof serve central roles in individuals' developing understanding of that proof and the ideas associated with it. Although our results are still in the preliminary stages, we believe that we are able to capture patterns in experts' developing understandings that might reflect different ways of coming to understand a proof, as well as other patterns that hold constant across participants.

[2] Betweenness was averaged across all elements of each type, and across total betweenness of each element for each individual.

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READING PERSPECTIVE ON LEARNING MATHEMATICS PROOFS

Kai-Lin Yang & Fou-Lai Lin

Department of Mathematics, National Taiwan Normal University, Taiwan

Reading perspective was seldom adopted to research on understanding of mathematics proofs. On the basis of a conceptual and verified model of reading comprehension of geometry proofs, this paper tries to move one step towards a model for learning mathematics proofs. A new study area of reading perspective on learning mathematics proofs is constructed by analyzing nature of mathematics proofs and analogizing to reading in second language. Research issues of reading perspective on learning mathematics proofs are formulated and integrated into four phases beyond an original scope. Some initiative studies are provided as accesses to this study area.

INTRODUCTION

The goals of learning mathematics proofs include constructing a valid proposition, convincing others and understanding mathematics proofs. Many approaches are proposed to reach the learning goals. For learning to construct, investigating or verifying a conjecture could be adopted to inspire the need for proving (Boero, Garuti & Mariotti, 1996; Reiss & Renkl, 2002). For learning to convince others, discourse activities could be implemented to articulate developmental argumentation, from naïve, informal to formal representations and norms (Sfard, 2000). For learning to understand argumentation and proof, validating proofs could be transformed for unpacking the logic of mathematical statements (Selden & Selden, 2003). Reading mathematics proofs from texts either printed in books or written on blackboard is experienced by most students in practice, yet reading is seldom counted to be an important and necessary learning activity.

Why is reading being less emphasized in learn mathematics proofs while either listening and speaking or doing and writing activities interest many researchers to investigate? Is it possible that these activities can cover all of the goals of learning mathematics proofs whereas reading is not really necessary? Listening, speaking, writing and doing activities have the limitation for distinguishing the logic value from the epistemic value of mathematics knowledge (Yang & Lin, 2008). Reading to learn and learning to read activities should not left out while learning mathematics proofs. We will argue that such activities could enhance not only students' construction of knowledge but also their adaptive reasoning in understanding concepts with different representations, different contexts. Students may realize through reading that mathematics proof comprises a sequence of logical arguments which can valid or refute a conjecture.

Recent research has made substantial progress in characterizing reading comprehension of word problem texts. The related factors or research methods in problem~~s~~ solving could be adopted to mathematics proofs learning. However, the difficulty and cognitive behaviors in reading problems should be different from those in reading mathematics proofs. For example, understanding of problems focuses on formulating situational and mathematical models, and understanding of mathematics proofs focuses on identifying logic value and catching proof ideas. Alternative understanding of the problems may facilitate efficient or effective answers, but alternative understanding of mathematics proofs may make contradiction or logic errors.

Accordingly, this paper tries to identify and elaborate a new study area of reading perspective on learning mathematics proofs. Firstly, our research experiences in reading comprehension of geometry proof are reflected to find limitation. Secondly, the nature of mathematics proofs is analyzed to evaluate the extension of reading comprehension of geometry proofs. Thirdly, the importance and related issues of reading in second language is analogous to reading in mathematics proofs. Afterwards, research issues are proposed and integrated into four phases.

Beyond Reading Comprehension of Geometry proof

Yang & Lin (2008) conceptualized a model of reading comprehension of geometry proof (RCGP) based on a real-world situation of what and how mathematicians read proofs and verified it by students' RCGP. Their study is just an initial start but a scope of reading perspective on learning mathematics proofs is waiting for enlarging.

One of the difficulties in discussing research on reading comprehension is the confusion over terminology. For this paper, reading is also viewed as an active construction process, which means that "Reading is a receptive language process. It is a psycholinguistic process in that it starts with a linguistic surface representation encoded by a writer and ends with meaning which the reader constructs. There is an essential interaction between language and thought in reading. The writer encodes thought as language and the reader decodes language to thought." (Goodman, 1998). In other words, readers play active roles for comprehending printed materials, and thinking is required by reading.

