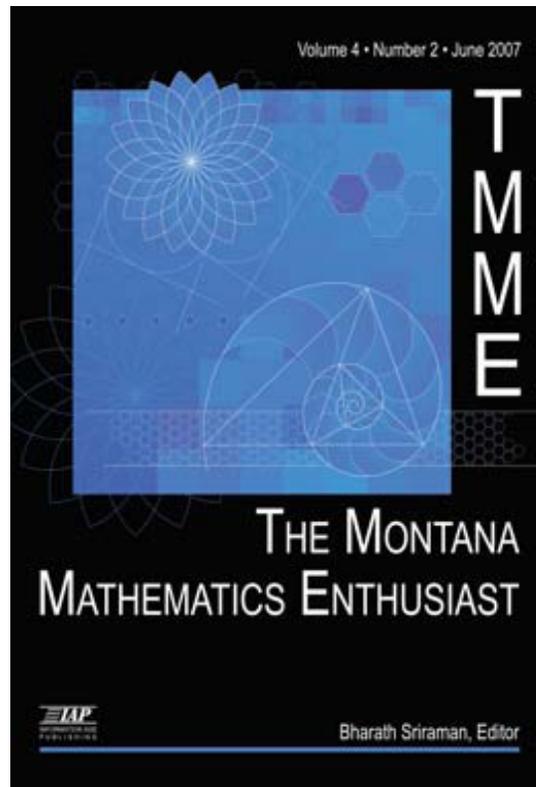


The Montana Mathematics Enthusiast

ISSN 1551-3440

VOL. 6, NOS.1&2, January 2009, pp.1-294



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The journal also includes a monograph series on special topics of interest to the community of readers. The journal is accessed from 110+ countries and its readers include students of mathematics, future and practicing teachers, mathematicians, cognitive psychologists, critical theorists, mathematics/science educators, historians and philosophers of mathematics and science as well as those who pursue mathematics recreationally. The 40 member editorial board reflects this diversity. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor in APA style. The typical time period from submission to publication is 8-11 months. Please visit the journal website at <http://www.montanamath.org/TMME> or <http://www.math.umt.edu/TMME/>

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THE MONTANA MATHEMATICS ENTHUSIAST

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TO PUBLISH OR NOT TO PUBLISH- the Editorial conundrum

Bharath Sriraman, The University of Montana

This editorial began in my mind (a mental blog if you will) as I was making my way from Tromsø (Norway) to Montana late in December. As 2009 slowly rolls in, I am reminded of the 18th century Scottish bard Robert Burns' famous poem "Auld Lang Syne" for several reasons.

*Should auld acquaintance be forgot,
And never brought to mind ?
Should auld acquaintance be forgot,
And days o' lang syne ?*

This poem, typically sung on New Year's eve, has served as the backdrop for many important events all over the world. Most recently it was played when the Pakistani president Pervez Musharraf stepped down as the Army Chief, signaling a transition to an era of civilian government in Pakistan. The heinous terrorist incidents that followed in Mumbai (Bombay), which partly can be attributed to the turmoil caused by the artificial borders carved by the British Raj in the wake of their departure from the Indian subcontinent, served as a reminder to the tenuous nature of "change". Yet we are hopeful that things are changing in a positive direction in spite of the mess caused by post colonial geopolitics. After all politics and radicalism need not be the lowest common denominator for communication between sides that share thousands of years of common heritage, language and history (Yes we Can!).

What role, if any, does mathematics and mathematics education have in all this? If we claim to live in a world where any two people can theoretically meet within 24 hours, or communicate in real time thanks to the advances in information technology, then it only makes sense that education instill in future generations of students a sense of shared heritage despite superficial differences based on the Bismarckian notion of a nation-state.

The history of Central Asia, the Indian sub-continent, the Persian-Greco world and numerous other regions when analyzed from the viewpoint of trade and the exchange of mathematical ideas reveals an intricate shared heritage. The current day turmoil in the world based on ideology, religion and artificially drawn post-colonial borders can very well serve as a focal point to examine how culturally based studies of mathematics could serve as a vehicle for promoting peace and discourse instead of economies that flourish under the politics of division and the export of weapon's technology. I envision one of our goals should be to revisit fundamental notions of what constitutes a culturally appropriate math curriculum, in a globally linked world with shared problems and a collective future. For the last few decades many mathematics educators have emphasized the place of critical mathematics education in order to better understand problems plaguing society. The global fall out resulting from the unchecked greed of Wall Street and the corporate world/mentality in general in numerous parts of the world, serves

as an important context to promote the basic principles of mathematics and the necessity to revisit prevalent notions of consumerism and materialism in the West, which come at the expense of other regions of the world. However as well intentioned an analysis of local socio-economically and politically situated problems may be through the lens of critical mathematics education, it is equally important to better educate young minds in critical history and geography. That is, not boring details and facts such as how high a mountain is, or how long a river is (Dewey, 1927 as cited by Howlett, 2008, p.27), but a global awareness of peoples, cultures, habits, occupations, art and societies contributions to the development of human culture in general (Dewey 1939, as cited by Howlett, 2008, p.27) in addition to the contiguous contributions of all cultures to the development of mathematics and science.

Edward Said (1935-2003), the Palestinian American literary /critical/cultural theorist redefined the term *Orientalism* to describe a tradition, both academic and artistic, of hostile and deprecatory views of the East by the West. The curricula used in many parts of the world today is still shaped by the attitudes of the era of European imperialism in the 18th and 19th centuries and conveys in a hidden way prejudiced interpretations of colonized cultures and peoples, particularly indigenous peoples. These biases become apparent in the popular media's simplistic and dichotomous view of problems in post colonial Asia (including the Middle East) where oversimplification is often done on religious, nationalistic and ethnic terms, such as Hindu versus Muslim, Arab versus Jew, Sunni versus Shia, Kurd versus Turk, Turk versus Greek, Irani versus Iraqi, etc. This perpetuates the patronizing and overtly patriarchal view of colonized peoples and indigenous cultures to justify external meddling in their political affairs.

What is the role of a math journal in all this? *The Montana Mathematics Enthusiast* aims to publish critically oriented articles relevant for mathematics education in addition to striving to represent under-heard voices in the larger debates characterizing mathematics education. The journal is thriving with submissions from all parts of the world and we are delivering on our promise to help non-English speaking authors from under-represented regions, to the extent we can to publish their work, by finding appropriate reviewers and other means of support. The present issue contains 22 articles with numerous authors from South America [Argentina, Brazil, Uruguay] in addition to contributions from authors in Central Europe (Hungary) and the Mediterranean (Cyprus, Greece, Turkey). Many of these articles are developed from papers presented at the International Conference on Teaching Statistics in Brazil (ICOTS-7). Other voices from Australia and New Zealand lend a nice representation to mathematics education developing in the Southern hemisphere. As usual there is a nice synthesis of articles focused on mathematics content, and those that focus on research of teaching, learning and thinking issues in mathematics education, as well as a Montana feature on Book X of Euclid's Elements.

In 2009, the journal will publish its normal 3 issues in addition to publishing special supplementary issues on inter-disciplinarity, mathematics talent development and at least three new monographs! This hopefully answers the rhetorical question, to publish or not to publish...

References

Howlett, C. F. (2008). John Dewey and peace education. In M. Bajaj (Ed). *Encyclopedia of Peace Education* (pp. 25-32). Information Age Publishing, Charlotte, NC.

TEACHER KNOWLEDGE AND STATISTICS: WHAT TYPES OF KNOWLEDGE ARE USED IN THE PRIMARY CLASSROOM?

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Abstract: School curricula are increasingly advocating for statistics to be taught through investigations. Although the importance of teacher knowledge is acknowledged, little is known about what types of teacher knowledge are needed for teaching statistics at the primary school level. In this paper, a framework is described that can account for teacher knowledge in relation to statistical thinking. This framework was applied in a study that was conducted in the classrooms of four second-year teachers, and was used to explore the teacher knowledge used in teaching statistics through investigations. As a consequence, descriptions of teacher knowledge are provided and give further understanding of what teacher knowledge is used in the classroom.

Keywords: cKc; elementary schools; mathematics teacher education; statistical investigations; statistical thinking; teacher knowledge

INTRODUCTION

Statistics education literature in recent years has introduced the terms of statistical literacy, reasoning, and thinking, and they are being used with increasing frequency. Wild and Pfannkuch's (1999) description of what it means to think statistically has made a significant contribution to the statistics education research field, and has provided a springboard for research that further explores and contributes to an understanding of statistical thinking and its application. Increasingly, it is recognised that statistics consists of more than a set of procedures and skills to be learned. School curricula, including New Zealand's, advocate for investigations to be a major theme for teaching and learning statistics.

Debate about teacher knowledge and its connections to student learning has had a long history. An important question arises as to what knowledge is considered adequate and appropriate. Although much is known about teacher knowledge pertinent to particular aspects of mathematics, the situation for statistics is less clear. Arguably, the mathematical knowledge needed for teaching and the statistical knowledge needed for teaching do share some similarities. Yet, there are also differences (Groth, 2007), due in no small way to the more subjective and uncertain nature of statistics compared with mathematics (Moore, 1990). Pfannkuch (2006, personal communication) claims that, because of the relatively brief history of statistics education research in comparison with mathematics education research, there is still much that is unknown about the specifics of teacher knowledge needed for statistics.

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This paper reports on a framework that was proposed and applied in a study that investigated teacher knowledge needed and used by teachers during a unit in which primary school students investigated various multivariate data sets. The focus here is on justifying the need for such a framework in relation to teaching statistics, and on providing descriptions of teacher knowledge as revealed in the classroom in relation to the framework for teacher knowledge that combines statistical thinking components with categories of teacher knowledge. Examples from the classroom are provided to support the knowledge descriptions in relation to some of the components from the teacher knowledge framework. Finally, the conclusions consider some of the implications of this research, particularly for teacher education, both preservice (or initial teacher education) and inservice (or professional development).

LITERATURE REVIEW

Research on teacher knowledge is diverse. The thread of research from that of Shulman (1986) who defined pedagogical content knowledge (as one category of the knowledge base needed for teaching) provides a useful way of examining teacher knowledge. Shulman claims that a teacher's pedagogical content knowledge goes beyond that of the subject specialist, such as the mathematician. Subsequent research has attempted to clarify the differences between categories of teacher knowledge, either using Shulman's categories, or others developed from Shulman's categorisation.

Much of this research, although conducted with teachers, has not been conducted in the classroom, the site in which teacher knowledge is used. Cobb and McClain (2001) advocate approaches for working with teachers that do not separate the pedagogical knowing from the activity of teaching. They argue that unless these two are considered simultaneously and as interdependent, knowledge becomes treated as a commodity that stands apart from practice. Their research focused on the moment-by-moment acts of knowing and judging. Similarly, Ball (1991) discusses how teachers' knowledge of mathematics and knowledge of students affect pedagogical decisions in the classroom. For instance, the subject matter knowledge of the teacher determines to a significant extent which questions from students should or should not be followed up. Similarly, subject matter knowledge enables the teacher to interpret and appraise students' ideas. Ball and Bass (2000) argue strongly that without adequate mathematical knowledge, teachers will not be in a position to deal with the day-to-day, recurrent tasks of mathematics teaching, and as such, will not cater for the learning needs of diverse students.

A focus on the knowledge of content that is required to deliver high-quality instruction to students has led to another model of teacher knowledge, which involves a refinement of the categories of subject matter knowledge and pedagogical content knowledge. Hill, Schilling, and Ball (2004) claim that teacher knowledge is organised in a content-specific way, rather than being organised for the 'generic tasks of teaching', such as evaluating curriculum materials or interpreting students' work. Two sub-categories of content knowledge are further clarified by Ball, Thames, and Phelps (2005): *common knowledge of content* includes the ability to recognise wrong answers, spot inaccurate definitions in textbooks, use mathematical notation correctly, and do the work assigned to students. In comparison, *specialised knowledge of content* needed by teachers (and likely to be beyond that of other well-educated adults) includes the ability to analyse students' errors and evaluate their alternative ideas, give mathematical explanations, and

use mathematical representations. Ball et al. (2005) also subdivide the category of pedagogical content knowledge into two components, namely *knowledge of content and students*, and *knowledge of content and teaching*. These two parts of teacher knowledge bring together aspects of content knowledge that are specifically linked to the work of the teacher, but are different from specialised content knowledge. *Knowledge of content and students* includes the ability to anticipate student errors and common misconceptions, interpret students' incomplete thinking, and predict what students are likely to do with specific tasks and what they will find interesting or challenging. *Knowledge of content and teaching* deals with the teacher's ability to sequence the content for instruction, recognise the instructional advantages and disadvantages of different representations, and weigh up the mathematical issues in responding to students' novel approaches.

Although statistics is considered to be part of school mathematics, there are some significant differences that have implications for the teaching and learning of statistics. In mathematics, students learn that mathematical reasoning provides a logical approach to solve problems, and that answers can be determined to be valid if the assumptions and reasoning are correct (Pereira-Mendoza, 2002), that the world can be viewed deterministically (Moore, 1990), and that mathematics uses numbers where context can obscure the structure of the subject (Cobb & Moore, 1997). In contrast, statistics involves reasoning under uncertainty; the conclusions that one draws, even if the assumptions and processes are correct, are 'uncertain' (Pereira-Mendoza, 2002); and statistics is reliant on context (delMas, 2004; Greer, 2000), where data are considered to be numbers with a context that is essential for providing a meaning to the analysis of the data. It becomes necessary when teaching statistics, to encourage students to not merely think of statistics as doing things with numbers but to come to understand that the data are being used to address a particular issue or question (Cobb, 1999; Gal & Garfield, 1997).

Statistical literacy, reasoning, and thinking have featured in the statistics education literature in recent years. Ben-Zvi and Garfield (2004) provide some clarity for these terms, although with regard to statistical thinking, Wild and Pfannkuch's (1999) paper provided a model for statistical thinking. Wild and Pfannkuch describe five fundamental types of statistical thinking: (1) a recognition of the need for data (rather than relying on anecdotal evidence); (2) transnumeration – being able to capture appropriate data that represents the real situation, and change representations of the data in order to gain further meaning from the data; (3) consideration of variation – this influences the making of judgments from data, and involves looking for and describing patterns in the variation and trying to understand these in relation to the context; (4) reasoning with models – from the simple (such as graphs or tables) to the complex, as they enable the finding of patterns, and the summarising of data in multiple ways; and (5) the integrating of the statistical and contextual – making the link between the two is an essential component of statistical thinking. Along with these fundamental types of thinking are more general types that could be considered part of problem solving (but not exclusively to statistical problem solving). Wild and Pfannkuch's dimension of 'types of thinking' is one of four dimensions that explain statistical thinking in empirical enquiry. The other three dimensions are: the investigative cycle (problem, plan, data, analysis, and conclusions – these are the "procedures that a statistician works through and what the statistician thinks about in order to learn more from the context sphere" (Pfannkuch & Wild, 2004, p. 41)); the interrogative cycle (generate, seek, interpret, criticise, and judge) – this "is a generic thinking process that is in constant use by

statisticians as they carry out a constant dialogue with the problem, the data, and themselves” (Pfannkuch & Wild, 2004, p. 41); and dispositions (including scepticism, imagination, curiosity and awareness, openness, a propensity to seek deeper meaning, being logical, engagement, and perseverance), which affect or propel the statistician into the other dimensions. All these dimensions constitute a model that encompasses the dynamic nature of thinking during statistical problem solving, and is non-hierarchical and non-linear.

This model for statistical thinking was developed through reference to the literature following interviews with statisticians and tertiary statistics students as they performed statistical tasks (Wild & Pfannkuch, 1999). Although it was developed as a model applicable to the statistical problem solving of statisticians and tertiary students, it has subsequently been used in a variety of other studies, such as an examination of the thinking of primary students (Pfannkuch & Rubick, 2002) and pre-service primary teacher education students (Burgess, 2001), through a professional development workshop with secondary teachers (Pfannkuch, Budgett, Parsonage, & Horring, 2004), and an investigation into how statistical thinking of learners can be encouraged through a teaching activity (Shaughnessy & Pfannkuch, 2002).

The Framework

Teacher knowledge frameworks from the mathematics education domain are inadequate for examining teacher knowledge for statistics because of the differences between statistics and mathematics, as discussed earlier. The development of a teacher knowledge framework that takes into account the particular needs of statistics teaching and learning is therefore required. Such a framework must be specific to statistics, since teacher knowledge is organised in content-specific ways (Hill et al., 2004). Consequently the framework on which this study is based draws heavily on the statistical thinking model of Wild and Pfannkuch (1999). The categories of teacher knowledge that are described by Hill, Schilling, and Ball (2004) and Ball, Thames, and Phelps (2005), namely mathematical content knowledge and pedagogical content knowledge, and each of these with two sub-categories, provide a good starting point for examining statistics content knowledge as enacted in classroom teaching.

A matrix for a conceptual framework, against which statistical knowledge for teaching can be examined, is shown in Table 1.

Table 1: The framework for teacher knowledge in relation to statistical thinking and investigating.

		Statistical knowledge for teaching			
		Content knowledge		Pedagogical content knowledge	
		Common knowledge of content (ckc)	Specialised knowledge of content (skc)	Knowledge of content and students (kcs)	Knowledge of content and teaching (kct)
Thinking	Need for data				
	Transnumeration				
	Variation				
	Reasoning with models				
	Integration of statistical and contextual				
Investigative cycle					
Interrogative cycle					
Dispositions					

The columns of the matrix refer to the types of knowledge that are important in teaching. These four types are: common knowledge of content (ckc); specialised knowledge of content (skc); knowledge of content and students (kcs); and knowledge of content and teaching (kct). Hill, Schilling and Ball (2004) and Ball, Thames, and Phelps (2005) describe the features of these four categories of teacher knowledge in relation to number and algebra. These descriptions arise from a consideration of the question, “What are the tasks that teachers engage in during their work in the classroom, and how does the teachers’ mathematical knowledge impact on these tasks?” From those researchers’ close examination of teachers’ work, it is apparent that much of what teachers do throughout their teaching is essentially mathematical.