Other difficulty for research on reading comprehension of mathematical proofs is the implicit products of reading. To assess students' abilities of writing proofs seems easier than of proofs. The products of reading are difficult to observe because not all of them are conscious or definite. Another difficulty for research on reading comprehension of mathematical proofs is the lack of a comprehensive framework to investigate its relationship to the learning of mathematical proofs.

Nature of Mathematics Proofs

Mathematics proofs of high school curricula could be classified into algebra and geometry. Focused on the proof modes or methods, the nature of algebraic proofs and geometric proofs is different. Algebraic proofs could uniquely rely on the logical rules because symbols are operated according to their regulations. On the contrary, acceptable geometric proofs could be substituted for rigorous proofs because geometry requires perceived figures or intuition to constitute concept definitions or theorems. Therefore, a model of RCGP could not be directly extended to reading comprehension of algebraic proofs.

On the other hand, mathematics proofs which may be used to denote valid or invalid arguments, formal or informal deduction, and supporting or refuting examples are beyond the specified proofs in Yang and Lin's study. One of the obstacles to understand proofs is due to the multidimensionality of the meaning of arguments, which includes content, epistemic and logic value (Duval, 2002). Gray, Pinto, Pitta & Tall (1999) further pointed out that the new cognitive difficulty results from "*the didactic reversal – constructing a mental object from 'known' properties, instead of constructing properties from 'known' objects*". However, we argue that this reversal is inevitable while learning mathematics proofs. Students has learnt or acquired concept definitions by building concept images and properties, and they construct understanding of mathematics proofs (objects) from epistemic value of proof content.

Reading in Second Language

Reviews of research (e.g. Ellerton & Clarkson, 1996) often note the importance of language factors in learning and teaching mathematics. Research on reading in second language is helpful and could be analogous to research on reading in mathematics proofs because mathematics proofs are like second language, which is not obvious in our daily life. Specifically, proof and proving is a second language among the field of mathematics because of its multidimensional~~ity of~~ meaning. Reading comprehension is a main goal of learning second language, hence a main goal of learning mathematics proofs.

We elaborate the importance of reading mathematics proofs by giving purposes for reading in different contexts. If taught with an emphasis on constructing proofs, reading conjectures is necessary. If taught with an emphasis on acquiring proof ideas, reading relevant texts is necessary. If taught with an emphasis on applying procedures, reading instructions is necessary. If taught with an emphasis on validating proofs, reading statements logically is necessary.

ISSUES OF READING PERSPECTIVE

Like reading in second language, a comprehensive framework and a broadened scope of mathematics proofs and multiple views are the first two phases for advancing research on learning mathematics proofs with reading perspective. Based on distinct operational frameworks, which and how factors affect reading comprehension of mathematics proofs could be well studied. For developing effective approaches to leaning mathematics proofs, coordinating reading and writing is proposed as a transitional stage. Some initiative researches with respect to each phase are provided as accesses to this new study area.

Reading Comprehension of Mathematics Proofs

The necessary of symbols to algebraic proofs is like figures to geometric proofs. Obviously, the roles of symbols are different from the roles of figures in proofs.

Symbols are a mode of representations which could be logically operated; on the contrary, figures are a mode of representations which could be intuitively visualized. Therefore, the implicit structures of symbols emerging from legitimate operations trivially rely on logic value; the implicit structures of figures emerging from visualization mainly rely on epistemic value and require to be verified.

The facets of RCGP are assumed to be complete by mapping into Bloom's taxonomy of cognition (see Yang & Lin, 2008). Thus, we suggest that re-formulating the meaning of each facet regarding the symbolic essence of algebraic proofs is proper for constructing a hypothetical model of comprehending algebraic proofs and further verify it by students' performance instead of re-formulating facets.

On the other hand, reading comprehension of mathematics proofs should be investigated from students' perspectives. How do students think and evaluate their reading comprehension of mathematics proofs? What is the difference between the facets of reading comprehension formulated from professionals and students? How do students' perspectives influence their mathematics proofs learning?

Teachers' and Students' Views on Reading Mathematics Proofs

What results in the consecutive ignorance of reading activities for learning mathematics proofs? Teachers with different beliefs or views about reading and writing mathematics have different intention and competence to try or modify the alternative teaching approaches emerged with reading perspective. On the other hand, some students believe that reading mathematics or preview strategy is not beneficial to learn mathematics (Yang, in preparation). These related phenomena are under investigation. Furthermore, reading perspective and its emerging approaches could be revised based on teachers' and students' views.