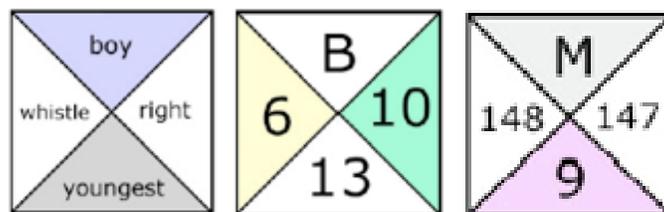
Just as Ball et al. (2001) claim that many of the everyday tasks of the teacher of mathematics are essentially mathematical, it is suggested that much of what a teacher engages in during the teaching of statistical investigations essentially involves statistical thinking and reasoning. Consequently, the four teacher knowledge categories are examined in relation to statistical thinking. The main feature that sets this framework apart from those offered for the mathematics

domain is the inclusion of the elements of statistical thinking and empirical enquiry (Wild & Pfannkuch, 1999), which are listed as the rows of the matrix.

THE STUDY

Since teacher knowledge is acknowledged to be important in relation to what and how students learn and is dependent on the context in which it is used (Ball & Bass, 2000; Barnett & Hodson, 2001; Borko, Peressini, Romagnano, Knuth, Willis-Yorker, Wooley et al., 2000; Cobb, 2000; Cobb & McClain, 2001; Fennema & Franke, 1992; Foss & Kleinsasser, 1996; Friel & Bright, 1998; Marks, 1990; Sorto, 2004; Vacc & Bright, 1999), it is argued that research should therefore take place in the classroom. Also, research on teacher knowledge must acknowledge and accommodate the dynamic aspects of teacher knowledge (Manouchehri, 1997), and be based on an understanding of how knowledge evolves. A post-positivist realist paradigm (Popper, 1979, 1985) was chosen because of the explanations about where knowledge comes from and how it grows in a dynamic fashion. Popper argued that knowledge develops through trial and elimination of error, and the logic of learning model (Burgess, 1977) was proposed as being appropriate for examining learning in classroom settings (Swann, 1999).

Using this post-positivist realist paradigm, case study research was undertaken with four inexperienced primary teachers (all in their second year of teaching), Linda, John, Rob, and Louise (all pseudonyms). The four classes were in the Year 5 (about 9-10 years old) to Year 8 (about 12-13 years old) level of primary school. The teachers were given a teaching unit that required students to investigate some multivariate data sets. The teachers developed their teaching based on this unit. The data sets generally consisted of 24 cases, each with four variables (or attributes). The first set used by each teacher included four category variables, while the other sets included at least two numeric variables along with the category variable(s). Each case was presented on a data card (see examples below from three different data sets), so that the students could easily manipulate and sort the cards in order to discover interesting things in the data.



Each lesson was videotaped, then edited by the researcher in order to focus on interesting episodes from the lesson. The edited videotape was shown to the teacher, and the discussion between the teacher and the researcher was audiotaped. The videotapes and the audiotapes from the post-lesson discussions were analysed in relation to the cells of the framework. Segments from the lessons or the discussions were identified in relation to the categories of teacher knowledge and the components of statistical thinking that were in evidence.

This paper reports on the results pertinent to the following research question:

What are the features of teacher knowledge in relation to aspects of statistical thinking that are used in the classroom?

DESCRIPTIONS OF THE FRAMEWORK

An understanding of the *need for data* on which to base sound statistical reasoning, instead of relying on and being satisfied with anecdotal evidence, is important in the development of statistical thinking. This corresponds to the first row of the framework. Classroom investigations can be conducted through two different approaches. First, an investigation can start with a question or problem to be solved and move onto data collection, which requires an understanding that data needs to be collected in order to solve the question or problem. The second approach is to start with a data set and generate questions for investigation from that data. By adopting this second approach for this study, teachers and students were not faced with the issues pertinent to establishing the need for data to help solve their questions. Consequently the *need for data* did not feature in this research. As such, the *need for data* is not described in relation to the four categories of teacher statistical knowledge for the framework.

Dispositions (corresponding to the final row of the framework), as another component of statistical thinking, did not emerge specifically in relation to the individual components of teacher knowledge but in a more general way. Teachers' statistical dispositions were apparent in the classroom. For example, inquisitiveness and readiness to think in relation to data along with an anticipation of what was to come was evident when Linda asked the students what they had started to notice when filling in their own data cards. She justified this question in the subsequent interview by saying that it was "to give them a hint of what was to come ... to see if the students had the inclination to start making their own conclusions already."

Common knowledge of content

As described by Ball, Thames, and Phelps (2005), *common knowledge of content* refers to what the educated person knows and can do; it is not specific to the teacher. They describe it as including the ability to recognise wrong answers, spot inaccurate definitions in textbooks, use mathematical notation correctly, and do the work assigned to students.

Wild and Pfannkuch (1999) describe *transnumeration* as the ability to: sort data appropriately; create tables or graphs of the data; and find measures to represent the data set (such as a mean, median, mode, and range). In general, transnumeration involves changing the representation of data in order to make more sense of it.

For teaching, *common knowledge of content: transnumeration* includes the knowledge and skills described above, along with the ability to recognise whether, for instance, a student gave the correct process or rule for finding a measure, had created a table correctly, or had sorted the data cards appropriately. Evidence of this category (as well as others involving common knowledge of content) was not often observed because the teachers generally used other types of teacher knowledge in relation to *transnumeration*. However if, for example, a teacher asked questions that led the students towards sorting the data in a particular way, it was assumed that the teacher

also had the *common knowledge of content* of how to do this for him or herself. There were instances where the researcher verified that this was indeed the case by asking the teacher during the interview to sort the cards, calculate a measure, or something similar. Consequently, *common knowledge of content: transnumeration* was subsumed within other categories of knowledge.

Consideration of variation in data is an important aspect of statistical thinking (Wild & Pfannkuch, 1999). It affects the making of judgments based on data, as without an understanding that data varies in spite of patterns and trends that may exist, people are likely to express generalisations based on a particular data set as certainties rather than possibilities.

The knowledge category of *common knowledge of content: variation* manifests itself in the classroom when the teacher gives examples of statements about data that acknowledge variation through the language used. Some of the more common situations that were observed related to inferential statements. Such statements were either about the actual data set and based on it, or generalisations about a larger group (population) from the smaller data set (sample). Such language included words and phrases such as “maybe ...”, “it is quite likely that ...”, and “there is a high probability that ...”. In addition, when the teacher talked about another sample being similar, but not identical, to the first sample, *common knowledge of content: variation* was evidenced.

For people to be able to make sense of data, statistical thinking requires the use of models. At the school level, appropriate models with which students could reason include graphs, tables, summary measures (such as median, mean, and range), and as used in this research, sorted data cards. If teachers demonstrated evidence of *common knowledge of content: reasoning with models*, it would be through making valid statements for the data, based on an appropriate use of a model.

Wild and Pfannkuch (1999) describe the importance of continually linking contextual knowledge of a situation under investigation with statistical knowledge related to the data of that situation. The interplay between these two enables a greater level of data sense and a deeper understanding of the data, and is therefore indicative of a higher level of statistical thinking.

The component of *common knowledge of content: integration of statistical and contextual* is characterised by the ability to make sense of graphs or measures, and by an acknowledgement of the relevance and interpretation of these statistical tools to the real world from which the data was derived. For example, John gave some possible reasons to support the finding that all the youngest students could whistle. He suggested that the older siblings could have taught the younger ones to whistle. This shows thinking of the real-life context in association with what the statistical investigation had revealed; such integration of the two aspects can sometimes enable the answering of ‘why might this be so’ that is being illustrated by the data.

One of the four dimensions of statistical thinking, as defined by Wild and Pfannkuch (1999), is the *investigative cycle*. This cycle, characterised by the phases of ‘problem, plan, data, analysis, and conclusions’, is what someone works through and thinks about when immersed in problem solving using data. If a teacher can fully undertake and engage with an investigation, then that teacher would be demonstrating *common knowledge of content: investigative cycle*. The teacher would be able to: pose an appropriate question or hypothesis, or set a problem to solve; plan for

and gather data; analyse that data; and use the analysis to answer the question, prove the hypothesis, or solve the problem.

For example, Linda discussed how data might be handled with an open-response type of question in a survey or census. Linda had considered, at the problem-posing phase of the investigation, how the responses from such an open-response type question would present a challenge at the analysis stage. This clearly indicated that Linda had some knowledge of the phases of the investigative cycle. She was able to maintain an awareness of a later stage of the cycle (analysis) while dealing with an early stage (planning data collection), and consider how decisions at that early stage could impact on the later stages.

A teacher would have *common knowledge of content: interrogative cycle* if it was evident that possibilities in relation to the data were considered and weighed up, with some possibilities being subsequently discarded but others accepted as useful. Engaging with data and being involved in 'debating' with it would be evidence of such knowledge. Likewise, developing questions that the data may potentially be able to answer is an aspect of *common knowledge of content: interrogative cycle*. Teachers who had immersed themselves with a data set prior to using it in teaching, so that they were aware of some of the things that might be found from the data, would be showing *common knowledge of content: interrogative cycle*. Such teachers would be prepared for knowing what their students might find in the data and what conclusions might be drawn from that data.

Specialised knowledge of content

A teacher requires *specialised knowledge of content: transnumeration* to analyse whether a student's sorting, measure, or representation was valid and correct for the data, particularly if the student has done something in a non-standard and unexpected way. It includes the ability to justify a choice of which measure is more appropriate for a given data set, or to explain when and why a particular measure, table, or graph would be more appropriate than another. Some of these skills, although considered part of statistical literacy (Ben-Zvi & Garfield, 2004), are still currently beyond what many educated adults can undertake. As such they are considered to be part of *specialised knowledge of content: transnumeration* rather than *common knowledge of content:transnumeration*.

Specialised knowledge of content: transnumeration was identified for all the teachers in the study. For example, Linda attempted to follow a student's description of how she had sorted the data and converted it into an unconventional table involving all four variables. The table consisted of: four columns labelled G, B, G, B; four rows with labels on the left to account for two more variables; labels on the right for three rows to account for the fourth variable; but no numbers or tally marks in the cells of the table to represent the sorted data. To determine the statistical appropriateness of that particular representation, Linda had to call on her *specialised knowledge of content: transnumeration* as she tried to make sense of the table. In another example in relation to some students deciding which measure or measures they should calculate for the data set (out of the mode, median and mean), Rob recognised that the mode would not be the most appropriate measure to use for the numerical data in question, and was able to give some justification regarding the inappropriateness of the mode.

Making sense of and evaluating students' explanations around whether it is possible to generalise from the data at hand to a larger group involves *specialised knowledge of content: variation*. For instance, when Linda asked whether there would be many boys who watched a particular programme on TV based on the class data that showed only a small proportion of such boys, a student answered, "Don't know; she hasn't asked all the classes yet." The teacher had to evaluate whether that was a reasonable response in relation to understanding of variation; Linda explained that there are factors that might affect the validity of this generalisation, but that the student's justification (about not having the data from the population so therefore it was not possible to make such a generalisation) was not a good reason for not generalising from the class data.

Specialised knowledge of content: reasoning with models is needed to interpret students' statements to determine the validity or otherwise of those statements. Students often struggled with making sensible and valid statements about the data based on a particular model they were using, and as a consequence it was not always straightforward for the teachers to make sense of the students' statements. Consequently, this category is seen as being quite distinct from *common knowledge of content: reasoning with models*.

Specialised knowledge of content: reasoning with models was a very commonly occurring component of teacher knowledge, especially as the focus of the unit was on finding interesting things in multivariate data sets, and making statements about these data sets. In many cases, students justified their statements through reference back to the model and as such, the teachers needed *specialised knowledge of content: reasoning with models* to help check the veracity of the students' statements. For example, the following interaction, initially between Linda and one student but later extended to the whole class, exemplifies the challenge for teachers to listen to and make sense of students' statements:

Student: That most girls can write with their right hand, ... most girls write with their right hand ... [inaudible].

Teacher: Sorry, I didn't catch what you said. Can you say that again for me? Slower this time.

Student: Most girls can write with their right hand are the youngest in ...

Teacher: Hang on. Most ... what are you saying? Most girls who produce their neatest handwriting with their right hand can whistle. ... [pause]. Okay ... [pause]. How many girls who produce their neatest handwriting with their right hand can whistle? ... [pause] Is that what you have got in front of you? [pointing at the cards on the desk] ... How many is that? [Student can be seen nodding as he counts cards] ... Is that these ones?

Teacher: So there are 5? ... These ones can whistle as well? But are they right handed? Okay. So what are you comparing that with? You said "most." So most compared with what? [No response from student.] In comparison with the right handed boys or in comparison with the left handed girls?

Student: Left handed girls.

Teacher: Okay... [pause] So R and J have taken that a step further and they have got ... [teacher moves to the whiteboard and starts drawing a type of two-way table – see Figure 1] ... here right-handed girls and right-handed boys and they have taken just this square [lower right] and sorted those people [the right handed girls] into different piles, into whistlers and non-whistlers. And they have found that there are more whistlers who are girls who are right handed than non-whistlers who are girls who are right handed. I think that is what they are trying to say.

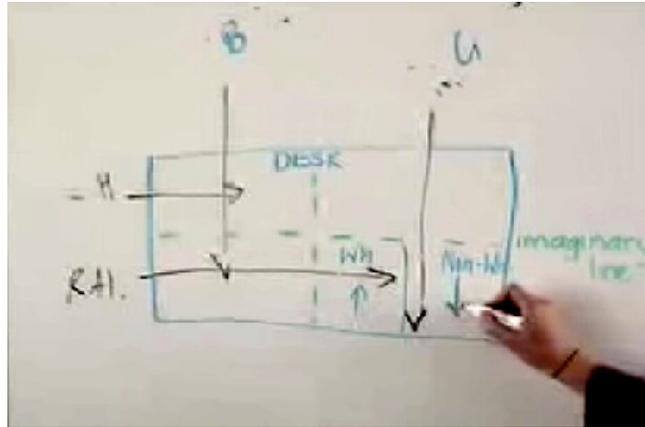


Figure 1: Diagram drawn by Linda to help students make sense of the statement from a student.

The interaction indicates the use of *specialised knowledge of content: reasoning with models* by the teacher, involving initially the model of sorted data cards on the student's desk, followed by the model on the board that she created from transnumeration of the data cards.

Being able to evaluate a student's explanation based on both statistical data and a knowledge of the context under investigation is one aspect of the category of *specialised knowledge of content: integration of statistical and contextual knowledge*. There were a number of situations in which the teacher prepared the students to gather data. Data collection questions had been suggested, such as, "What position are you in the family, youngest, middle or eldest?" When the students were considering the question prior to the actual data gathering, Linda was asked:

- Does it count if you have half brothers or sisters?
- What if your sister or brother has died?
- What if your brother or sister is not living at home?
- What would you put if you were an only child?

Each of these questions, and others involving the definition of family, were unexpected by Linda. She had to decide 'on the spot' how to respond to each question from students. She was required to weigh up the statistical issues related to answering such a data gathering question with the contextual issue of interpretation of 'family'. Her answers indicated that she was able to do so satisfactorily and therefore were evidence of her having *specialised knowledge of content: integration of statistical and contextual*.

A teacher needs *specialised knowledge of content: investigative cycle* when dealing with students' questions or answers in relation to phases of the investigative cycle, or when discussing or explaining various phases of the cycle and how they might interact. When thinking about suggestions for what could be investigated in a data set, the teacher needs to be able to evaluate the suitability of the problem/question, and whether it needs to be refined to be usable and suitable, in relation to the subsequent analysis.

So what does *specialised knowledge of content: interrogative cycle* look like, as distinguished from *common knowledge of content: interrogative cycle*? When a teacher has to consider

whether a suggestion from a student is viable for investigating within that data, the teacher requires *specialised knowledge of content: interrogative cycle*. Also, it involves determining whether a student's suggested way of handling and sorting the data would be useful to enable the later interpretation of results in relation to the question at hand.

Knowledge of content and students

The *knowledge of content and students: transnumeration* component includes: knowledge of the common errors and misconceptions that students develop in relation to the skills of transnumeration (including sorting data, changing data representations such as into tables or graphs, and finding measures to summarise the data); the ability to interpret students' incomplete or 'jumbled' descriptions of how they sorted, represented, and used measures to summarise the data; an understanding of how well students would handle the tasks of transnumeration; and an awareness of what students' views may be regarding the challenge, difficulty, or interest in the tasks of transnumeration.

There were situations in which students, when handling the data cards and sorting them, tried to consider too many variables at once and could not manage the complexity in the sorting of the cards and in making sense of what the cards showed. Linda was aware of this difficulty and guided the students to sort the cards 'more slowly'. She suggested sorting by one variable, and then splitting the groups by a second variable; she knew how many groups of data there would be from sorting by three variables and therefore that it needed to be simplified for the students. In general, the teachers did not realise how much the students would struggle with sorting the data cards, especially when the students were looking at numeric data such as arm spans, heights, and so forth. The teachers were surprised that the students did not naturally order the numeric data but simply grouped the data cards into piles. Furthermore, sorting data cards to check for and show relationships between two data sets was difficult for students, and most of the teachers underestimated the level of challenge that students would therefore face with sorting to show relationships in the data.

Knowledge of content and students: variation includes knowing what students may struggle with in relation to understanding variation, and to predict how students will handle tasks linked to variation. Whether students can appreciate and think about variation in data while looking for patterns and trends in the data is something that a teacher needs to listen for in students' explanations and generalisations. Although all the teachers posed questions as to whether it was possible to generalise from the class data to a wider group, there was no significant evidence of *knowledge of content and students: variation* being used by the teachers. It may be that for the investigations being conducted, such teacher knowledge of *variation* was not called on because the students were not ready for this inferential-type thinking. Since it was something new for the teachers to teach, they had not considered the statistical implications relevant to the students' readiness for thinking in relation to *variation*.

If a teacher can anticipate the difficulties that students might have with reasoning using models, or can make some sense of students' incomplete descriptions, then the teacher would be showing evidence of *knowledge of content and students: reasoning with models*. In one example of such knowledge, Rob described how he worked with a group of students who had made a statement

from the data cards comparing the number of boys with the number of girls who were right or left handed. Rob knew that the students were capable of proportional thinking so he encouraged them to consider proportions. He did so because the numbers of boys and girls in the data cards were different, and therefore using proportions for the comparison would be more appropriate than using frequencies. Rob knew these students sufficiently to encourage them to reason with a proportional model, which two of the students handled particularly well.

Can a teacher anticipate that students may have difficulty with linking contextual knowledge with statistical knowledge? Are students, through focusing on statistical knowledge and skills, likely to ignore knowledge of the real world, that is, contextual knowledge, or vice versa? Such aspects would give an indication of a teacher's *knowledge of content and students: integration of statistical and contextual*.