Central Factors Influence Reading Comprehension of Mathematics Proofs

The reader, the text, and the context are considered to be the three central and interrelated factors that affect reading (Lipson & Wixson, 1991). In language, there is a substantial research base suggesting that the reader factor - prior knowledge, interest, motivation, or reading strategies, the text factor - the organizational structure of a text, or the genres of texts, and the context factor - purposes for reading or instructional strategies, are critical variables that influence reading comprehension (e.g. Armbruster, Anderson & Ostertag, 1987). These factors and their affiliated variables could also be manipulated to investigate their effects on reading comprehension of mathematics proofs for short-term or long-term experiments.

Regarding the factor of reader, Lin and Yang (2007) found that prior knowledge and reasoning ability predict a significant and considerable amount of variance in RCGP. Regarding the factor of text, Yang, Lin & Wang (2008) had found that the effect of written formats on students' understanding of geometry proof is not significant. However, the generality of their finding is limited to the task of proof texts without their corresponding propositions, and understanding is classified into micro, local and global types. Regarding the factor of context, Yang and Lin (in submission) designed innovative worksheets of geometric proofs by reading strategies and found the score of the delayed posttest of two experimental groups was significantly higher than the control group.

Effective Approaches to Learning Mathematics Proofs

Reading perspective is suggested to improve the learning of mathematics proofs, however, single perspective is not our preference. Students seem to learn better while reading and writing instruction are integrated (Stevens, 2003). The reader becomes a writer while the source text is transformed into a new text, and the writer becomes a reader while the constructing text is reviewed. That is to say, reading and writing literacy are related. Coordinating reading and writing seems a feasible approach for meaningful and constructive mathematics proofs learning.

Yang and Wang (in submission) have analyzed potential of statement-posing tasks for linking the learning of reading and writing mathematics proofs. Moreover, coordinating reading and writing is viewed as a process of learning mathematics instead of an end. Afterwards, a holistic perspective of listening, speaking, reading, writing and doing mathematics proofs should be constructed for mathematics proofs learning for all students of different ages.

SUMMARY

This is a position paper for identifying and elaborating a new study area of reading perspective on learning mathematics proofs. Reading approaches are argued to connect epistemic and logic value for learning mathematics proofs. Research issues are integrated into four phases, re-formulation of reading comprehension of mathematics proofs, involvement of subjects of instructional practices, exploration of factors related to this construct, and going beyond reading via coordinating writing (further speaking, listening and doing). Research on reading perspective of learning mathematics proofs creates opportunities for planning alternative instructions of learning mathematics proofs, and makes mathematics proofs more accessible to most students.

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HIGHER ORDER REASONING PRODUCED IN PROOF CONSTRUCTION: HOW WELL DO SECONDARY SCHOOL STUDENTS EXPLAIN AND WRITE MATHEMATICAL PROOFS?

Oleksiy Yevdokimov

University of Southern Queensland, Australia

The culture of mathematical explanations and writings based on conceptual understanding in proof construction is on the focus of the paper. We explore students' attempts to explain construction of mathematical proofs after reading them and write mathematical proofs after working out their own constructions. Two examples of proofs, by induction and by contradiction, are discussed in detail to highlight students' difficulties in proving and possible ways for their resolving.

INTRODUCTION

Despite a consensus on the importance of proof in any mathematically related activities, from the children's first logical reasoning in primary school to mathematicians' research work, its role in the teaching and learning of mathematics, in particular secondary mathematics, has traditionally been neglected in curricula documents for long time. However, recently this situation has changed dramatically. Probably the most demonstrative formal evidence took place in the U.S., where the status of proof has been significantly elevated in the Standards document (NTCM, 2000) with respect to the previous one (NTCM, 1989). Proof has also received a much more prominent role throughout the entire school mathematics curriculum. Evidence of similar actions can be also seen in many other countries throughout the world. The conception of proof seems to be a bridge that connects mathematical research work and teaching of mathematics. Metaphors on the role of proof in mathematics that directly relate to mathematics education (Hanna, 2000; Hanna and Barbeau, 2008; Manin, 1992; Rav, 1999) emphasise the importance of the teaching of proof in school mathematics. Reviews of research on the teaching and learning of proof (Battista & Clements, 1992; Tall, 1991; Yackel & Hanna, 2003) have informed and inspired more recent studies of proof and proving in mathematics education. Nevertheless this area is still not being developed to its maximum potential, and still not enough is known about how students can best be taught proof and proving skills. In one of the latest surveys on the teaching and learning of proof (Harel & Sowder, 2007) the authors stated that

> overall, the performance of students at the secondary and undergraduate levels of proof is weak… Whether the cause lies in the curriculum, the textbooks, the instruction, the teachers' background, or the students themselves, it is clear that the status quo needs, and has needed, improvement. (p.806)

This paper is an attempt to investigate the links between students' abilities in proof construction and their conceptual understanding of mathematical content they deal with. The paper is divided into two parts: the first part elaborates a theoretical model based on Weber's idea (2005) to consider proof construction as a problem-solving task, and the second part presents examples of proofs produced by secondary school students as well as examples of proofs proposed to the same group of students to work on them; and discusses the influence they [examples] may have upon development of students' conceptual understanding and structural knowledge.

ABOUT THE THEORETICAL MODEL OF PROOF CONSTRUCTION

Hanna (1995) emphasised that

> the most important challenge to mathematics educators in the context of proof is to enhance its role in the classroom by finding more effective ways of using it as a vehicle to promote mathematical understanding. (p.42)

We address this challenge in specific conditions, where secondary students possess higher order mathematical thinking and reasoning. We consider these questions with respect to a special group of students, who, for several years, were invited to sit Australian Mathematical Olympiad, which is the highest level of mathematics competitions for school students in Australia. Most high-profile students regularly participate in numerous mathematical competitions and, for them to achieve the best results, their training should be grounded on a comprehensive theoretical base, where the role of proof and proving hardly can be underestimated. In this paper we explore students' attempts to explain construction of mathematical proofs after reading them and write mathematical proofs after working out their own construction. Mathematical reading provides a challenge to understand a text and work up a strategy resolving a given task (Mamona-Downs & Downs, 2005). Mathematical explanations are used to highlight a more general approach that can be applied and elaborated beyond a given task, e.g. to check writing of student's own proof as well as reading of the given proofs. Mathematical explanations allow the reorganisation of the activity of proof construction according to functions of proof (Balacheff, 1988; Bell, 1976; de Villiers, 1990, 1999; Hanna, 1990; Hanna & Jahnke, 1996; Hersh, 1993). Hanna noted (2000) that even for practising mathematicians understanding is more important than rigorous proof, i.e. "they see proofs as primarily conceptual, with the specific technical approach being secondary" (p.7). We consider the mentioned above group of students as potentially prospective candidates, at least some of them, to become professional mathematicians in the future. Therefore, we understand the role of proof in work with gifted students as transitional from the teaching and learning mathematics, at the one hand, to inquiry work in mathematics, at the other hand, i.e. the role which combine both kinds of activities. To analyse this we use a method of simultaneous investigation of both: (1) influence, which proof construction in common, and specific examples in particular, may have upon development of students' abilities to understand proofs in the proper way, and (2) perception of proving process by individuals, which may or may not contribute towards conceptual understanding of mathematical content. We call this method a model of mutual convergence, keeping in mind that mutual impact of both components of the method on each other requires further clarification.

According to Weber (ibid.) proof construction is a mathematical task in which a desired conclusion can be deduced from some initial information (assumptions, axioms, definitions) by applying rules of inferences (theorems, previously established facts, etc). Weber (2001) noted that there are dozens of valid inferences in most proving situations, but only a small number of these inferences can be useful in constructing a proof. Our special interest was analysis of the situations in proof construction, where students didn't know how to proceed (in the sense of both kinds of activities, students' mathematical reading with explanations that followed and their own attempts in proving, including writing). The hypothesis was in existence of non-linear complicated dependence between (1) and (2), which under certain conditions may lead to a significant extension of learning opportunities (Weber, 2005) affordable for students as a result of proof construction.