Whereas Linda's students' questions which related to the data question of position in the family (as discussed above) were unexpected, John anticipated such possible difficulties for his students and pre-empted their questions by asking the class how each child from a four-child family might answer the question, "Are you youngest, middle, or eldest in the family?" John's question encouraged the students to think about the data question (the statistical) in association with their knowledge of particular families (the contextual). This helped the students understand that statistics is not performed 'in a vacuum', removed from real issues, but deals with numbers that have a context (delMas, 2004).

Knowledge of where students might encounter problems or particular challenges in an investigation, and whether students will find an investigation interesting or difficult, are aspects of *knowledge of content and students: investigative cycle*.

One teacher predicted that students could have a problem with knowing how to interpret a data collection question so had to consider how he would deal with this potential problem within an early phase of the investigative cycle. The analysis phase of an investigation was predicted to present challenges for students in relation to them deciding on the form to present the data.

Some teachers were aware that students would be challenged within the investigative cycle with moving from the analysis stage to the drawing of conclusions or the answering of questions that had formed the basis of the investigation. Such awareness meant that those teachers had thought about how to address the students' difficulties.

Knowledge of how students would handle the development of appropriate questions for investigating the data, and the extent to which they might engage with the data and be prepared to question and consider various possibilities, are elements of *knowledge of content and students: interrogative cycle*.

There were a number of instances when teachers became aware that students, rather than fully engaging with the data and seeking possibilities, were focusing on a narrow aspect of the data, such as individual data points. The students then used this narrow focus to argue for or justify a particular position. Teachers who had *knowledge of content and students: interrogative cycle* were able to consider ways in which this tendency amongst students could be mitigated. Such considerations led to knowledge of content and teaching: interrogative cycle being utilised.

Knowledge of content and teaching

Knowledge of content and teaching, as far as mathematics is concerned, includes the ability to appropriately sequence the content for teaching, to recognise the instructional advantages and disadvantages of particular representations, and weigh up the mathematical issues in responding to students' unexpected approaches. So what are the features of knowledge of content and teaching with regard to statistics?

The ability to plan an appropriate teaching sequence related to transnumeration data, to understand which representations are likely to help or hinder students' development of the skills of transnumeration, and to decide from a statistical point of view how to respond to a student's answer, are all aspects of *knowledge of content and teaching: transnumeration*. All the teachers displayed this component of knowledge. Some examples of its use included suggestions: for how the data cards might be arranged on the desk when sorting; to spread the data cards within each group so that all the data cards could be seen, which helped with noticing patterns or irregularities within the data and then making statements about what had been found; and for creating a two-way table of frequencies as another useful representation of the sorted data cards.

How to structure teaching for understanding variation is the main component of *knowledge of content and teaching: variation*. Teachers intentionally modelled appropriate explanations and generalisations, through the use of language that acknowledged the existence of variation, and their questioning encouraged the students to consider whether various generalisations were appropriate. Students were challenged to consider the presence of variation in the data and therefore how it would affect statements that could be made about the data.

An example of a teacher using *knowledge of content and teaching: variation* arose when Linda challenged a student who claimed that, although all boys in the class could whistle, not all boys could whistle. She asked: "Why not? We have just found that all boys in this class can whistle. Why wouldn't it be the same everywhere else?" Linda justified this question as encouraging the students to think about "the bigger picture ... This was data for our class. It was just a sample of maybe everyone in our school". Another teacher, Rob, posed a question for the students to consider: "Will the things that we found out from the data squares yesterday be similar or different to our class?" This question was designed to encourage the students to consider variation; the challenge for students was to consider and account for similarities along with differences at the same time. Louise also posed a question that encouraged students to consider variation in data between samples; she asked how many boys in the school might have the same data square (i.e., respond identically to the four data questions), given that there were four boys in the class with that particular data square. When one student answered, "I don't know the right answer but there could be four in every class," Louise pushed the students' variation thinking further by asking whether there were other possible answers. By using her *knowledge of content and teaching: variation* in this way, she was encouraging the students to develop their conceptual understanding of variation.

Beyond asking questions such as in the examples above, the teachers did not know how to further develop the students' thinking about variation. Teaching the relatively sophisticated and

complex concept of variation and inference was new for these teachers. Therefore it is not surprising that evidence of *knowledge of content and teaching: variation* was relatively limited.

How should a teacher structure the teaching to encourage students' statistical thinking in relation to reasoning with models? This question is at the heart of the teacher knowledge category of *knowledge of content and teaching: reasoning with models*. A teacher with sound knowledge in this category would have considered various approaches to teaching this aspect, could justify a particular approach that was taken and maybe why other approaches were rejected, and could consider any statistical issues that might arise from students' statements or explanations.

John commented that because the students had tended to focus on only one variable at a time and make frequency-based statements for comparisons, he would structure the next lesson differently. He intended to encourage the students to consider two variables simultaneously, and would do this by posing some questions to focus the students, as well as suggest to them ways of sorting the data cards to enable the questions to be investigated. John's *knowledge of content and teaching: reasoning with models* and *transnumeration* developed as a result of becoming aware of a difficulty that the students had with reasoning with models, that is, as a result of a development of his *knowledge of content and students: reasoning with models*.

Knowing how to encourage students to consider the relevance of contextual knowledge in relation to the statistical investigation being undertaken is part of a teacher's *knowledge of content and teaching: integration of statistical and contextual*. The situations described above for *specialised knowledge of content: integration of statistical and contextual* (in relation to the definition of family and unusual cases) required the teacher to weigh up, prior to answering each student's query, the extent to which such interpretations of 'family' might affect the reliability of the data obtained. Linda commented:

Everyone has their own definition of what a family is ... so I decided that the children could, if they wanted to, include their half brothers and sisters.

Also John decided on an approach to teaching that involved asking students a question based on a 'what if ...' scenario, as he had anticipated a possible difficulty that students might have with interpretation of the question for a particular family. Louise encouraged her students to integrate the statistical and the contextual when she asked them to think of situations involving various aspects of statistics (such as graphs and summary measures of data), and what these are used for. These examples show that each teacher demonstrated some *knowledge of content and teaching: integration of statistical and contextual*.

Being able to encourage students to think about each phase of the investigation and to consider how these phases link to one another (i.e., to deal with the parts without losing sight of the whole) are components of the *knowledge of content and students: investigative cycle*.

Earlier, an example was given of a teacher predicting that students may have problems interpreting some data questions. John, based on this knowledge of students, considered how to approach his teaching so as to prevent the students from having such problems. He handled it in two ways: on one occasion, he discussed an example with the students about their experience of having students from another class gather data from them, and how they had found some of the data questions difficult to answer; on another occasion John asked the students about how they

would answer a particular data question, knowing that different interpretations were possible. By structuring the teaching in this way, based on *knowledge of content and students: investigative cycle*, the teacher successfully utilised *knowledge of content and teaching: investigative cycle*.

The strategies a teacher might use to address students' tendency to ignore a wide range of possibilities and, instead, be content with a narrow, restricted focus in their investigation of data, constitutes a part of *knowledge of content and teaching: interrogative cycle*. Being able to consider, from a statistical point of view, how such limited views of the data might impact on an investigation is another component of this category of teacher knowledge.

Linda decided that to assist the students to examine possible relationships in the data, it was important to spend some time discussing with the students what are relationships. Following this, Linda brainstormed with the class some possible relationships that might be investigated in the data. She considered that this was time well spent, as it enabled the students to focus quite quickly on the data and engage with it meaningfully from the outset. It was quite common for teachers to ask the students to think about what might be found in the data, once the students had an idea of what the data set contained (in terms of the variables), but prior to seeing the complete data set. Again, this teaching strategy helped the students to engage quickly with the data as they had already started to think about the data and had developed an interest in it. These examples are evidence of the teacher having *knowledge of content and teaching: interrogative cycle*.

SUMMARY AND CONCLUSIONS

The framework proved a useful tool for identifying aspects of teacher knowledge in relation to statistical thinking. These aspects were obtained from classroom episodes or interviews with the teachers that re-examined those episodes.

Generally, within one cell of the framework, it was found that there is a diversity of teacher knowledge pertinent to statistical thinking. Consequently, evidence of teacher knowledge as related to statistical thinking for one cell does not imply thorough and complete knowledge for those aspects in relation to the desirable knowledge associated with the lesson.

Some examples of each category of teacher knowledge are listed below. These examples have been derived from the study's data and discussion, and are by no means intended as a complete list of the knowledge that was observed in use or shown as needed in the teaching of investigations. Because so many statistical concepts were covered in the investigative process (from the posing of questions for investigation, consideration of data collection questions, analysis through sorting and other transnumerative processes, and concluding statements), the examples given are a small sample covering a wide variety of statistical concepts. Many classroom episodes resulted in multiple coding as more than one 'cell' of the framework was in evidence. Consequently, some of the examples given below show more than one type of knowledge and/or aspect of statistical thinking.

Examples of common knowledge of content (ckc)

Able to find the three measures of average (mode, median, mean)	ckc: transnumeration
Can explain why mode is not useful in certain instances	ckc: transnumeration
Considers the effect of sample size on generalising	ckc: reasoning with models
Knows that larger sample size leads to statement of greater confidence	ckc: reasoning with models
Changing the order of wording in conditional statement changes the group total and therefore the fraction (e.g., right handed whistlers or whistling right handers)	ckc: reasoning with models
Able to make a generalisation to a population	ckc: variation ckc: reasoning with models
Suggests reasons why the youngest child in a family is likely to be able to whistle	ckc: integration of statistical and contextual

Examples of specialised knowledge of content (skc)

Ability to make sense of students' data based statements, with reference to sorted data cards	skc: reasoning with models
Determines whether suggested data collection question is suitable	skc: investigative cycle
Recognition of inappropriate comparison of unequal sized groups	skc: reasoning with models
Ability of evaluate appropriateness of inferential statement	skc: reasoning with models and skc: variation
Explains why measures such as mean or median are used as appropriate summary of data	skc transnumeration:
Ability to link student's question about 'unusual cases' in relation to data collection question with contextual knowledge	skc: integration of statistical and contextual

Examples of knowledge of content and students (kcs)

Recognise the need for data collection questions to be closed, with only 2-3 possible responses otherwise students will	kcs: transnumeration
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struggle to sort and group data

Ability to anticipate students will struggle with making accurate inferential statements	kcs:reasoning with models and kcs:variation
Recognise that students will have difficulty with sorting to explore relationships between two variables	kcs: transnumeration
Recognise that students may find some data collection questions ambiguous	kcs: investigative cycle
Need to encourage students to examine the data, continually looking for patterns, interesting aspects	kcs: interrogative cycle
Recognise need for students to make links between what is found in the data with what they know about the real world	kcs: integration of statistical and contextual
Recognise that students find difficulty with making valid statements from data	kcs: reasoning with models

Examples of knowledge of content and teaching (kct)

Uses discussion with students to evaluate suitability of data collection question, and how to refine the questions to make them unambiguous	kct: investigative cycle
Can pose suitable questions to encourage inferential thinking	kct: reasoning with models and variation
Encourages students to predict what might be found in data, and revisits those predictions after sorting data and making data based statements	kct: investigative and interrogative cycles, and reasoning with models
Uses 2x2 table as suitable representation for helping make statements from data	kct: transnumeration and reasoning with models
Considers the statistical implications for data collection from student's questions about 'unusual' family situations (e.g., how would you answer the data collection question about your position in family if you have ½ brothers/sisters, if a brother/sister has died, ...)	kct: investigative cycle and integrating statistical and contextual
Gives examples of statements involving two variables that would be suitable for investigating to help encourage students with posing conjectures to investigate	kct: reasoning with models, interrogative cycle, and investigative cycle
Shows students a way to sort data by two variables, and suggests possible statements that can made from such a representation	kct: transnumeration and reasoning with models

This study, by conducting research on teacher knowledge in the classroom in which that knowledge is used, has provided a significant contribution to the research field. Literature searches have been unable to locate any other research in statistics education that both focuses on teacher knowledge at the primary school level and is classroom based. This study therefore provides important insights to what knowledge a teacher needs for teaching statistics, based on the reality of the classroom context.

With regard to the classification of knowledge through the framework, the category of *specialised knowledge of content* provided challenges in differentiating it from *common knowledge of content*. It is not possible, with what is known or not known about the common statistical knowledge of the 'typical' educated person, to be certain about the boundaries between *common knowledge of content* and a teacher's *specialised knowledge of content*. The research literature documents a considerable amount about statistical misconceptions, and the general need for a greater level of statistical literacy in today's world (e.g., Ben-Zvi & Garfield, 2004). This study's classification of and distinction between these two categories of teacher knowledge may need redefining. The suggested differences between these two categories are therefore tentative until proven inadequate.

This study focused on the knowledge for teaching statistics of teachers early in their teaching careers. As the teachers were all in their second year of teaching, some of their current knowledge could be attributed to development from the teachers' teaching experience or from knowledge that developed prior to their initial teacher education. This study's findings can provide guidance for what particular aspects of knowledge development should be the focus of initial teacher education programmes. As most initial teacher education students have not had the advantage of learning statistics through investigations, their *common knowledge of content* should be developed through immersing the students in investigations. As their *common knowledge of content* develops, their *specialised knowledge of content*, particularly for listening to and making sense of students' responses (such as through the use of video of students), will develop.

Knowledge of content and teaching (e.g., teaching sequences, advantages and disadvantages of various alternative data representations, and knowing how to respond from a statistical viewpoint to students' ideas, especially the unconventional ones) is dependent on *knowledge of content and students* (e.g., understanding the aspects of investigating data that present particular challenges for students, knowing the common misconceptions, or errors that students are liable to make). Consequently, these two categories of knowledge should also be a focus in initial teacher education programmes. Overall, all aspects of teacher knowledge must be targeted, as the connections between the categories of knowledge mean that individual categories of knowledge cannot operate in isolation.

Teaching statistics through investigations is a recent development in school statistics curricula. As most experienced teachers would have had little opportunity to teach statistics in this way, there are also implications for teacher professional development, irrespective of the length of teaching experience of the teachers. Targeting teachers' professional development in relation to

knowledge of content and students and teaching simultaneously with building their own *common knowledge of content* through investigations (and consequently also *specialised knowledge of content*) is considered an optimum approach.

REFERENCES

- Ball, D. L. (1991). Teaching mathematics for understanding: What do teachers need to know about subject matter? In M. M. Kennedy (Ed.), *Teaching academic subjects to diverse learners* (pp. 63-84). New York: Teachers' College Press.
- Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83-104). Westport, CT: Ablex Publishing.
- Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433-456). Washington, DC: American Educational Research Association.
- Ball, D. L., Thames, M. H., & Phelps, G. (2005). Articulating domains of mathematical knowledge for teaching. Retrieved May 13, 2005, from http://www-personal.umich.edu/~dball/Presentations/RecentPresentations/041405_MKT_AERA.pdf
- Barnett, J., & Hodson, D. (2001). Pedagogical content knowledge: Toward a fuller understanding of what good science teachers know. *Science Education*, 85(4), 426-453.
- Ben-Zvi, D., & Garfield, J. B. (2004). Statistical literacy, reasoning and thinking: Goals, definitions, and challenges. In D. Ben-Zvi & J. B. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 3-16). Dordrecht, The Netherlands: Kluwer.
- Borko, H., Peressini, D., Romagnano, L., Knuth, E., Willis-Yorker, C., Wooley, C., et al. (2000). Teacher education does matter: A situative view of learning to teach secondary mathematics. *Educational Psychologist*, 35(3), 193-206.
- Burgess, T. (1977). *Education after school*. London: Gollancz.
- Burgess, T. A. (2001). Assessing the statistics knowledge of pre-service teachers. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and Beyond (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia - Sydney)* (pp. 114-121). Sydney: MERGA.
- Cobb, G. W., & Moore, D. S. (1997). Mathematics, statistics, and teaching. *American Mathematical Monthly*, 104(9), 801-823.
- Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning*, 1(1), 5-43.

- Cobb, P. (2000). The importance of a situated view of learning to the design of research and instruction. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 45-82). Westport, CT: Ablex Publishing.
- Cobb, P., & McClain, K. (2001). An approach for supporting teachers' learning in social context. In F.-L. Lin & T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 207-232). Dordrecht, The Netherlands: Kluwer.
- delMas, R. C. (2004). A comparison of mathematical and statistical reasoning. In D. Ben-Zvi & J. B. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 79-96). Dordrecht, The Netherlands: Kluwer.
- Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.
- Foss, D. H., & Kleinsasser, R. C. (1996). Preservice elementary teachers' views of pedagogical and mathematical content knowledge. *Teaching and Teacher Education, 12*(4), 429-442.
- Friel, S. N., & Bright, G. W. (1998). Teach-stat: A model for professional development in data analysis and statistics for teachers K-6. In S. P. Lajoie (Ed.), *Reflections on statistics: Learning, teaching, and assessment in grades K - 12* (pp. 89-??). Mahwah, NJ: Lawrence Erlbaum.
- Gal, I., & Garfield, J. B. (1997). Curricular goals and assessment challenges in statistics education. In I. Gal & J. B. Garfield (Eds.), *The assessment challenge in statistics education* (pp. 1-38). Amsterdam: IOS Press.
- Greer, B. (2000). Statistical thinking and learning. *Mathematical Thinking and Learning, 2*(1), 1-9.
- Groth, R. (2007). Towards a conceptualization of statistical knowledge for teaching. *Journal for Research in Mathematics Education, 38*(5), 427-437.
- Hill, H. C., Schilling, S., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal, 105*(1), 11-30.
- Manouchehri, A. (1997). School mathematics reform: Implications for mathematics teacher preparation. *Journal of Teacher Education, 48*(3), 197-209.
- Marks, R. (1990). Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Teacher Education, 41*(3), 3-11.
- Moore, D. S. (1990). Uncertainty. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 95-137). Washington, DC: National Academy Press.