ANALYSIS OF SOME EXAMPLES AND METHODS USED IN PROOF CONSTRUCTION

Below we present two examples of proof construction and discuss them with respect to students' explanations either on the base of their reading or writing. We use Weber and Alcock (2004) terminology of procedural, syntactic and semantic proof productions as components of proof construction.

Proof by mathematical induction

Mathematical induction is an important part of knowledge on proof construction. Many students perceive mathematical induction as a procedural proof production. We observed no difficulties in students' work with direct proofs. Therefore, mostly we focused on the situation, where the procedural or syntactic part of proof was completed, but proof itself wasn't. The following extract (as mathematical reading activity) proposed to students to get their views and explanations, gives a good example of the case. Text in bold italic was unavailable for students.

Example 1 (Euler)

Prove that for each positive integer $n \geq 3$, a number 2^n can be represented as $2^n = 7x^2 + y^2$ where x and y are both odd numbers.

Proof

The beginning of this proof is syntactic.

We prove this statement by induction. For $n = 3$ it is true. Assume that the property is true for a certain *n*, i.e. $2^n = 7x^2 + y^2$, where x and y are both odd numbers.

Semantic part of proof begins here. Direct application of induction doesn't work and informal interpretation of the components of inductive process needs to be done.

Then, for pairs

$\left\{A=\frac{1}{2}(x-y),\ B=\frac{1}{2}(7x+y)\right\}$ and $\left\{C=\frac{1}{2}(x+y),\ D=\frac{1}{2}(7x-y)\right\}$ we have

$2^{n+1}=7A^2+B^2$ and $2^{n+1}=7C^2+D^2$, respectively.

The first gap within semantic part of proof is below. Since it depends on understanding of a certain concept or theorem and may lead to (in)correct application in the construction of proof we call this gap as a conceptual one.

A and B are either odd or even simultaneously. Indeed, if $A=\frac{1}{2}(x-y)=l$ is odd, then $B=\frac{1}{2}(7x+y)=\frac{1}{2}(7y+14l+y)=4y+7l$ must be odd. If A is even, then B is even, respectively. The same property is valid for C and D.

Another conceptual gap follows.

Moreover, if $A=\frac{1}{2}(x-y)$ is odd, then $C=\frac{1}{2}(x+y)$ is even, and vice versa. This means that both numbers are odd in one of the pairs Q.E.D.

Our observations show that students may fail to provide explanations of proof construction because of limited understanding of the relationships between mathematical objects involved.

Proof by contradiction

Proof by contradiction is a complex activity, where students may experience significant difficulties. The following example was supposed for students' own attempts to construct a proof and provide explanations in writing.

Example 2

Natural numbers from 1 to 99 (not necessarily distinct) are written on 99 cards. It is given that the sum of the numbers on any subset of cards (including the set of all cards) is not divisible by 100. Prove that all the cards contain the same number.

Analysis of Example 2 and students' writings

The first part of proof construction (syntactic one) is easy to follow – to assume the opposite, which means that at least two cards contain distinct numbers, e.g. $n_{98}\neq n_{99}$ using standard notation, where n_i is a number written on the card i. The next step is to identify and apply a method (technique) that leads to a contradiction. The main idea of the semantic part of proof is to investigate different remainders x_i of $n_1+n_2+...+n_i$ upon division by 100, which guarantees the result that all x_i must be distinct for $i=1,2,\dots,99$. After that, making comparison of the sum $n_1+n_2+\cdots+n_{97}+n_{99}$ (just one of the two distinct numbers needs to be omitted) with another sum having the same remainder (conceptual gap) gives three possible results, each of which leads to a contradiction.

Our observations show that students may have difficulty with their own approach and explanations of proof construction due to lack of understanding of which mathematical objects can be used. Consequently, some invalid conceptual gaps (we call them pseudo-conceptual gaps) within semantic part of proof may appear in writing. It leads to the vague construction of a proof, where actual information about mathematical objects may be replaced with desirable property.

CONCLUDING REMARKS

We observed that in writing their own explanations on proof construction students are more aware about the gaps between different parts of proof, i.e. syntactic and semantic ones, than in the case of explanations based on reading. It can be connected with students' perception of mathematical reading as more instructional and prescriptive part of learning activities than writing. At the same time representation of formal mathematical concepts as components of proof makes reading more beneficial than writing, if students can identify some conceptual gaps properly (those gaps that often constitute the style and culture of formal mathematical texts used in textbooks and monographs). We suggest that focusing teacher's actions on such transitional and conceptual gaps within proof construction will influence the ways in which students attempt to construct proofs. In other words, transitions between different parts of proof in Weber's terms together with local components of semantic part of proof are the places, where significant learning potential can be accumulated. It may lead to further positive impact on development of conceptual understanding and optimization of learning process in the context of proof construction.