- Pereira-Mendoza, L. (2002). Would you allow your accountant to perform surgery? Implications for education of primary teachers. In B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics (ICOTS 6) - Cape Town* (pp. CD Rom). Voorburg, The Netherlands: International Association for Statistical Education.
- Pfannkuch, M., Budgett, S., Parsonage, R., & Horrying, J. (2004). Comparison of data plots: Building a pedagogical framework. Retrieved October 12, 2004, from <http://www.icme-organisers.dk/tsg11/>
- Pfannkuch, M., & Rubick, A. (2002). An exploration of students' statistical thinking with given data. *Statistics Education Research Journal*, 1(2), 4-21.
- Pfannkuch, M., & Wild, C. J. (2004). Towards an understanding of statistical thinking. In D. Ben-Zvi & J. B. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 17-46). Dordrecht, The Netherlands: Kluwer.
- Popper, K. R. (1979). *Objective knowledge: An evolutionary approach*. Oxford: Clarendon Press.
- Popper, K. R. (1985). The growth of scientific knowledge (1960). In D. Miller (Ed.), *Popper selections* (pp. 171-180). Princeton, NJ: Princeton University Press.
- Shaughnessy, J. M., & Pfannkuch, M. (2002). How faithful is Old Faithful? Statistical thinking, a story of variation and prediction. *Mathematics Teacher*, 95(4), 252-259.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Sorto, M. A. (2004). *Prospective middle school teachers' knowledge about data analysis and its application to teaching*. Unpublished doctoral thesis, Michigan State University.
- Swann, J. (1999). The logic-of-learning approach to teaching: A testable theory. In J. Pratt & J. Swann (Eds.), *Improving education: Realist approaches to method and research* (pp. 109-120). London: Cassell Education.
- Vacc, N. N., & Bright, G. W. (1999). Elementary preservice teachers' changing beliefs and instructional use of children's mathematical thinking. *Journal for Research in Mathematics Education*, 30(1), 89-110.
- Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, 67(3), 223-265.

WHAT MAKES A “GOOD” STATISTICS STUDENT AND A “GOOD” STATISTICS TEACHER IN SERVICE COURSES?

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Abstract: Statistics is taught within a diverse array of disciplines and degree programs at university. In recent research we investigated international educators' ideas about teaching and learning 'service' statistics. This paper investigates what these educators think are important attributes, knowledge and skills for learners and teachers of statistics. Results show that educators are in agreement about qualities of 'good' statistics students, such as curiosity and critical thinking. An emerging issue was the role mathematics plays in learning statistics as a service subject with some academics postulating mathematics as the basis of statistical learning, others proposing it has limited or little importance in learning service statistics or even that it presents obstacles, detrimental to students' statistical thinking. The features of statistics teachers that were highlighted in the data were knowledge of statistics and its applications, empathy with students and knowledge about teaching and enthusiasm for it. Respondents had practical suggestions on how to help students become competent learners of statistics. We extend a theoretical framework for synthesizing the findings.

Keywords: teaching statistics at university, service courses, statistics in mathematics

1 Introduction

What are educators' experiences of teaching statistics as a service subject at university? This question was the topic of a recent investigation conducted through email interviews with statistics educators in many countries. In this paper we report findings from this qualitative research project that focus on what the participating educators think are the important attributes, knowledge and skills for students and teachers in their courses and how educators address the challenges of teaching students who lack the motivation or skills to be 'good' learners of

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statistics.

Statistics education has been transformed by the availability and widespread use of technology—especially software and the internet, the changing needs of society and the diversity of the contemporary student cohort. Historically, statistics as a tertiary subject was based on mathematics, and was taught as a topic in mathematics to students training to be professional statisticians. Today, the study of statistics and its applications to data analysis is part of the curriculum for students in many, diverse disciplines and these students have differing knowledge bases, a range of professional interests and goals and variable mathematical skills. There are resulting tensions between teaching statistics as a discipline in its own right, as a branch of mathematics, or as “methodology serving some other field” (Moore, 2005, p. 206). Issues that affect teaching and learning statistics at the tertiary level include training statistics students and teachers to work with other disciplines, encouraging students to solve problems collaboratively—in a team—and to communicate well with others both in writing and verbally (Nicholls, 2001). Statistics students are not homogeneous in ability, educational background or discipline specialisation. Indeed statistics students may be regarded as a microcosm of the diversity found in contemporary universities. Hence, as Latterell (2007) highlights for mathematics educators, statistics teachers need to understand their students, including being aware of how students’ cultures and mathematics backgrounds differ from their own experiences (summarised, perhaps, as less algebra, more EBay). This is similar to the first recommendation of the Mathematical Association of America’s (2004) *Committee on the Undergraduate Program in Mathematics Guide* (which includes Statistics). That is, to understand students and the world in which they live.

Teaching is arguably the most important factor that affects the quality of students’ learning (Kember & Gow, 1994; Ramsden, 2003; Prosser, Ramsden, Trigwell & Martin, 2003). Further, as Watson et al. (2007) observe, it cannot be assumed that content knowledge in a specialised subject is sufficient for effective teaching. As an extension of this idea, Hodgkinson (2005) asks to what extent it makes sense to think of learning as specific to a particular discipline such as mathematics?

Lindblom-Ylänne et al. (2006) show that approaches to teaching are relational—affected by both discipline and teaching context. These researchers found differences in whether approaches were student focused or teacher focused, in the self-efficacy beliefs of teachers in different disciplines and the contextual effects on their teaching. Studying teaching in service statistics provides a singular array of contexts to investigate learning and teaching. Statistics is unusual as a discipline as it is taught in a range of environments and in the context of a host of other disciplines—as disparate as business management, engineering, psychology and biology.

Teaching statistics as a service subject has special challenges: students studying statistics as part of their degree program do not necessarily have an interest in the subject and may not wish to engage with any study perceived as mathematical (Gordon, 2004). Further, studying subjects in statistics or quantitative methods can generate anxiety for some student groups (Onwuegbuzie & Wilson, 2003). There is no one method of teaching or learning that fits all statistics courses. Further, according to Northedge (2003), increasing levels of student diversity in higher education mean that educators cannot persist with transmission models of teaching nor replace these with

unfocused, student-centred approaches that do not offer students genuine opportunities in the realities of the classroom. This observation is particularly relevant to statistics pedagogy, as students will need statistical knowledge for their professional work as well as for their informed citizenship within a knowledge economy. Hence a study of statistics educators' insights about their students and their teaching approaches, including investigating the challenges presented by unmotivated or 'stats-phobic' students and ways of tackling these challenges, has the potential to stimulate reflection and debate in this important area of pedagogy.

2 Method

The investigation consisted of a three-phase series of e-mail interviews with statistics educators. Participation was invited through an electronic request to the membership list of the IASE (International Association for Statistics Education) and Faculty bulletin boards of Australian universities. Thirty-six IASE participants took part in the first email interview, with 32 completing the full series of three interviews, an indication of the engagement of respondents with the project. The remaining four carried out partial interviews before our cut-off date for data collection. The IASE statistics educators were from many countries: Argentina, Australia, Belgium, Brazil, Israel, Italy, Netherlands, New Zealand, Slovenia, Spain, Uganda and the USA. Most interviews were conducted in English (one was bilingual with English questions and Spanish responses). An additional nine interviews (seven completed, two partial) were conducted with Australian educators who responded to requests through departmental bulletin boards. The resulting interview transcripts of over 70 000 words formed the raw material of our study.

The interview protocol consisted of an initial series of six questions, reflecting our research focus on educators' ideas about teaching and learning statistics as a service subject. After studying the initial reply, we sent a second interview with questions following up and probing each participant's responses. Finally, a third interview was sent with further questions to elicit clarification and in-depth explanations of the responses given as well as a request to evaluate the e-mail interview method.

The initial questions included one on the specifics of the educators' backgrounds: *What country do you work in? What type of institution do you teach in? What level of students do you teach statistics? What discipline areas do you teach statistics in?* Responses to this question showed that participants taught service statistics at universities in a range of contexts. The respondents taught at the full range of levels, from pre-degree and first year to postgraduate, using various teaching methods including traditional, large-group lecturing, tutorials and small research groups, problem-based learning and distance education; in some universities, statistics teachers pooled their strategies and resources to work as a team. Many participants reported teaching service statistics to student groups in several disciplines, within programs ranging from the traditional areas of physical, health and social sciences, business, economics and management, engineering, psychology and education, through to less common areas such as theology and liberal arts.

All other questions were posed in a deliberately open way to enable the participants to explore their own ideas rather than we, the researchers, eliciting responses in a specified direction. Two of these questions, below, are the focus of this paper.

What do you think makes a good statistics student?

What are the attributes of a good statistics teacher at university?

The follow-up questions explored the thread of thought that was prompted by the original question and so depended on the individual response, for example: *Are there qualities specific to a good student in statistics? What do you feel you can do with a student or a group of students who don't display these qualities? Can you explain how mathematical ability can help students but can also blinker them? How do you go about teaching students to communicate statistics?* The follow up questions also generated discussion on how to encourage those students who were not 'good' to achieve the desired qualities.

The interview process was a written version of the usual face-to-face interview, with the modification that at each point in the process the respondent had a record of all previous communication including their own responses, and both interviewers and respondents could continue the dialogue in their own time. This iterative e-mail interview provided the participants with an opportunity to reflect on and expand their initial responses to questions. We found that the responses were well considered, and, at times, participants clarified and refined previous statements.

We have critically reviewed this method of e-interviewing (*Authors*, in press). This review includes data from the participants evaluating the methodology, thus positioning respondents as co-researchers. We have also previously written about other aspects of participants' experiences from the data: including educators' views of the importance of communication skills for their statistics students, educators' conceptions of teaching service statistics (*Authors*, 2005, 2007a), recognising and developing professional expertise in statistics pedagogy (*Authors*, 2006) and the range of tools, teaching strategies and approaches utilised by the respondents (*Authors*, 2007b).

In this paper we focus on participants' responses to the two initial questions concerning 'good' statistics students and teachers, and the follow-up discussion in the interviews. Pseudonyms were chosen by the participants themselves and included unusual choices such as 'Henry VIII' and 'QMmale' or the names of famous statisticians. Excerpts from interviews in this paper are reported under these self-chosen pseudonyms.

3 Results: What makes a good statistics student?

Many participants interpreted the notion of a 'good' student in terms of personal qualities. The attributes of good students commonly mentioned included critical thinking, curiosity or an active mind, a preparedness to work hard and try to understand and a willingness to overcome maths phobia. There were also more individualistic ideas such as a sense of humour, willingness to take responsibility or to play with abstraction.

Table 1 summarises the qualities that respondents reported were important with illustrative extracts from the interview transcripts.

Table 1: Qualities of Good Statistics Students

Quality	Illustrative excerpt
Critical thinking	Samuel: <i>A student who asks why, what is the connection to the context, have I met a problem like this before //A student who is systematic, who begins in an orderly way, asks the right questions and applies the appropriate techniques. A person who is self critical, always questioning themselves and checking their conclusions.</i>
Curiosity	Henry VIII: <i>The good student is the one who is curious to see what statistics has to offer, and how it relates to their careers (whatever their backgrounds, and their knowledge of maths). The bad student (by far, the majority) is the one who wonders how he can pass the examinations with the least effort and the least pain.</i>
Motivation	QMmale: <i>So there are two, opposite directed, mechanisms in play: level of prior education, and motivation. In the long run, the last one is dominating; in the short, the first one.</i>
Numeracy/literacy skills	John: <i>Just as enthusiasm alone will not do it for the teacher, motivation alone will not be the making of a good student. A good statistics student needs to have reasonable levels of numeracy literacy/skills, comprehension, interpretation and writing skills.</i>
Maturity	Maria: <i>The application of statistical techniques and interpretation of results require more maturity than the one we encounter in students in the first years at the university. Thus, I think that statistics should not be taught in the first years of a degree.</i>
Interest and diligence	Alice: <i>Students who attempt questions themselves as well as attending lectures and tutorials, students who read the learning guide, students who practice using exercises and students who continually ask why are we doing this!</i> Leigh: <i>One who wants to know! One who takes responsibility for his/her own learning.</i>
Same as a good student anywhere	Rose: <i>Same as a good student anywhere I guess //good sense of humour, open, sense of wonder about the world, appreciation of the beauty, of the inter-connectedness of things in general (and ideas specifically), grounded in the world but also willing to 'play' in the world of abstractions.</i>

The ability to see statistics as a tool that could be applied to their home discipline—to see the connections—was considered important by many educators. Natalie proposed that a good student: *is able to think about situations rather than just doing calculations and analyses*. At a more advanced level of study Sjeſke considered that: *a good statistics student knows how to formulate research questions. He/she knows how to get from constructs to variables and can take*

the measuring level of the variable in mind and the conditions when deciding which statistical technique to use to process the data.

Communication skills—the ability to write coherently as well as verbal skills—were addressed by surprisingly few (less than one third) of the respondents. Those that did write about this stressed various aspects including the importance of a good command of language, the ability to communicate statistical terms in plain, everyday language, to use language to teach their fellow students and to write reports (discussed further in *Authors*, 2005).

As with communication skills, we did not specifically ask respondents about the importance of mathematics for statistics students. However, many of the educators expressed views on how mathematics affected students' learning. We outline these in the next section.

4 Results: The role of mathematics in learning statistics

Responses showed mathematics in statistics to be a controversial topic with some educators of the opinion that mathematics is the basis of statistical knowledge, others that mathematical knowledge was not necessary for a 'good' statistics student in service subjects, or even detrimental to statistical thinking. Many educators were concerned about math-phobia and its effect on students' attitudes to learning statistics.

For those endorsing the role of mathematics in learning statistics, the abstraction, rigour and power of mathematics to solve problems were seen as fundamental to gaining skills in statistics. Daria maintained that: *Every "scientist" should have a minimum knowledge of calculus. The mathematical background helps in developing the ability to solve problems and in the process of generalization. [Do students without a maths background have a different approach to problem solving, or are they just less successful at it?] My impression is that they are less successful at it.*

Margaret qualified the idea of mathematics as fundamental to statistics, differentiating between students studying statistics as a major subject and statistics as a service subject. *(A good statistics student needs) a strong mathematical background and a methodical way of thinking. I am answering this in terms of a student who will go on to be a statistician. //For students in other disciplines who must take one or more statistics courses for their degree program, a good student is one who understands the theory in their own field of study enough so that they can ask appropriate questions and apply the statistical techniques in meaningful ways.*

Maria agreed, saying that for some introductory subjects the emphasis on mathematics was different. *Students' previous knowledge is one of the most important factors to influence learning. Therefore I consider that a 'good statistics student' should have a mathematical background and computer skills. [Is there a minimum amount or level of mathematics that students need in order to be "good" students?] It depends on the statistics course. In the course I teach ... students that have a stronger background in calculus and linear algebra have less difficulties in following the introduction of statistical techniques and their application. They also have less difficulties in abstract reasoning and computing. But if the statistics course is introductory with an emphasis on exploratory analysis applications, a student with a weak mathematics background (basic mathematical concepts and computation) may be a 'good' statistics student.*

Sjefke was more direct in stating an opposite view that: *in our competence based approach mathematics will play no role in teaching statistics to our (psychology) students. //Now we rather concentrate on interpretation and content than on mathematics.*

Cara spoke for many in reporting that: *statistics is much hated and feared by students. //I'd say there are two major reasons: hatred of maths dating back to the primary and secondary school + the way statistics is taught at most European universities: 'ex katedra' lectures in huge groups (sometimes as many as 300 students), lack of personal contact with the lecturer; it's fairly easy to lose track of what is going on up front + study materials are usually rather dull + the propaganda of senior classes (if you pass stats you've practically made it into the next year of studies). Natalie observed that people who have a fear of mathematics would bring this fear into their learning of statistics: Most people don't see statistics as any different from mathematics. And Kay added: A sizable minority of the students I meet have had someone in their background-either in K-12 (school) or as an undergraduate, who told them they were poor in math. Students are fearful mostly of what they see as the mathematics involved in statistics.*

Vivian felt that a lack of interest in mathematics was compounded by anxiety. *My experience is that most psychology students choose psychology because they want to help people who have psychological problems and not because they want to find answers for research questions. They want to work with people, and not sit in a room and do sums. //Because they already had low grades (at school) they don't see statistics as something they are good at and therefore they are anxious.*

However, Horace explained that one should take care with the assumption that many students have negative attitudes to mathematics in statistics. *An important issue I need to watch out for is that some students, maybe doing maths or even maths stats concurrently, do want to talk about formulas and assumptions and formalisms. So I need to be careful not to put down them or their formal approach, indeed to encourage them to see how that body of theory is essential for the software and all I'm doing, and that they therefore have a privileged insight, even though my main aim is to present in ways accessible to as wide a range of students as possible.*

Some educators put forward the idea that the philosophy underpinning statistical thinking is quite different to the thinking for mathematics. Leigh reported that: *It is a mistake to call the subject mathematics, at least the way we teach at this level. //In this class we are addressing questions about the real world through collecting and looking at data. Ford Prefect felt that mathematical ability could even 'blinker' the students as: Students will be looking for (right answers). //When students are taught maths at school and at university, there is a lot of the old theoretical QED type stuff. Students are graded for the correct answer and method. //In stats there is no guarantee that they get the right answer even if they do the right calculations and use the best method. That is a super tough concept to overcome. Part of the problem with teaching statistics in a traditional form is we sometimes don't get that concept across.*

Horace insisted that appropriate assessment was essential to avoid such deterministic thinking: *Don't just give marks for correct calculations otherwise we are encouraging them to think in exactly this way. Instead, always hold marks aside for interpretations and understanding of*

alternative approaches and the extent to which we might be wrong. //You will never overcome it for all students. Let's face it, stats is hard.

Andrew expanded on the differences in mathematical and statistical thinking explaining that each had its own beauty. *Some students just love the logical way in which mathematics develops and also enjoy the great generality which it develops. I suspect at the beginning calculus is quite exciting and also the problems that can be solved. But in reality these are quite simple problems that do not reflect real life without a lot more work. //Statistics on the other hand is a bit "dirty" at the beginning in comparison when it deals with numbers, graphs, variability etc.*

I feel the real beauty of statistics does not come from the beauty of mathematics. It comes from the fact that you are actually able to solve problems and answer questions for researchers. And you can see clearly the fruits of your work.

Given the diversity of the student cohort studying service statistics and the range of teaching contexts of our respondents, we would expect considerable variation in perceptions of good teaching. We next report participants' views on the knowledge, skills and qualities considered essential for teaching service statistics.