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PROOF AND EXPERIMENTATION: INTEGRATING ELEMENTS OF DGS AND CAS

Nurit Zehavi **Giora Mann**

Weizmann Institute of Science, Israel *Beit Chanan, Isreal*

Realization of the educational potential of computer algebra systems (CAS) led us to develop CAS-based curricular materials within an ongoing curriculum project. Teachers, who participated in professional development courses, have been our partners in the research and development of curricular materials. Influenced by dynamic geometry (DGS), we implemented slider bars in CAS to dynamically explore the behavior of tangents to conic sections. Teachers' views of the need for proof of unfamiliar geometric results, obtained by animation, led us to study the types of proofs teachers produced by using the expressions that created the animation. Duval's classification of transforming representations has been utilized for analyzing the proofs presented by the teachers.

THE MAIN DIFFERENCES BETWEEN DGS AND CAS

While the algebraic infrastructure that enables constructions and animation in dynamic geometry is hidden, CAS users need to develop the algebraic expressions in order to produce constructions and animation. These expressions can be used for experimentation, and then, for justifying visual results.

Duval's classification of the various registers of semiotic representations and their transformation (Duval, 2006) can help in analyzing users' perception of what is displayed on the screen. Duval contends that the learning of mathematics is supported by *treatment* within the same register (representation) and by *conversion* between registers. He argues that conversion, and not treatment, is basically the deciding factor for learning (p. 103). In dynamic geometry, the actions of the user and the software are all in the same representation. For example, in order to draw tangents to a curve from a point, one needs only to click on a command in the toolbar and to point to the specific point and to the curve; there is no infrastructural feature in CAS that enables one to do so. Therefore, in order to obtain tangents to a curve from a point, one has to find the equations of the tangent lines and then plot the associated graphs. The significance of this difference between the two types of software is that on the one hand, in CAS we lose the intimate relation (within the geometric register) between the geometric actions of the user and the software; on the other hand, an essential conversion between the algebraic register and the geometric register is possible. Of special interest is the implicit plotting feature in CAS, which enables conversion between an implicit expression (equation, or inequality) and the graphical representation of the expression.

Colette Laborde reacted to a paper on CAS and curriculum (Cuoco and Goldenberg, 2003) by 'playing a game' of replacing everywhere in the paper the words Algebra and CAS by Geometry and DGS (Laborde, 2003). For example: CAS technology can be used to experiment with expressions; DGS, through the *drag* mode, enables to experiment with figures. However, no counterpart in DGS was found for the question (raised by Cuoco and Goldenberg), "how can learners come to flexibly use the two ways to think about algebraic expressions - as algebraic functions and as algebraic forms?" We shall address this 'flexibility' issue in the last section.

PROOF AND EXPERIMENTATION

Following Duval's theory, we identify a special register that we term as *parametric register*. A parametric register is implemented in some mathematical software - not exclusive to CAS - in the form of slider bars that enable demonstrating, in a dynamic way, how changing a parameter in an algebraic expression affects the shape of the related graph. The slider bar enables to identify differences between two geometric representations associated with two values of the parameter. Moreover, it can facilitate a dynamic direct connection between the algebraic parameter and its geometric interpretation. Drijvers (2003) focused on the concept of parameter in investigating learning algebra in a computer algebra environment. Students explored the dynamic effects of expressions in specific algebraic or real-life context by means of a slider tool. He realized, however, that experimenting with the slider tool did not motivate the search for proving the results (probably because of the convincing effect of the slider tool).

Traditional mathematical publications, in general, present theorems and their proof the deductive way. The advent of computer technology opened up opportunities to include experimental mathematics in research and in education. The border line between the experiment and the proof seems to get blurred (in contradiction to the traditional disclaimer: "don't base your proof on drawing").