5 Results: What makes a good teacher of service statistics?

The most commonly mentioned requirement for 'good' teachers was solid knowledge of statistical theory and practice. Anette wrote: *I am absolutely sure that good statistics teacher should have at least some practical experience – I mean she has to be involved in real research projects as a data analysis expert.* Maria believed that: *Her/his background should be related to the statistics subjects she/he teaches. She/he should be involved in research about this subject.* And Samuel put it this way: *Sound background, the teacher needs to see the material being taught in the context of the wider picture of statistical theory and practice.* Ford Prefect gave more justification of this view: *Statistics has its own issues in that it is a subject that is taught in a variety of contexts and disciplines, each of which has its own complexity. This forms part of our problem, because approaches we take in business are different to psychology and are different to mathematics. We need to have a good understanding of our own discipline area. This is one of the dangers (I believe) of trying to centralise statistics teaching in maths / science departments at some universities.*

Some respondents noted that knowledge of applications was needed to "motivate students" and that teachers needed the ability "to show the basic statistics ideas and concepts without resorting to complicated maths". Natalie explained. *They need to realise that statistics is a really practical and useful interdisciplinary subject that will most likely be invaluable for them in the future. By broadening their concept of statistics from "doing exercises in the book" to seeing a wide range of applied statistics examples, they will hopefully be more motivated and inspired to learn statistics and not be plagued by the question: "why are we doing this?"*

Sjefke stated that: *Statistics is a way of describing psychology in another language; it is not performing calculations at all!* He went on to issue a warning: *against another 'possible failure' or pitfall: mathematics teachers constitute the overwhelming majority of teachers in the field of statistics education. But I fear that most mathematics teachers are not interested in the*

(psychological) content, that they can't really bridge the gap between theory and empirical reality, because empirical data are not ideal data for teaching sums and models.

Henry VIII summed up these ideas: *I think the good teacher is the one who is able to show the students that statistics is the science that provides the researchers with tools for dealing with the uncertainties of the real world, and not just a set of boring formulas and procedures that the students have to fit somehow in their research papers in order to have them accepted*

The next theme was the importance of a range of personal attributes. We summarise these in Table 2.

Table 2: Qualities of good statistics teachers

Quality	Illustrative excerpt
Curiosity	RON FISHER: <i>Curiosity about the world around them. //Some teachers view their field (whatever it may be) as “finished” in a real sense, and their role as just reporting on what was done. //I am constantly updating my examples, and looking for new applications that will interest my students. Not only do I do this for the students’ sake, it also makes the class much more interesting for me, since I am interested in the world around me.</i>
Enthusiasm	John: <i>Enthusiasm, enthusiasm, enthusiasm!!!</i> Janet Cole: <i>I looked at experts in the field of statistics education and identified a kind of ‘magic’ that each of them had in the classroom. I am not sure how to qualify this ‘magic’ in any other word. Passion and enthusiasm are a must. A good statistics teacher is not someone who is teaching only so that s/he can do research or just as a job – a good statistics teacher wants her/his students to learn and be excited about learning.</i>
Empathy with students	Cesar talked of: <i>Capacidad para comprender los códigos culturales de los jóvenes que ingresan a la universidad. Es decir, la habilidad de relacionarse positivamente con los estudiantes. [Capacity to understand the cultural codes of the young people who enter university. That is, the ability to relate well to students.]</i>
Confidence	Leigh: <i>Very confident about the subject matter. Use real data and make it interesting and relevant.</i>
Willingness to learn	Rose: <i>Willingness to learn (imperative!!!), ability to listen, excellent communication skills, capacity to respond with rather than react to, flexibility, sense of humour, healthy sense of self.</i>
Patience	Lizzie: <i>Patience!!!! //These are very nervous students who need a lot of encouragement and I need to be patient enough to say the same thing in as many different ways as I can think of until the student indicates their understanding.</i>
“All the usual things”	Leigh: <i>All the usual things: patience, availability, etc etc.</i>

There was consensus among the respondents that verbal and written communication skills were essential for teachers in service statistics courses: “such that the lecturer can communicate new concepts at an appropriate level”. Joanna amplified this. *Often lecturers can lose sight of the fact that they are teaching students who are new to statistical concepts and they unnecessarily complicate their presentations thereby alienating the students.* Anette described this skill as being a “vivid story-teller”. *I think that to keep students interested and motivated to learn and to give them the knowledge and skills they can apply, the teacher has to tell the “stories” (imaginary or real) about what has happened or could happen in real research projects //(including) what can go wrong and how to learn from the errors the other people have made.* Alice explained the necessity of being able to promote discussion. *Discussion is helpful in that students can see how others understand something (or don’t—which can be helpful in not feeling alone if they are confused!). It also helps clarify misconceptions.*

Heintje summarized her student-centred approach: *A good statistics teacher will stimulate students to take their own initiative, to become confident about themselves in doing statistics, exploring data, discussing subjects with other students or teachers. A good statistics teacher will help students to overcome their statistics anxiety and will take care for the process that they get familiar with the discipline step-by-step, embedded in a psychological context. A good teacher will also be a good listener and will seriously consider student evaluations as a means to improve the educational design.*

The above reports show that many of our respondents were passionate about their subject and desired to imbue students with curiosity about statistics and interest in its applications. However, as is well documented, not all students are motivated to study service statistics. Hence our interview conversations, in many cases, led to discussions about ways of engaging students with their study of statistics.

6 Results: Can you teach a student to be a good statistics student?

Some educators felt there was “not a lot” they could do with students who were unmotivated to study statistics. However, others were more positive and proactive in their approach.

Andrew felt the key was to access examples from recent consultation projects and not just to use textbook case studies. *I would like to think I can help turn them into a good student. //I believe it helps by being able to reference recent consulting examples by discussing them with the students and also getting them to work on project data, possibly in small groups.*

Kay stated that her first job, if there were stats-phobics in her class, was to reduce anxiety: *I address it directly by talking about anxiety; telling the class that people have actually survived the course before; that stats started with law and business and NOT mathematics; that stats has an underserved bad rep; by using humour or what passes for it with me.* She offered a practical approach to helping students who were struggling: *They get paired up with a group of other students, so they have peers to talk to about statistics who can help them. These students also get direction to plan on spending more time on the course than their friends might—and to spend some time each week with the instructor or the graduate teaching assistant. These students get extra worksheets with examples, they get extra help.*

Heintje acknowledged that a student who was *perhaps less talented and less motivated might be an excellent student in other domains. //It might be the case that such a student knows a lot of psychological treatments or practices, is a good writer or has excellent social or communicative skills. All these (different) skills and talents might be of great importance for the group assignment or group task. The accents of this student's contribution will be different, he or she will play a different role in the discourse, but his or her social, creative or writing contribution has its own value in the social construction of knowledge.* Like Kay, Heintje observed that often these students needed more active support and encouragement in statistics and methods. *In most cases the group itself takes this student by the hand in order to survive the struggle with statistics and methods.* Heintje pointed out that: *no academic institute is able to provide one recipe for one approach, to involve those students. But, supported by their peers, these students often are able to complete the program with a satisfying result.*

Respect for students' diverse abilities was also the basis for Janet Cole's approach to helping students learn statistics. *Any student has the potential to be a good statistics student—it is our job to motivate a desire to learn and to provide an environment that is conducive to learning and that is an environment where safety and respect for all are cherished. //Since I believe in students constructing their own knowledge, I tailor my questions and explanations to their needs. For those students who have not tried or have given up, I give them some direction, send them away with an assignment, and ask them to return to talk again.*

A key factor in helping students become motivated learners of statistics was to help them see the relevance of the discipline to their future professional work. Henry VIII explained his approach for medical students. *What I try to show medical students is that, even if they don't ever intend to do any research, they still need some basic knowledge of stats in order to be able to fully understand the concepts of "statistical patterns" and "typical values", and the probabilistic nature of the decisions they have to make every moment during their practice. I try to do this by highlighting, through examples, the probabilistic nature of the patterns and decisions, and by trying to steer them away from the sort of deterministic thinking they are exposed to during most of the other courses they attend at college.*

Moore (2005) describes views of statistics that emphasise different dimensions of the discipline—as mathematics, data analysis or a tool in the service of other disciplines. In resonance with this conceptualisation, Margaret reflected that there are three different sets of students: *(1) theoretical, mathematical statisticians, (2) applied statisticians, and (3) students in other fields who take some statistics courses.* Each group needed different skills and so different teaching. In the case of the first group, their needs were for: *a strong mathematical background and a methodical way of thinking. For #2, such students need to be able to learn how to manage not only the statistical techniques, but to appreciate what is and is not doable from practical perspective in the fields where they are applying their stats. They need to learn consulting skills so that they can appropriately interact with experts in other fields and work together to get a satisfactory solution to a given problem. They need to develop good teaching skills, too, because at least some of their interactions with clients will require subtle teaching. And, they need to be able to know how to learn from their clients, so that they can better assist them. For #3, I think*

our goal should be to give them enough information to appreciate the intricacies of good statistical work so that they can more successfully interact with type #2 above.

7 Discussion

The participants in this study found themselves in different sorts of contexts: disciplinary, undergraduate, postgraduate, servicing, mode of delivery and many other variations impacting on their teaching. These contexts provided a means for our respondents to interpret and reflect upon their experiences. Our data show that educators of service statistics have a range of ideas about what makes a good statistics student in service courses, including qualities such as critical thinking and curiosity, a diligent approach to learning and numeracy and literacy skills. Many of these attributes could be transferred to describing competent learners of many disciplines at tertiary level. In contrast, respondents' views about the role and even value of mathematical knowledge to learners of statistics diverged. Some participants acknowledged the historical embedding of statistics in mathematics and the necessity for learners to understand and appreciate mathematics as the basis of scientific thinking. Others contested the "deterministic thinking" students may learn from studying mathematics, describing it as antithetical to the uncertainty and complex interaction of context and content surrounding statistical problems, which students need to understand to appreciate the discipline.

Not surprisingly, discipline knowledge and experience in applying statistics to their research or professional work were prominent in our respondents' reports about what makes a good statistics teacher. Many of these educators appeared to concur with Moore (2005, p. 206) that a relevant introduction to statistics must include all the areas of project design, exploration of data and statistical reasoning "in the context of work with real data in real problem settings". Respondents collectively expressed qualities needed to teach service statistics. In general, these effectively matched the qualities they reported about good learners. Many educators offered practical suggestions on how to overcome 'stats-phobia' and to assist students who were struggling.

We have used the voices of the teachers themselves to illuminate their experiences of teaching and learning service statistics and focussed on responses to questions on what makes a good learner of service statistics, on the one hand, and a good teacher on the other hand. However, underpinning these questions is a more fundamental and broader issue: what is the basis of effective learning and teaching in service statistics? In expressing their responses to the interview questions our respondents were directly or indirectly articulating their positions and "embodied" theories (Hodkinson, 2005, p. 116) on this more complex issue.

It is tempting to suggest that there could be a match between espoused theories of good teaching and espoused theories of good learning. The educators' ideas about the qualities of 'good' teachers and 'good' students, summarised above, provide examples of familiar practices that are seen as successful (such as: *seeing a wide range of applied statistics examples*), and by comparison, or omission, practices that may be less conducive to learning (*doing exercises in the book*). Remarkably, the notion of 'good' was usually tied to some emotive personal quality, such as enthusiasm (in the teacher), motivation (in the student). The implication here is that enthusiastic teachers perhaps generate and support motivated learners. Yet, experienced teachers know that the broader contexts of learning impact hugely on the outcomes of learning. In this research project, we have shown that the role of mathematics for statistics provided a

spontaneous outpouring of tension. Other broader contexts included students' backgrounds and preparedness for their statistical study.

In a previous paper (*Authors, 2007a*) we explicated a model for the phenomenon of teaching statistics as a service subject that serves as a lens for the positions revealed in this paper. There we found evidence for three conceptions—qualitatively different ways of looking at service statistics teaching. The first of these conceptions was labelled 'Teacher' and focused on the qualities, expertise, resources and strategies brought to the classroom by the teacher. Students were 'acted upon' and the teacher used his/her expertise to decide on the important aspects of statistics to teach. A broader conception was labelled 'Subject' and represented a change of focus to the course content or subject matter itself. The role of the teacher changed to illuminating the material and helping students to understand it. The broadest conception was called 'Student', and highlighted the voices, perspectives and concerns of the students: the teacher was certainly part of the overall learning context, but not the sole part, nor even the most privileged part.

The characteristics of 'good' students of statistics and 'good' teachers of statistics, reported in this paper, could be viewed as positions along this theoretical continuum of Teacher – Subject – Student. We emphasise that we are not attempting to categorise any individual view. However, if we review participants' collective responses, we can locate views about aspects of the findings, outlined in sections 3, 5 and 6, along the continuum. Firstly, what makes a good statistics student? Focus on Teacher – personal attributes such as willing to listen and work hard; focus on Subject – personal attributes such as curiosity about the subject, ability to think through the issues rather than just do the technical bits; focus on Student – personal attributes such as willingness to see what role statistics has in their own discipline and how it can be useful.

Next, what makes a good teacher of service statistics? Teacher – the discipline knowledge and experience to motivate students; Subject – show students that statistics can illuminate their own discipline areas, ability to show that statistics is practical and useful; Student – willingness to engage with students' lives, contexts and problems, to put themselves into the background.

Finally, can you teach a student to be a good statistics student? Teacher – no, nothing much can be done; Subject – yes, if you can show them the interest in statistics and give them the right examples of its application; Student – even those who don't appear to be good students initially are likely to have other strengths and be able to contribute these strengths to a group approach to learning.

The theoretical lens we developed in *Authors (2007a)*, and expanded above, also provides a means of placing the collective (and divergent) conversations about the role of mathematics in statistics into a framework. A Teacher focus could fit with a view that students should work hard on developing their abilities in order to get an entrée into the world of statistics (as their teacher already has). Hence mathematics would emerge as very important, maybe even essential, along with an emphasis on other aspects of training or experience undertaken by statistics educators. This is illustrated by Daria's assertion that: *every "scientist" should have a minimum knowledge of calculus*. With the Subject conception, mathematics could be viewed as part of the development of the statistical theory and hence an important aspect of the learning. An example is given by Horace's comment showing that he encourages mathematics students (in his statistics

class) to see how mathematical theory is essential, *and that they therefore have a privileged insight*. With the Student conception, the view could be that mathematics is not particularly important unless the student wants to move towards studying mathematical statistics—a small minority, particularly in service subjects. For most statistics students the role of mathematics is essentially optional: one can discuss the meaning of statistical ideas and approaches without the intermediary of mathematics. This idea is encapsulated in Sjefke's assertion that for psychology students, mathematics plays no role—*we rather concentrate on interpretation and content than on mathematics*.

8 Conclusion

The empirical findings of this study provide an opportunity for statistics educators to become aware of and evaluate a range of ideas about learners and teachers, and to locate their own positions in the theoretical framework discussed above. More generally, the study alerts us to the complexity and diversity of educators' views about successful teachers and students and, by inference, about effective teaching and learning practices in service statistics.

Lindblom-Ylänne et al. (2006) showed that both discipline and teaching context impact relationally on approaches to teaching. In accord, Hodkinson (2005, pp. 117-118) answers his own question about whether learning is general or discipline specific by summing up that although “our broad conceptualizations of learning can be fairly general, understanding how these conceptualizations can be applied in practice requires attention to the specifics of each location”. By presenting data on ‘good’ teachers and learners within a specific discipline area and applying the findings to expand our theoretical model on experiences of teaching service statistics, we hope to add to scholarly conversations about teaching and learning.

Qualities, knowledge and skills of good learners and teachers—however important—cannot in themselves promote effective learning in statistics. It is in the interactions-in-practice of teachers, students and their environments that the many dimensional and dynamic ‘life’ of teaching and learning plays out. We offer a provocation to statistics educators to acknowledge, reflect on and build upon these complex interactions to inform pedagogy and practice in service statistics.

Acknowledgements

We gratefully acknowledge the insights of the participants of this project.

References

- Authors* (2005). How important are communication skills for ‘good’ statistics students? - An international perspective. In L. Weldon, and B. Phillips (Eds.), *Proceedings of the ISI/IASE Satellite Conference on Statistics Education and the Communication of Statistics*. Sydney: ISI, Voorburg, The Netherlands (On CD). Available online: (*http to be supplied after refereeing*).
- Authors* (2006). Recognising and developing good statistics teachers (invited paper). In Rossman, A. and Chance, B. (Eds.), *Proceedings of the Seventh International Conference on Teaching Statistics, ICOTS7*. Salvador, Brazil: ISI, Voorburg, The Netherlands (On CD). Available online: (*http to be supplied after refereeing*).
- Authors* (2007a). Teachers' conceptions of teaching service statistics courses. *International Journal for the Scholarship of Teaching and Learning*, 1(1).

- Authors* (2007b). Tools, artefacts, resources and pedagogy-stories of international statistics educators. In *Australian Association for Research in Education 2006 Conference Papers*, compiled by P. L. Jeffery, AARE, Adelaide. Online at: <http://www.aare.edu.au/xxxxx.pdf>.
- Authors* (2008, in press) Research interviews in cyberspace. *Qualitative Research Journal*.
- Gordon, S. (2004). Understanding students' experiences of statistics in a service course. *Statistics Education Research Journal*, 3(1), 40–59.
- Hodkinson, P. (2005) Learning as cultural and relational: Moving past some troubling dualisms. *Cambridge Journal of Education*, 35(1), 107-119.
- Kember, D. & Gow, L. (1994). Orientations to teaching and their effect on the quality of student learning. *Journal of Higher Education*, 65(1), 58-74.
- Latterell, C. (2007). Today's Mathematics Students. *The Montana Mathematics Enthusiast*, 4(1), 66-72.
- Lindblom-Ylänne, S., Trigwell, K., Nevgi, A. & Ashwin, P. (2006). How approaches to teaching are affected by discipline and teaching context. *Studies in Higher Education*, 31(3), 285-298.
- Mathematical Association of America (2004). *CUPM Curriculum Guide: Undergraduate programs and courses in the mathematical sciences. A report by the Committee on the Undergraduate Program in Mathematics*. Available online at: www.maa.org/cupm.
- Moore, D. (2005). Quality and Relevance in the First Statistics Course. *International Statistical Review*, 73(2), 205–206.
- Nicholls, D. F. (2001). Future directions for the teaching and learning of statistics at the tertiary level. *International Statistical Review*, 69(1), 11-15.
- Onwuegbuzie, A.J. & Wilson, V.A. (2003). Statistics Anxiety: nature, etiology, antecedents, effects, and treatments—a comprehensive review of the literature. *Teaching in Higher Education*, 8 (2), 195–209.
- Northedge, A. (2003). Rethinking teaching in the context of diversity. *Teaching in Higher Education* 8 (1), 17–32.
- Prosser M., Ramsden P., Trigwell K. & Martin E. (2003). Dissonance in Experience of Teaching and its Relation to the Quality of Student Learning. *Studies in Higher Education*, 28(1), 37-48.
- Ramsden, P. (2003). *Learning to Teach in Higher Education* (2nd edn). London: Routledge.
- Watson K; Steele F; Vozzo L and Aubusson P. (2007). Changing the subject: Retraining teachers to teach science. *Research in Science Education*, 37(2), 141-154.

STUDENTS' CONCEPTIONS ABOUT PROBABILITY AND ACCURACY

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Abstract: College students' conceptions about probability and accuracy were explored. Both qualitative and quantitative analyses were done by means of two tests applied at two different moments. We show the results referring to the beliefs and conceptions about probability, margin for error, accuracy, certainty, truth and validity. Previous misconceptions about science may cause difficulties in the interpretation of scientific models. So, to find out students' beliefs about science and technology, a Likert scale type test was made and presented to part of the sample. Although most of the people who answered the survey accredited the incidence of probability in the results of a physical experiment, they also gave it accuracy and truth values which are not inherent. It is also remarkable that only a very low percentage has a posture that is coherent with the scientific vision of the terms.

Keywords: student beliefs; probability; teaching and learning statistics

1. Introduction

Probabilistic models are more often used in different disciplinary fields. For this reason basic concepts of probability and data analysis stretch to be introduced at high school and in some cases also at elementary level. Although, as teachers, we observe that freshmen not always can clearly recognize a random phenomenon or an actual situation with possibilities of representation with a statistical model. Reality and knowledge perceptions implicate on the individuals different attitudes in front of the randomness concept then also facing the probability ideas. This has a strong influence on understanding, develop possibilities and statistical-mathematic models applications.

Several papers have shown that misconceptions and erroneous beliefs about science and technology bring about misinterpretations of scientific models (Aikenhead et al, 1987; Aikenhead et al, 1992; Azcárate et al, 1998). Students who start studying Engineering at Universidad Nacional de La Matanza (UNLaM) show very little scientific and technologic knowledge at the moment they enter the university. Quantitative and qualitative analyses were made in order to make a test with the results of the answers students gave in the test (Alvarez et al, 2004; Sacerdoti et al, 2004). We present the results concerning the students' conceptions and beliefs about probability, margin for error, accuracy, certainty, truth and validity.

With the release of "*Curriculum and Evaluation Standards for School Mathematics*" (NCTM, 1989), it is proposed that primary and secondary school students have to study

probability, and also explore situations actively, experimenting and simulating probability models. In Argentina the study of probability has been included on the curricula from General Basic Education (EGB) to Polimodal since the Federal Law of Education was passed. But as Batanero (2002) argues, most schools do not deal with statistics owing to the length of the syllabuses. Thus, students start university without the expected knowledge of the subject.

Another aspect Fischein (1975) pointed out is the exclusive deterministic nature that the mathematics curriculum has had up to some years ago, and the need to show students how to face facts more realistically: "In our contemporary world, scientific education cannot be merely reduced to a certain, deterministic interpretation of events. An efficient scientific culture demands an education in statistic and probabilistic thought". The same idea can be applied to the teaching of Physics, when there is an abuse of the explanation of deterministic models, such as Newton's, ignoring in many cases the uncertainty of experimental results and the differentiation between model and reality (Gilbert et al, 1998). On the other hand, the fact that some phenomena we want to model have results that depend on chance rather than on a deterministic nature, makes it necessary to use probabilistic models.

In the present study we show the results referring to the beliefs and conceptions about probability, margin for error, accuracy, certainty, truth and validity of engineering students at the Universidad Nacional de La Matanza (UNLaM) at Argentina.

2. Previous research

In a previous study (Sacerdoti et al, 2004, abstract only) to search for the students' conceptions about probability and accuracy a semi structured poll was submitted to a 60 UNLaM engineering student sample. No directions on the poll subject were given to the students. Based on Azcarate et al. (1998) we generated a tool from their proposed test adding some items that allow us to disclose "ways of saying".

Conceptions about random events and the meaning of the words certainty, accuracy, uncertainty and probability were analyzed. A portion of the questionnaire used in that work is shown in Appendix I. The first question concerns the randomness of different experiences (referring to chance, the occurrence timing, the physical phenomena, meteorological events and health). The second question refers about the type of knowledge that is possible to obtain on the above experiences given as options certainty, accuracy, margin of error and probability. The last question has open answers; it is requested to define the previous phonemes.

Among the main conclusions of the previous study we can quote that many students associate randomness with event timing. This is shown by connecting, with higher frequency, randomness to future events and with less frequency to past events.,

Other remarkable issue concern the approach to the events, making a difference among *chance*, *everyday life* and *scientific*. Students considered a higher degree of randomness in the events related to chance while in those events related to the weather, where the everyday life predominates, a more unlike and subjective opinions showed up.

Finally, in those events related to the scientific,--physical models of planetary or missile movement -- physics perceptions as a very accurate and in some cases exact science were found.

Answering the last question the polled students should show what is the meaning of certainty, accuracy, margin of error and probability. From the answers it is noticed the words association. Among the answers the following are remarked:

The term *probability* was mainly associated with different, possible, results (it will rain, it will not rain) but it is not connected with values obtained from measurements of a magnitude. Probabilistic quality and incidence of chance in the values are ignored. Instead the students

prefer to use “margin of error” that is related to “rank”, “approximation”, “uncertainty”, “not reliable”, “low knowledge”. These conceptions should be taken into account when teaching subjects as uncertainty or confidence gap.

The term accuracy is associated with “precision”, “unique solution” and “no errors”. Also the term certainty is associated with “safety”, “correctness”, “predictable”, “true” and “valid”. The students admit that some information are given “using probability terms” but it seems that the meaning is not clear to them. Intrinsic temporary nature of science knowledge is not recognized by the students, on the contrary they express that science must be true or looking for the truth. Some students’ beliefs do not agree with Díaz, E. (1997): “Science is temporary in two aspects: first, the fact that an observational enunciation reveals itself as true, does not authorize to declare that the law from which it was derived is also true. In a second sense, temporary quality is shown in the emergence of rival theories not generated because an empirical disproof but originated in a determination of the scientific community”.

Nine phrases were selected and worked out from the analysis of answers (Appendix II). These phrases were integrated in a Likert type scale. In a next stage the nine phrases were included in a more general scale allowing a treatment with the principal component analysis (PCA) methodology.

3. Methods

With the aim of building an instrument to find out beliefs and conceptions of students who could start studying Engineering at UNLaM, 103 phrases were chosen from a first test with open ended questions. Almost two hundred students were asked to mark their agreement with each of the open ended questions, in a 1 – 5 scale. The students were picked at random among those who started to study Physics I in careers in Engineering at UNLaM. In order to reduce the number of phrases and integrate them on a new multidimensional scale, principal component analysis (PCA) with Varimax rotation was used, thus allowing us to choose those phrases which showed the greatest variety of answers, associating those representing the same idea in the same component. Afterwards, these phrases were analyzed with the aim of looking for testees’ profiles. As for the application of PCA, validity requirements were verified: KMO (Kaiser-Meyer-Olkin measure of sample adequacy) and Bartlett test. The following conditions were observed to select the number of components (Hair et al, 1999). **a.** to choose the components corresponding to self-values higher than 1; **b.** to include in each component items with factor loadings higher than 0.4 and high communality; **c.** to admit the items which are theoretically coherent with the component in which it is found, and **d.** to consider the number of components necessary to explain a minimum percentage of 60% of the total difference.

Using these criteria, at the beginning 22 main components were obtained (Giuliano et al, 2005), among which three referred to the character of validity of scientific knowledge, including terms such as probability, margin for error, accuracy, certainty, truth value, validity in the context of science of a specific physical phenomenon. These three components grouped nine phrases from the original 103, and were used to analyse conceptions in the testees. The PCA methodology was also applied to the sub-group of nine phrases released from the first test and coherent results with the analysis of the test of 103 phrases were obtained (Giuliano et al, 2006).

4. Results

PCA was performed over the sub-group of the 9 phrases (see Appendix II), and three equivalent components were found which satisfactorily passed the requisites of validity. Each component explains more of the 17% of the total difference, and the whole explains 56%, moreover, each was interpreted in the light of the phrases they were made of. Table I shows the resulting components.

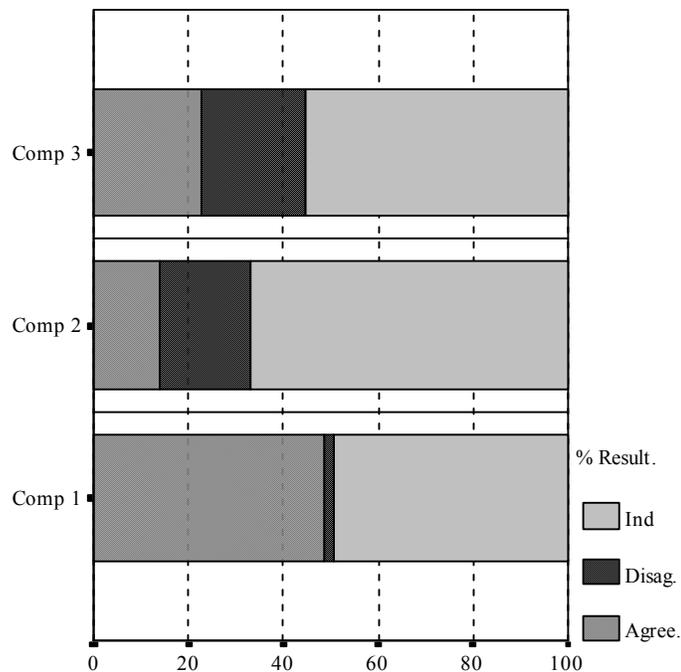
Component	Interpretation	% Var. Explicate
Comp 1	It is possible to know with a certain probability or margin for error the exact location of the place a missile will hit.	20.4
Comp 2	It is possible to know certainly and accurately the exact location of the place a missile will hit.	17.7
Comp 3	Affirmations of science cannot be defined as true nor be formulated as completely accurate.	17.5

Table I: Interpretation of the resulting components and % of the explained difference

The tipicity index was estimated for each one of these components, as an average of the phrases it was made of pondered by its factor loading. The typical quality of each component was considered in an ordinal way grouping the values in three equal intervals classified as disagreement, indifference and agreement. The results are shown in Figure 1.

Figure 1: Percentage of responses in agreement, disagreement and indifference in each component.

As can be seen in Figure 1, approximately half of the responses are in the indifference area for all three components. In component 1 a high percentage of agreement with the probabilistic idea can be seen, although most of them also agree with the idea of certainty, and this implies ambiguity in the



interpretation of the concepts. The equal distribution of agreement and disagreement in both components 2 and 3 should not be taken as similar interpretations seeing that with the method of building of the components their co-relationship is low. This implies that it is not the same testees who do not agree or disagree with both factors.

The combinations of answers within factors show diverse postures distributed among all possible combinations of these three components, showing that only 10% of the testees show a posture which is

scientifically adequate, i.e., agrees with components 1 and 3 and disagrees with 2. Nine % of the testees admits probability and certainty in scientific phenomena, i.e. agrees with component 1 and disagrees with component 3, while only 16% agrees with both components.

5. Discussion

It is worrying to note that students show diverse interpretations of terms such as probability and accuracy and that these do not coincide with the scientific meaning of the terms.

Half of the testees do not have definite postures regarding the phrases under discussion, what represents a high percentage considering their simplicity. This sample, taken among a group of students interested in studying Engineering, reveals a poor knowledge of the topics analysed. It is feared that major deficiencies may be found in students who have finished the secondary school and have different interests.

It is highly important in our role of teachers at the basic level of engineering to acknowledge the deficiencies our students may have in order to help them to improve, taking into account that similar words may carry different meaning to teachers and students. The teaching of physics at any level should take into account modeling and probability. It is advisable to do activities which aim at surpassing ingenuous beliefs about deterministic models apart from including probabilistic models as from pre-university levels, not only in Maths but also in Physics.

Present results allow us to think about meaningful aspects of the students integral training. It is important that they develop an ability to recognize that there are different ways of reality modeling; among them the probabilistic and determinist models are relevant. Science looks for results prediction but cannot obtain exact values, however the probability concept should not be assigned to the phenomenon but to the information amount that can be obtained.

6. Acknowledgments

This paper has been made within the following projects: PICT 04-13646-BID 1728/OC-A and C054 UNLaM.

7. References

- Aikenhead, G.S., Fleming, R. W., Ryan. (1987). A. High school graduates beliefs about science, technology and society I. Methods and issues in monitoring students. *Science Education*, 71 (2), pp.145-161.
- Aikenhead, G.S., Ryan, A. (1992). The development of a new instrument: views on Science– Technology – Society (VOSTS). *Science Education*, 76 (5), pp.477-491
- Alvarez, M., Giuliano, M., Nemirovsky, I., Pérez, S., Romero, C., Sacerdoti, A., Santorsola, M.V., Vázquez, S. (2004). El análisis multivariado ACP como método para agrupar concepciones sobre la naturaleza de la ciencia en alumnos de ingeniería. *Memorias del 7^a Simposio de Investigación en Educación en Física*, La Pampa, Argentina.
- Azcárate, P., Cardeñoso, J:M. Porlán, R. (1998). Concepciones de Futuros profesores de primaria sobre la noción de aleatoriedad. *Revista Enseñanza de la Ciencias*, 16 (1) pp 85-97.
- Batanero, C. (2002). Los retos de la cultura estadística. *Jornadas Interamericanas de Enseñanza de la Estadística*, Buenos Aires. *Conferencia inaugural*.
- Diaz, E.(1997). *Metodología de las ciencias sociales*. Biblos, Bs.As
- Fischbein (1975). *The intuitive sources of probabilistic thinking in children*. Dordrecht: Reidel
- Gilbert, J., Boulter, C. Rutherford, M. (1998). Model in explanations (Part 2). *International Journal of Science Education*, 20 (2): 187-203.
- Giuliano, M., Pérez, S., Álvarez, M., Nemirovsky, I., Sacerdoti, A. (2005) Análisis estadístico de actitudes tecno-científicas de alumnos que ingresan a carreras de Ingeniería. *Memorias de la XIV Reunión Nacional de Educación en Física*, Bariloche, Argentina.

- Giuliano, M.; Nemirovsky, I.; Concari, S.; Pérez, S.; Alvarez, M.; Sacerdoti, A. (2006). Conceptions about probability and accuracy in Argentine students who start a career in engineering. *Proceedings of ICOTS7, International Conference on Teaching Statistics*, Brazil.
- Hair, A., Tatham, B. (1999). *Análisis Multivariante*. 5ta edición. Prentice Hall. Iberia, Madrid.
- NCTM (1989). Curriculum and evaluation standards for school mathematics. Reston, Va: N. C. T. M.
- NCTM (1991). Estándares curriculares y de evaluación para la educación matemática. Sevilla: Sociedad Thales (traducción española del original publicado en 1989 por la National Council of Teachers of Mathematics)
- Sacerdoti, A.; Giuliano, M. (2004). La Probabilidad como Medida de la Información. Resumen publicado en *Memorias IV Conferencia Argentina de Educación Matemática*. Buenos Aires.

Appendix I

- 1) Have the following experiences a random behavior? Why?
 - a) A seed germination
 - b) A number obtained from throwing a dice.
 - c) To guess correctly the number of a thrown dice that you cannot see.
 - d) Number of right sides obtained with 100 coin throws.
 - e) On November 5, 1933 it was raining in Buenos Aires.
 - f) Tomorrow it will rain in Buenos Aires.
 - g) In a month it will rain in Buenos Aires.
 - h) The place where Mars will be on Nov. 15, 2005.
 - i) The place where a missile will hit.
 - j) On next winter you will catch a flu.
- 2) Mark with an X possible knowledge types associated with each experience:
 - Certainty
 - Accuracy
 - Margin of error
 - Probability
- 3) Write down what you understand for: certainty, accuracy, margin of error, probability.

Appendix II

- P1) Science can tell if something is valid or not.
- P2) Science assertions cannot be define as true.
- P3) Science cannot asserts anything with full certainty.
- P4) It is possible to know with certainty the place where a missile will hit.
- P5) It is possible to know with accuracy the place where a missile will hit
- P6) It is possible to know with certain margin of error the place where a missile will hit.
- P7) It is possible to know with certain probability the place here a missile will hit.
- P8) Scientific theories change according to discovering of errors.
- P9) Scientific theories change because others scientists can find errors, meet others colleagues and discuss its truthfulness.

UNDERGRADUATE STUDENT DIFFICULTIES WITH INDEPENDENT AND MUTUALLY EXCLUSIVE¹ EVENTS CONCEPTS

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Abstract: The concepts of disjunctive events and independent events are didactical ideas that are used widely in the classroom. Previous observations of attitudes in assessments given to students at university level who attended the introductory Statistics course helped to detect the confusion between disjunctive and independent events, and indicate the spontaneous ideas that students tend to elaborate about both concepts in different situations in which these appear. However the didactical relation between these ideas and their formal definitions is not known in detail. In this work, we analyze students' misconceptions, their persistence, and the process by which the student confronts his misconceptions by applying theoretical concepts. The aim is to improve the teaching of these topics.

Keywords: independent events; mutually exclusive events; probability; statistics education; undergraduate mathematics education

1. Introduction

Statistical education is not very well researched in Argentina. However Statistics is a science which gets more emphasis as we move from basic education to mathematics education in post-graduate levels onto career paths.

Therefore, in education we must ask ourselves: What happens with knowledge in Statistics? The problem starts from reality and for the student it is a real problem to associate reality with formal concepts. Ernesto Sanchez says: in education it is good to ask in which conditions and how an individual person changes a conception, a belief, an intuition or a spontaneous idea about a pre-determined situation, by virtue of using a scientific instrument.

This work is based on the analysis of answers to common problems encountered by university students going into mathematics and social science careers.

This study, carried out with students from Statistics I, a course for majors in Accountancy, is an analysis to the responses to a problem where there is the concept of mutually exclusive and independent events and the confusion they have about these concepts.

According to Sanchez E. (who wrote Ph.D. thesis about independent events³) the problem starts with:

¹ The term disjunctive is interchangeably used with the words mutually exclusive

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- the belief that independent and disjunctive events are the same;
- confusion between independent events and independent experiences.

Besides, it is understood that independence is only quantitatively proven by the product rule. These concepts are simple when they are defined. However it has been proven through interviews that the confusion persists in many university students attending Statistics I. The phenomenon appears in students with different mathematical backgrounds.