According to Hanna the *explanation* role of proof is to clarify why the statement is true by promoting explicit understanding of every link in the proof (Hanna, 2000). In professional development courses teachers realized that the CAS expressions encapsulate the relationships between the different parameters of the geometric figures. Unfolding these relationships by means of *symbol sense* can help in justifying unfamiliar results obtained by experimentation (Zehavi, 2004). In the next section we present a learning activity that was designed to study the inter-related roles of proof and experimentation.

A LEARNING ACTIVITY

The problem

Part (a):

(i) Given the hyperbola whose equation is $x \cdot y = 1$ (Fig. 1)

Is it possible to draw two tangents to the hyperbola from every point in the plane, which is not on the asymptotes?

Do two tangents to the hyperbola from specific points touch the same branch?

Justify your answers.

(ii) Open the CAS file *task-a*, and implement slider bars for the parameters X and Y, to move a pair of tangents to the hyperbola drawn from a point P(X, Y) (Fig. 2). Identify the loci of points in the plane from which:

No tangent can be drawn;

A single tangent can be drawn;

Two tangents to the same branch can be drawn;

Two tangents can be drawn, one to each branch.

(iii) Please rate (from 1 to 6) the need for students to prove algebraically the answers in (ii). What are the pedagogical arguments for your rating?

Part (b):

(i) The CAS file *task-b* contains the expressions that were used for creating the animation of tangents to the given hyperbola drawn from a general point P(X,Y).

Follow through the derivation of expressions in the file and complete the missing annotations.

The coordinates of the tangency points TP1 and TP2 are given by the following expressions:

$$\text{TP1} = \left[\frac{1-\sqrt{1-X\cdot Y}}{Y}, \frac{\sqrt{1-X\cdot Y}+1}{X}\right] \qquad \text{TP2} = \left[\frac{\sqrt{1-X\cdot Y}+1}{Y}, \frac{1-\sqrt{1-X\cdot Y}}{X}\right]$$

Use these expressions to prove *algebraically* the answers regarding the partition of the plane into four loci.

(ii) Please again rate the need for students to prove algebraically the answers.

What are the pedagogical arguments for your rating?

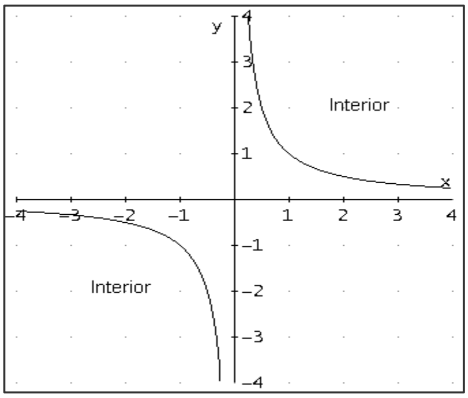

Figure 1. Graph of $x\cdot y=1$

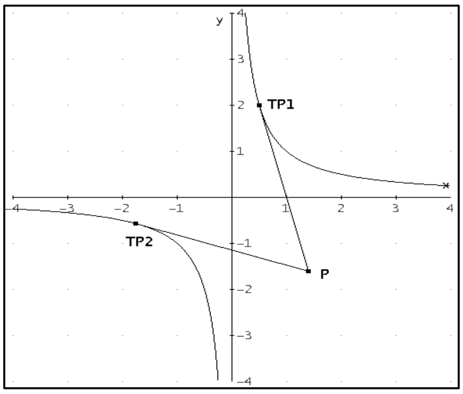

Figure 2. Dynamic tangents

The problem was given to 43 high school teachers who participated in a 36-hour professional development course. Most of them had only some basic experience in using CAS. These teachers came up with a variety of approaches and developed the relevant techniques. We present in the following the solution given by one of the teachers.

Note: In a previous study (Mann et al, 2007) teachers who had extensive experience with CAS explored the behavior of tangents to a 'more complicated' hyperbola, $\frac{x^2}{9}-\frac{y^2}{4}=1$. They almost automatically converted the representation by implementing implicit plotting of the inequality $x1 \cdot x2 < 0$ ($x1$ *and* $x2$ are the x-coordinates of the tangency points), and used the symbolic manipulator (treatment) to confirm the visual result. It yields $4X^2-9Y^2<0$.

The solution of Teacher LH

Rating the need for proof: Part (a) – 2, Part (b) – 6

Answer to Part (a):

"I drew in red several tangents to one branch and drew in blue several tangents to the other branch. From the graph we can see that:

Only tangents of the same color intersect in quadrants I or III; only tangents of different colors (if not parallel) intersect in quadrants II or IV.

Therefore, from a point in quadrants I or III, a pair of tangents to the same branch can be drawn; from a point in quadrants II or IV, each tangent touches one branch."

Arguments for rating (2) the need for proof after running the animation:

"There is no need for algebraic justification if one explains well the situation, including the case of the asymptotes. That is, only one tangent passes through an intersection point of a tangent-line to the hyperbola with the coordinate axes (the asymptotes).

I rated the need for algebraic proof '2' and not '1' because sometimes one cannot trust the software graphics."

Answers to part (b):

"The variables X and Y (in the expressions for the coordinates of the tangency points) are the coordinates of a general point P. The structure of these expressions is interesting: it contains factors of the type $(a+b)$ and $(a-b)$. This pattern calls for multiplication. When we multiply the two sub-expressions, we get: $(1-\sqrt{1-X \cdot Y})\cdot(\sqrt{1-X \cdot Y}+1)=X \cdot Y.$ Now if we multiply the x-coordinates of the tangency points $x1 \cdot x2$, we get the simplified expression $\frac{X}{Y}$. If $\frac{X}{Y}>0$, then $x1 \cdot x2>0$, meaning that the tangency points are both either in quadrant I or in quadrant III. If

$\frac{X}{Y} < 0$, then $x1 \cdot x2 < 0$, meaning that the tangency points are both either in quadrant II or in quadrant IV. This verifies my conclusion from observing the animation."

Arguments for rating (6) after providing a formal proof:

"I see now that the visual confirmation does not make the algebraic proof obsolete: the formal proof motivates reflection that enhances pattern recognition and thus develops symbol sense. The arguments of the formal proof integrate visual and algebraic representations in a meaningful way. The manipulations of symbols are not just expression processing - they require awareness."

PROFESSIONAL DEVELOPMENT WITHIN CURRICULAR R&D

LH designed an experimental activity by using two different colors to draw tangents to each of the two branches of the hyperbola. She rated (2) the need for an algebraic proof because she felt that her demonstration provides a good explanation of the situation. She was probably not aware that her experiment explains only one of the two inverse theorems involved in identifying a locus of points. However, she noticed the flaw while performing the animation for a point whose $X = 0$ or $Y = 0$, which lies on the asymptotes. The main arguments of LH for rating (6) the need for algebraic proof, in part (b), was that reflection on the expressions can make explicit the relationships between the different parameters in the expressions. These relationships are used in the steps of the proof.

Our findings show that half of the teachers did not change their rating after dealing with the algebraic expressions. The other half either increased their rating (12 teachers) or decreased it (10). Most of those who decreased their ratings wrote that students might have difficulties in producing a proof. As a group, the distribution of ratings for the need of proofs did not change much after working on part (b); for example, a rating of 5-6 was given by 20 teachers in part (a) and by 22 teachers in part (b). However, there were internal changes since only 12 teachers gave a rating of 5-6 in both parts. In part (b) we identified the 'focal' objects that led to the formal proof given by the teachers: 15 teachers did not provide a proof; for 13 teachers the focal object was the 'origin' of tangents P(*X, Y*); the expressions representing the coordinates of the tangency points were the basis of the proof of 11 teachers, and 4 teachers started their proof by converting the graphical observation into an algebraic inequality, $\frac{X}{Y} < 0$ or $x1 \cdot x2 < 0$.

The teachers' formal proofs demonstrate flexible use of *qualitative* exploration of the effect of changing the value of the parameter on the geometric representation, that is, viewing the expressions as algebraic functions, and *quantitative* explanation of the cause of the change, that is, viewing the expressions as algebraic forms (see the end of the first section).

Educators believe that the use of dynamic geometry has the potential to expand teachers' and students' understanding of proofs (for example: De Villiers, 2004).

We have realized that involving teachers in experimental mathematics in technological environment that integrates elements of DGS and CAS, helps in producing proofs; it also changes teachers perspectives regarding what constitutes a proof. Teachers' involvement in curricular R&D may help in the didactical *transposition* of mathematical proof into the classroom.

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