Are there techniques of teaching and learning good enough to take into account the spontaneous concepts of the probability notions while developing their formal knowledge?

Some studies about attitudes and responses in exams indicate that students use intuitive ideas to analyze independent events and mutually exclusive events in different situations where these notions play a role. But the relation between these intuitive concepts and the formal definitions is not known, at least in Argentina at the tertiary level.

2. Research Problem

A first course in Statistics typically covers the minimal contents required for a basic understanding of statistics in day to day use. Most text books include probability topics for introducing the following concepts: events, probability definition, conditional probability and independence, random variables and probability distributions. There are very few didactical studies on such topics or proposals on the teaching and learning of such topics. This study seeks to understand students confusions between mutually exclusive events and independent ones, with the goal of providing some recommendations to deal with this confusion and implementation in education.

3. Research topic

Historically according to the renowned probabilist Kolmogorov the concept of independence represents a crucial concept which provides probability theory numerous pedagogical illustrations. I will focus on the concepts of: mutually exclusive events and independent ones, because of general reasons such as:

Firstly, it is common the confusion which associates disjunctive to independent, and it is well known that only if one of them is empty both are confirmed, in the context of finite sample spaces.

Secondly independence is confused with individual experiences without an explanation about the difference between both ideas.

As Sanchez (1996) found in his thesis, this confusion is due to causality.

³ Sánchez, E. (1996) *Conceptos teóricos e ideas espontáneas sobre la noción de independencia estocástica en profesores de bachillerato : Un estudio de casos*. Tesis de Doctorado, Departamento de Matemática Educativa, Cinvestav-IPN, México.

3.1 General questions

The problem starts in probability which has always been considered, according to teachers experience, a topic of difficulty for most students. Even though the independent and mutually exclusive events concepts are apparently simple, people spontaneous ideas give rise to wrong answers. These misconceptions have become interesting for researchers not only in psychology but also in didactics. Therefore among the various questions posed by researchers, I have chosen the following:

What are the relations between subjective or intuitive conceptions and those which are transmitted in the classroom, and which compose the formal knowledge of probability?

Are there optimal teaching and learning techniques which consider the spontaneous conceptions that individuals hold about probability ideas while they develop their formal knowledge in courses?

In fact previous observations of attitudes have shown some spontaneous ideas which tend to elaborate about independent and mutually exclusive events in different situations where this idea is involved, but it is unknown what is the relation between these intuitive conceptions and the formal definitions they encounter? Sánchez also poses the following questions .

- What happens with an individual's misconceptions about independence in determined situations, when discussing independent and mutually exclusive event definitions in a probability course?
- What is the process via which an individual can confront their wrong conceptions?

4. Some Prior Findings

In a probability survey by Sánchez administered to 44 mathematics professors who had some probability and statistics knowledge , they were asked to answer the following question⁴:

A card from an American deck is taken by chance: It is "A" the event "it was taken a clover" and B "it was taken a queen". Are A and B independent events? Justify

It was expected that they would calculate the probabilities of A and B and $A \cap B$ and then check the product rule by performing $P(A \cap B) = P(A) \cdot P(B)$.

However, this simple solution was found by only 4 professors. Standard answers were:

Answer 1: Independence or mutually exclusive events are the same. They are not independent because there is a clover queen.

Answer II: Solution depends on an application of a succession, namely

If a card to verify the event A is taken and it is placed in the card deck to verify event B, so A and B are independent.

If it is taken to verify A and it is not placed again so they are dependent

⁴ Editorial note: The language of translated exercises, and responses have been left in the native (English) formulation of the author. The mathematical context allows for no misunderstandings.

5. Hypothesis to explain the mistakes

The mistakes in the answers noted above could come from various factors:

- They forgot independence concepts
- Terminology confusión
- We must consider that the problems which involve independent events notion are immersed in a wider range of problems, which leads us to the area that gives the most general idea of stochastic.
- Epistemological problems about the meaning

6. History and epistemology

The concept of independence emerges in the analysis of hazard games “without replacement” given by De Moivre (1718-1756) and by Bayes (1763). Before them Bernoulli used this concept to formulate his theory without realizing it.

There were no changes in the intuitive concept of independence with the improvement made by Laplace and Moivre. The concept of independence was understood only in the context of independent experiences as is shown with the definitions of classical authors such as De Moivre (1756):

“Two events are independent when there is no connection between them and what happens in one of them does not occur in the other one.”

“Two events are dependent when they are connected in such a way that the probability that one occurs is altered by the occurrence of the other”

Laplace does not define in any explicit way independent events and their properties. In this period drawing with and without replacement in successive trials was identified with independent and dependent events respectively.

At present numerous difficulties arise from these classical authors' concepts.

Some theorists like von Mises reject the formal definition of independence. He considers that in the axiomatic theory of Kolmogorov there are events that are independent but are not seen as independent one of each other, in the intuitive sense that “they do not influence each other”.

“When two characters are considered to influence each other or not, it is given a notion of independence. Nevertheless, a definition based on the multiplication rule is no more than the weak generalization of a concept full of meaning”.

This problem of the inversion of content and the mathematics definition plays an important role in the teaching process. In some books the deduction of the independence formula appears as a consequence of conditional probabilities.

These difficulties appear in the historical development of the concept. During the eighteenth century, the task was to legitimize and delimit the object of mathematical studies; most diverse methods were accepted to analyze this object. In the nineteenth century, the relation was inverted. The object became arbitrary, and the task was to confirm the methods and define strict procedures to allow the abstraction of the objects and so, an extension of the applications.

Feller (1983) comments:

Generally the correct intuition that certain events are stochastically independent is felt, because if it is not like that, the probability model would be absurd. Nevertheless [...] there exist situations in which the stochastic independence is discovered just from calculus. (p. 137)

Steinbring (1986) analyzes the historical development of stochastic independence from an epistemological perspective, to find elements for a didactic perspective. In the historical development there is an inversion of content of the concept and its mathematics definition.

- Firstly, there is an association of concrete representations of dependency with real facts.
- Secondly, the concepts have been defined formally in mathematics by the multiplication rule.

These statements are usually not connected properly. Consequently, it may produce a confusion about the concepts of independency (or dependency).

7. Theoretical framework

Some studies on student's misconceptions have shown that:

1-It seems that what characterizes students' misconceptions is its steadiness in time, its relative internal coherence and its acceptance among the larger community of students.

2-Ideas are not by chance but in relation with what they know and their thinking characteristics and abilities; that is, the idea that a child has, implies a determined knowledge about how things are, and happen, and a determined intellectual functioning, a way of explaining not only a particular concept but also others in relation with it.

3- The number of different conceptions that classroom students depict about a fact or a situation is not limited, whereas several common patterns among them are found.

Beliefs and conceptions are intuitively acquired by students from their experience of interaction with reality and with ideas, as a consequence of the learning processes which they are formally exposed. Cornu (1991) designates spontaneous conceptions of a mathematical idea to the group of intuitions, images and prior knowledge which are made in the individual person from their daily experience and from the semantic contexts in which these ideas arise.

According to Cornu these spontaneous conceptions are made before the formal learning processes, so it does not mean that they disappear after those processes, but are usually mixed in with the new ones. They are modified and they end up as hybrids. Impediments appear in the phylogenesis and ontogenesis of concepts (Sriraman & Törner, 2008) so it can be noticed in the historical developments of those concepts, and also in the individual construction processes of those.

8. Methodology

8.1 Economic sciences career

An exam was developed on the topics of probability and random variables. In the first exercise there were concepts of probability organized in items. One of them included the concepts of mutually exclusive and independent events. The students were asked if the proposition was false or true and to justify their answer. The statement was false.

The analyzed exercise is the following:

Let (S, P) be a probability space, A and B events in S so that $P(A) > 0$ and $P(B) > 0$. Decide whether the following statement is true or false. If it is true, justify your conclusion; if it is false, state the right expression.

“If A and B are mutually disjunctive, the probability of at least one of them occurring is: $P(A).P(B)$ ”.

8.1.1 Responses

Among the responses, the student is given the total mark for the exercise if he or she answers “false: and writes the correct expression: that is, the following:

“The probability that at least one of them occurs, when the events are disjunctive, is:

$$P(A \cup B) = P(A) + P(B).”$$

Table 1 shows the results of the 97 students.

This apparent easy response was answered correctly by 14 students, of whom 10 passed. As it is seen, 12 students do not answer, of whom only 4 passed. Incorrect, with response true, 14. Responses F but with wrong justification or without justification and had no total mark were 57, only 19 students passed.

Our target is to analyze the false answers which were wrongly justified.

Table 1

student	correct	no answer	F no Justified	F Justified wrong	F Justified regular	True Justified	True no Justified	Total
passed	10	4	1	14	5	3	0	37
failed	4	8	6	26	5	7	4	60
Total	14	12	7	40	10	10	4	97

8.1.2 Analysis of responses of the students

An analysis was made of the responses of each student, particularly of the 40 that considered F and justified incorrectly. Nearly half of them (17) justified it in this way:

$A \cap B = \emptyset \Rightarrow$ the probability of occurrences is given by $P(A).P(B)$; that is

$$P(A \cap B) = P(A).P(B).$$

This was the most repeated mistake in the justifications.

The confusion of the concepts followed the patterns proposed by Sanchez. The question was not direct, which can produce misunderstanding. This shows that the students make mistakes systematically because they do not have a clear concept of both notions.

A second type of response that was presented was:

Si A y B son me $A \cap B = \emptyset$ luego $P(A)= 1-P(B)$ ó $P(B) = 1 - P(A)$

entonces $P(A) \cdot P(B) = P(A \cap B) = \emptyset$

If A and B are disjunctive $A \cap B = \emptyset$ then $P(A)= 1-P(B)$ or $P(B) = 1 - P(A)$; then

$P(A) \cdot P(B) = P(A \cap B) = \emptyset$

The third answer that also was present with several students was:

$P(S) = P(A)+P(B)$.

We can say that the second and third responses are associated because the students consider that the sample space is formed by two sets. They do a Venn diagram including these sets in S. According to Duval's commonly known finding that there is a problem in the translation from graphs to symbols. Students represent one thing and write another.

This problem of symbol representations is repeated among the answers that we can associate because of the wrong symbolic representation. Half of the students wrote $P(A \cap B) = \emptyset$.

Other students wrote $P(A) \cup P(B)$, $A \cap B = 0$

They tend to misunderstand symbols. On the one hand they considerer union of probabilities. On the other hand they associate the empty set with the number 0 (zero). They equate the probability of intersection of disjunctive events to the empty set, and they equate the intersection set to zero. It is recurrent in students from all the careers.

8.2 Political Science case

The following conceptions were taken from probability surveys which were done by students in the humanities who are attending statistics.

Example: A behaviour test in a large number of drug addicts indicated that after their treatment re-incidence (relapse) occurred within the two years after treatment. This relapse could depend on the socio-economical level they belong to, as shown in the following contingency chart:

		Condition within the two years after the treatment	
		Reincidence(R)	No reincidence(NR)
Socio-Economical level	Superior (S)	10	50
	Inferior (I)	30	10

- Which is the probability that he gets back and belongs to a superior level?
- Whis is the probability that he belongs to I socio-economical level or not get back?

- c) Are the events R and S independent? Justify.
- d) If the chosen interviewed belongs to the superior socio-economical level, which is the probability he gets back?
- e) Are the events R and S mutually exclusive? Justify with the definition.

8.2.1 Analysis of responses of the students

These were the answers given by some students to questions c and e :

- 1) c) *No because two events are independent ,because when S happens it does not modify that R happens*
e) *They are not mutually exclusive because if S event happens it can not happen event R*
 $S \cap R = \emptyset$
- 2) c) *No because one events depends on the other one.*
e) *No because they have common elements*
- 3) c) *Justify with $P(S \cap R) \neq P(S) \cdot P(R)$*
e) *They are not mutually exclusive because they are different $S \neq R$*
- 4) c) *They are not independent because they are not the same*
e) *They are because they do not happen at the same time*
- 5) c) *No because the intersection is not empty*
e) *R y S are mutually exclusive because they can happen simultaneously*
 $S \cap R = \emptyset$
- 6) c) *They are not independent because the rule is not followed $P(S \cap R) = P(S) \cdot P(R)$*
e) *They are mutually exclusive because there is intersection between R and S*

Among the 54 students only 8 answered the items c) and e) properly well and used the form.

8.2.2 Observations:

- It was noticed that most of the students confused the product rule with the addition one trying to show the independence .
- Others generalized the product rule considering that the events are independent
 $P(A \cap B) = P(A) \cdot P(B)$
- It is curious that in any case they used the conditional $P(A/B) = P(A)$ to proved the independence.
- In the case of justifying mutually exclusive events it was detected the mistake
 $P(A \cap B) = \emptyset$

9. Discussion

The difficulty of the subject is depicted in students' answers. It was present not only in students coming from careers with a better background in mathematics (like Economics, Business, etc.) but also in those like Politics, Sociology, etc. It could be thought that students with some mathematical knowledge are less confused, however because of the concept complication it is not always the case. There is again a symbol and meaning problem.

10. Conclusions

The concepts of disjunctive events and independency persist in the students in a mistaken way. In a way, these come from games of chance but they have a more complex relationship in probability calculus. From the historic studies with De Moivre (1756) a wrong concept of independent events may be inferred, if it is not analyzed exhaustively: “Two events are independent when they have no connection to each other and what happens to one does not affect the occurrence of the other.”

When we say they have *no connection*, we are talking improperly; this persists at present. The difficulty in the case of independence is to place the concepts in opposition. On the one hand, there is a theoretical mathematical definition. On the other hand, there are numerous intuitive representations. The symbolic representation associated with the graph presents difficulties too. Although the students know the formal symbolic definition of each concept separately, in exercises such as the ones we have analyzed, they cannot distinguish one from the other. Teachers must be conscious that the idea of independence has a meaning only in a probability context while the one of disjunctive events may be considered with no knowledge of this.

Endnote:

The author is deeply grateful to Professor Bharath Sriraman for extensively rewriting portions of this article and editing the English language, syntax and verbage in this manuscript, in order to make it accessible and coherent to an English audience. Without his help this manuscript would not have been publishable.

References

- Cornu, B. (1991). Limits. In D. Tall (Ed) *Advanced Mathematical Thinking*.(pp. 153-166). Kluwer Academic Publishers.
- D'Amelio A., Diblasi, A (2006) “*Analysis of didactic situations suggested to distinguish disjunctive events and independent events*” ICOTS 7. Salvador de Bahía. Brasil
- Feller, W.(1983) *Introducción a la teoría de probabilidades y sus aplicaciones. Vol 1*. Limusa, México.
- Guzmán I.(1999) “*Fundamentos Teóricos de la Didáctica de las matemáticas*” Lecciones para un curso del Programa de Magister ECDIMAT (Magister en Enseñanza de las ciencias con mención en Didáctica de la Matemática).
- Hernández R. Joffre M . (2000) *Concepciones de los estudiantes de educación superior acerca de la noción de limite* .Tesis de Magister en Matemática,Mención Enseñanza de la Matemática República bolivariana de Venezuela Ucla-Upel-Unexpo
- Kilpatrick J., Gómez P., Rico, L. .(1995). *Educación Matemática* Grupo Ed. Iberoamericana.
- Sánchez, E (1996) *Dificultades en la comprensión del concepto de eventos independientes*. In F. Hitt (ed) *Investigaciones en Matemática Educativa*. (pp.389- 404)Grupo Ed. Iberoamericana. Departamento de Matemática Educativa, Cinvestav-IPN, México.
- Sánchez, E (2000) *Investigaciones Didácticas sobre el concepto de eventos independientes en probabilidad* . Recherches en Didactique des Mathématiques, Vol. 2 n°3,pp. 305 - 330 .2000.

D'Amelio

Sriraman, B., & Törner, G. (2008). Political Union/Mathematical Education Disunion : Building bridges in European didactic traditions. In L. English (Ed). *Handbook of international research in mathematics education* (2nd edition). (pp. 660-694 . Routledge, Taylor & Francis.

Steinbring, H. (1986) *L'Indépendance stochastique. Recherches en Didactique des Mathématiques*, 7(3), 99-118.

ENHANCING STATISTICS INSTRUCTION IN ELEMENTARY SCHOOLS: INTEGRATING TECHNOLOGY IN PROFESSIONAL DEVELOPMENT

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Abstract: The research discussed in this article comes from an ongoing multifaceted program for the teaching and learning of early statistical reasoning in Cyprus. The initial stage of the program was concerned with the design of a line of instructional materials for the development of statistical reasoning. Central to this design was the functional integration with existing core curricular ideas of the recently developed dynamic statistics software Tinkerplots[®], which provides young students with the opportunity to model and investigate real world problems of statistics. Next, professional development seminars for the teaching of statistics with the use of Tinkerplots[®] were designed and organized. The article discusses insights gained from the professional development seminars regarding the ways in which computer visualization tools can enhance teachers' content and pedagogical knowledge of statistics and how this, in turn, might impact student learning.

Keywords: Cyprus; elementary education; dynamic statistics software; professional development; statistical reasoning; technology; Tinkerplots; visualization

1. INTRODUCTION

New values and competencies are necessary for survival and prosperity in the rapidly changing world where technological innovations have made redundant many skills of the past (Ghosh, 1997). The Lisbon European Council of March 2000 placed the development of a knowledge-based society at the top of the Union's policy agenda, considering it to be the key to the long-term competitiveness and personal aspirations of its citizens. Statistics education has a

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crucial role to play in this regard. In a world where the ability to analyze, interpret and communicate information from data are skills needed for daily life and effective citizenship, statistical concepts are occupying an increasingly important role in mathematics curricula. Statistics education is becoming the focus of reformers in mathematics education as a vital aspect of the education of citizens in democratic societies (National Council of Teachers of Mathematics [NCTM], 2000).

Despite the larger place for statistics in school and university curricula, the research literature indicates that people continue to have poor statistical reasoning even after having formally studied the subject. Most college-level students and adults have little understanding of data beyond the simple – and often misleading – bar-charts and pie-charts encountered in the media (Rubin, 2002), and exhibit a strong tendency to attribute deterministic explanations to situations involving chance (e.g. Hirsch & O’ Donnell, 2001). While university level statistics instruction can indeed be successful in helping students improve their stochastic reasoning (e.g. Meletiου-Mavrotheris & Lee, 2002), poor intuitions and biases acquired early on can be extremely difficult to change (Fischbein, 1975). It is now widely recognized by leaders in mathematics education that the foundations for statistical reasoning should be built in the earliest years of schooling rather than being reserved for high school or university studies (NCTM, 2000).

Statistics has already been established as a vital part of the K-12 mathematics curriculum in many countries. However, instruction of statistical concepts is, similarly to the college level, still highly influenced by the formalist mathematical tradition. Deep-rooted beliefs about the nature of mathematics “as a subject of deterministic and hierarchically-structured knowledge” (Makar & Confrey, 2003) are imported into statistics, affecting instructional approaches and curricula and acting as a barrier to the kind of instruction that would provide students with the skills necessary to recognize and intelligently deal with uncertainty and variability. Intuition and mindset about data and variation are systematically ignored in mathematics classroom (Makar & Confrey, 2003). There is over-emphasis on center criteria and a tendency to underestimate the effect of variation in real world settings (Meletiου-Mavrotheris & Stylianou, 2003a). This tendency is related to the emphasis of the traditional mathematics curriculum on determinism and its orientation towards exact numbers. Since centers are often used to predict what *will* happen in the future, or to compare two different groups, the incorporation of variation into the prediction would confound people’s ability to make clean predictions or comparisons (Shaughnessy, 1997). The formalist tradition prepares students to search for the one and only correct answer to a problem – a condition that can easily be satisfied by finding measures of center such as the mean and the median. Variation though rarely involves a “clean” numerical response. Standard deviation, the measure of variation on which statistics instruction over-relies, is computationally messy and difficult for both teachers and curriculum developers to motivate to students as a good choice for measuring spread.

One of the most important factors in any educational change is the change in teaching practices. The direct relationship between improving the quality of teaching and improving students’ learning in mathematics is a common thread emerging from educational research (Stigler & Hiebert 1999). For it is what a teacher knows and can do that influences how she or he organizes and conducts lessons, and it is the nature of these lessons that ultimately determines what students learn and how they learn it. Statistics has been introduced into mainstream mathematics curricula without adequate attention paid to teachers’ professional development. There is substantial evidence of poor understanding and insufficient preparation to teach statistical concepts among both pre-service and practicing teachers (Carnel, 1997; Begg &

Edward, 1999). As Lajoie and Romberg (1998) point out, statistics may be as new a topic for teachers as for children. Most teachers are likely to have a weak understanding of the statistical concepts they are expected to teach and relatively deterministic epistemological sets, often sharing the same misconceptions regarding the stochastic as their students (Carnel, 1997). As a result, they tend to focus their instruction on the procedural aspects of probability and statistics, and not on conceptual understanding (Nicholson & Darnton, 2003; Watson, 2001).

The arid, context-free landscape on which so many examples used in statistics teaching are built ensures that large numbers of students never see, let alone engage in, statistical reasoning. In order to make statistical thinking accessible by all students, there ought to be fundamental changes to the instructional practices, curricular materials, tools and cognitive technologies employed in the classroom to teach statistical and probabilistic concepts. If the statistics classroom is to be an authentic model of the statistical culture, it should provide ample opportunities for exploration and experimentation with stochastic ideas in varied contexts. It should encourage statistical inquiry and data modeling rather than teaching methods and procedures in isolation (Lehrer & Schauble, 2004). The emphasis should be on the statistical process. The teaching of the different statistical tools should be achieved through putting students in a variety of authentic contexts where they need those tools to make sense of the situation. Rather than having students repeatedly practice how to calculate measures such as the mean and median, instruction should focus on helping them understand how one could use these measures in making comparisons, predictions and generalizations (Rubin, 2005). It is only through exploration and experimentation that students will appreciate the wide applicability and practical usefulness of statistical tools, and will come to view statistics as a powerful means for modeling and describing their physical and social world.

Advances of technology provide us with new tools and opportunities for the teaching of statistical concepts to young learners. These new technological tools are, in fact, designed explicitly to facilitate the visualization of statistical concepts by providing a medium for the design of activities that integrate experiential and formal pieces of knowledge, allowing the user to make direct connections between physical experience and its formal representations (Pratt, 1998; Meletiou-Mavrotheris, 2003; Papanistodemou & Noss, 2004). Having such a set of tools widely available to students has the potential to significantly change the curriculum—to give students access to new mathematical topics and insights by removing computational barriers to inquiry (Rubin, 1999). Students can experiment with statistical ideas, articulate their informal theories, use them to make conjectures, and then use the experimental results to test and modify these conjectures. There is evidence that use of such software in the statistics classroom promotes conceptual change in students and leads to the development of a more coherent mental model of key statistical and probabilistic concepts (Bakker, 2003; Hammerman & Rubin, 2003).

This article reports some of the insights gained from a case study of a group of teachers that participated in professional development seminars for the teaching of statistics with the use of the recently developed dynamic statistics software Tinkerplots[®]. This research is part of a multifaceted program for the teaching and learning of early statistical reasoning, which has as an overall aim to enhance the quality of statistics education offered in Cypriot elementary schools by facilitating professional development of teachers using contemporary technological tools and exemplary materials and resources.

2. BACKGROUND TO THE STUDY

In this section, we provide some information on Cypriot elementary schools with regards to the technology use and statistics teaching in mathematics classrooms, in order to help him/her appreciate what is described in the next sections. We also describe some of the main features of Tinkerplots[®], the dynamic statistics software employed in the study.

2.1 STATISTICS INSTRUCTION AND USE OF TECHNOLOGY IN CYPRIOT SCHOOLS

2.1.1. Technology use in Cypriot mathematics classrooms

Technology is still not central to mathematics teaching in almost all countries, and Cyprus is no exception. Almost in their entirety, Cypriot teachers at both the elementary and secondary school level report that they rarely use computers in their classrooms when teaching mathematics (U.S. Department of Education, National Center for Education Statistics, 2001; Meletiou-Mavrotheris & Stylianou, 2003b; Mavrotheris, Meletiou-Mavrotheris, & Maouri, 2004). Even when computers are used, this use is usually confined to performing routine calculations, practicing skills and procedures, and checking answers. Students rarely or never use technology to solve complex problems, discover mathematics principles and concepts, process and analyze data, produce graphical representations of data, or develop models through simulations (Mavrotheris, Meletiou-Mavrotheris, & Maouri, 2004).

A main factor limiting use of technology in the mathematics classroom is the lack of professional development opportunities for teachers. Professional development is a necessary condition for technology implementation. It includes both learning how to use the computer itself and learning how to effectively integrate technology into mathematics teaching and learning (Rubin, 1999). In Cyprus, the opportunities available to teachers are, for the main part, limited to the former. A large number of teachers have or are currently attending professional development courses, and though these programs are useful in building teachers' computer-literacy skills, they do not prepare teachers to apply technology in instruction. In a recent survey (Mavrotheris, Meletiou-Mavrotheris, & Maouri, 2004), eighty-two percent of elementary school and sixty-four percent of secondary school mathematics teachers noted that they never had any training on computer use in mathematics instruction.

There are several other underlying factors limiting technology use in mathematics instruction. One such factor is the lack of integration of technology into the curriculum. Although teachers' guides encourage teacher use of calculators and computers, there are no specific suggestions on how to integrate them in the teaching and learning process, or recommendations about what software to use. The following factors have also been rated by the majority of both elementary and secondary school Cypriot teachers (see Mavrotheris, Meletiou-Mavrotheris, & Maouri, 2004) as limiting computer use in math instruction to a large extent: lack of support by specialists regarding ways to integrate technology into the curriculum, an oversized curriculum, shortage of computers, shortage of suitable software, and lack of knowledge about suitable software. And, while, these conditions are not unique to Cypriot education (on the contrary, they describe the reality of many educational settings around the world), the fact remains that technology is yet to be integrated functionally in mathematics teaching in Cyprus at any level in K-12 education.

2.1.2 Statistics instruction in Cypriot schools

The elementary mathematics textbooks in Cyprus place strong emphasis on data analysis.

These textbooks present numerous tasks on statistical concepts and also often make connections with concepts from other mathematical domains. However, most of these tasks focus exclusively on low-level data analysis skills. The majority of items included in the textbook are of the type Konold & Khalil (2003) refers to as “encode/decode” items, i.e. items which ask students to convert raw data into a statistic or a tabular or graphical display, or to do the reverse — to determine from a data display or a statistic the corresponding frequencies or data values. There is lack of any real “doing statistics” tasks, that is completely open-ended tasks which develop higher-level skills. Key ideas in data analysis (e.g. choosing between different measures of center based on context, sampling, scaling, predicting and making data-based decisions, etc.) that should ideally be at the focus of statistics instruction are missing. Considering the central role that the textbook plays in the mathematics classroom – the majority of Cypriot teachers report that the textbook is the major source they use in deciding how to present a topic to their classes and in assigning tasks for homework (U.S. Department of Education, National Center for Education Statistics, 2001) – it becomes obvious that the curricula currently used do not adequately nourish the development of statistical literacy in students. Students are not given the opportunity to develop the necessary conceptual understanding for analyzing data using the statistical techniques they are taught (Bakker, 2003).

The problem of inadequate professional development opportunities for mathematics teachers is particularly serious when it comes to statistics instruction. The majority of Cypriot teachers have been trained in very traditional mathematics classrooms with little or no exposure to statistical concepts, and, as a result, have very limited knowledge of statistical content and its pedagogy. Many of the senior teachers have never formally studied statistics. Younger teachers may have taken an introductory statistics course at college, such a course however does not typically adequately prepare future teachers to teach statistics in ways that develop students’ intuition about data and uncertainty (Makar and Confrey, 2003). College-level statistics courses are often lecture-based courses that do not allow future teachers to experience the model of data-driven, activity-based, and discovery-oriented statistics they will eventually be expected to adopt in their teaching practices. As a consequence, teachers tend to have a weak understanding of the statistical concepts they are expected to teach and relatively deterministic epistemological sets, often sharing the same misconceptions regarding the stochastic as their students (Carnel, 1997).

2.2. FEATURES OF THE DYNAMIC STATISTICS SOFTWARE TINKERPLOTS®

Use of technology is essential to learning data analysis/statistics. Technology can illuminate key statistical concepts by allowing students to focus on the process of statistical inquiry – on the search and discovery of trends, patterns, and deviations from patterns in the data, and on the communication of findings to others. Choosing the right software however is of paramount importance. Ben-Zvi (2000) argues that statistics instruction ought to employ the use of technological tools which support active knowledge construction, provide opportunities for students to reflect upon observed phenomena, and contribute to the development of students’ metacognitive capabilities. Most existing tools for young students do not possess these qualities. They are basically simplified versions of professional tools that have been developed top-down (Biehler, 1995), i.e. from the perspective of the statistician rather than bottom-up from the perspective of statistical novices. They provide a subset of conventional plots and are simpler than professional tools only in that they have fewer options (Konold, 2002).

In response to the need for statistics software specifically designed to meet the learning needs of younger students, Konold and his team recently developed Tinkerplots® (Konold, 2005), a

dynamic statistics data-visualization software package intended primarily for elementary and middle grades. Tinkerplots[®] is a tool designed “from the bottom up”, building on the foundation of what young learners already understand (Konold, 2002). The design of the software drew on current constructivist theories of learning as well as five years of academic research about the way young students learn and process statistical concepts and the main difficulties they face.

Tinkerplots[®] offers an easy-to-learn interface that encourages student activity. Although a complete data analysis tool, Tinkerplots[®], unlike any other software, allows students to create their own graphs and plots “from the ground up” (Ben-Zvi, 2000). Using Tinkerplots[®], young learners can start exploring data without having knowledge of conventional types of graphs, or of different data types. Rather than choosing from a menu of ready-made plot types, the software allows students to progressively organize their data using a construction set of intuitive operators. Through performing simple actions such as ordering data according to the values of a variable or sorting data into categories, children can develop a wide variety of both standard graphical displays (e.g. pie charts, histograms, boxplots and scatterplots), but also unconventional data representations of their own invention (Ben-Zvi, 2000). They can progressively organize data to answer their questions.

Tinkerplots[®] aims at genuine data analysis with multivariate data sets from the start, by beginning with students’ own ideas and working towards conventional statistical notions and graphs (Bakker, 2002). The software’s design allows even young students to use what they already know to search for and detect group differences and trends. By using features such as differences in icon size, color, and sound (e.g. the user can highlight information by the value of an attribute), students can detect subtle relationships in multivariate data in powerful and intuitive ways.

Tinkerplots[®] belongs to the new family of educational software in the teaching of statistics that came to be known as dynamic software, which offer an environment that permits the construction and flexible usage of multiple data representations. All of the software’s objects are continuously connected and, thus, selection of data in one representation means the same data are selected in all representations. Changes in a data point in one representation are reflected in all related representations. Thus, students can interact with the data and see the immediate impact that their actions will have on the different representations of the data on the screen.

The software was built upon the foundation of Fathom[®] Dynamic Statistics, a highly acclaimed educational software for students in secondary higher education. Both Tinkerplots[®] and Fathom[®] are based on the same design principles, encouraging interactivity and empowering students through exploration, simulation, and dynamic visualization of data, to investigate and understand abstract statistical concepts. Both packages share a common code base and allow young learners using Tinkerplots[®] to make a natural progression to more advanced learning with Fathom[®].

Dynamic statistics software such as Tinkerplots[®] and Fathom[®] do much more than producing fancy graphs; they facilitate the discovery of patterns through exploratory data analysis. Strong research support exists for the efficacy of dynamic computer graphics as instructional media that support active construction of knowledge by learners rather than forcing them to accept information provided by the computer without deep processing (e.g., Yu & Behrens, 1994; Hoyles & Noss, 1994; Meletiou-Mavrotheris, 2003; Bakker, 2003; Paparistodemou & Noss, 2004). Attributes like the ability to link multiple representations, to provide immediate feedback, and to transform a whole representation into a manipulable object, have affordances towards more constructive pedagogical approaches than traditional packages. The direct manipulation of

mathematical objects and synchronous update of all dependent objects facilitates learning by allowing users to ask “what if...?” questions, make conjectures, and then easily test and see these conjectures in action (Ben-Zvi, 2000). Technology goes far beyond the role of mere means for data display and visualization to become a tool for thinking and problem solving.

3. METHODOLOGY

3.1 CONTEXT AND PARTICIPANTS

At an initial stage, we designed a line of research-based instructional materials for the development of overall statistical reasoning that meet curriculum objectives for elementary school. Central to this design was the functional integration of technology with existing core curricular ideas, and specifically the integration of the dynamic statistics software Tinkerplots[®]. We designed data-centered activities, in contexts familiar to children, which would provide them with opportunities to model and investigate real world problems of statistics using technology. Most of the activities were embedded in the existing elementary school mathematics curriculum and aimed to enrich it using technology.

Next, we designed and organized professional development seminars for the teaching of statistics with the use of technology. The design of the seminars was based on current pedagogical methodologies utilizing statistical investigation, exploration with interactive problem-solving activities, and collaboration. Acknowledging the fact that teachers are at the heart of any educational reform effort, the program aimed to offer high-quality professional development experiences to elementary school teachers that would enable them to effectively integrate technology into their teaching of statistical concepts and ideas.

Twelve in-service elementary school teachers (9 females, 3 males) participated in the professional development seminars which lasted for three weeks (15 hours in total). Teachers varied in their level of comfort with computers. Some had knowledge of only the basic computer applications, while others were very proficient with technology. The teachers also had varied background in statistics. Some had very limited exposure to statistics and had never formally studied the subject, while others had taken a university-based statistics course. The teachers were all experienced educators who had taught mathematics for several years.

During the professional development seminars, we worked with the teachers to help them see how their teaching and, subsequently, their students' learning of statistical concepts could be enhanced using a technological tool like Tinkerplots[®]. We adopted Tinkerplots[®] as the software we would use during the professional development seminars, because we aimed at helping teachers see how the use of a powerful dynamic software could improve their students' learning opportunities and empower them as data analysts. We hoped that by giving teachers exposure to an inquiry-based environment that captures learners' interest, we would encourage them to adopt teaching practices that would allow their students to develop their data literacy skills and competencies through their own thinking and exploration rather than receiving it predigested from teachers and books (Rubin, 1999). Additionally, we wanted to show teachers how dynamic statistics environments could also be effectively integrated into the teaching of general mathematics topics, as well as subjects outside mathematics (geography, science, etc.). We wished them to experience some of the ways in which dynamic statistics software could bring data analysis into the mathematics classroom in meaningful, relevant and accessible ways that could help convince students of the usefulness of statistics; how it could, through a data-driven

perspective, help students internalize key mathematical concepts across the school curriculum while at the same time developing data literacy skills.

The emphasis during the seminars was on enriching the participants' content and pedagogical knowledge of statistics by exposing them to similar kinds of learning situations, technologies, and curricula to those they should employ in their own classroom. Teachers worked in group activities to explore a variety of data sets using Tinkerplots[®]. Through computer-based practice and experimentation, intensive use of simulations and visualizations, feedback from each other and reflection, we aimed at helping teachers to gain better understanding of some of the bedrock concepts in probability and statistics that should be integrated into the mathematics curriculum. In addition to computer activities, there were also discussions focusing on children's learning and what is required to involve them in learning about statistics. We explored a broad range of topics of interest to the statistics teacher, including curriculum issues (e.g. role of statistics in the national and international mathematics curricula) and statistics education research (development of statistical reasoning in children, common student misconceptions, etc.). Teachers brought in examples based on their own experiences and suggested ways in which their students' learning could be improved through using the tools provided by Tinkerplots[®].

3.2 THE STUDY DESIGN

A case study design was employed in the research project. It was judged that this research strategy was well suitable to exploring, discovering, and gaining insight into teachers' perceptions, actions and interactions with the dynamic software Tinkerplots[®]. The study was exploratory in nature, and thus its purpose was not to prove or disprove hypotheses, but rather to generate descriptions based upon in-depth investigation of teachers' interactions with the technological tool, and of the impact this might have on their content and pedagogical knowledge of statistics. These descriptions, while of limited generalizability, may be used to understand similar situations (Stake, 1995) and can inform future research.

3.3 INSTRUMENTS, DATA COLLECTION AND ANALYSIS PROCEDURES

Consistent with the case-study methodology, the research team collected and analyzed a wealth of data on the development of the teachers' confidence and ability to work with the topic of statistics using technology. One videocamera was used to record all group activities and whole-class discussions taking place during the seminars, with different groups of teachers being filmed at different times. Other data consisted of participant observation, mini-interviews with teachers, and samples of teacher work.

The videotapes of group activities were first globally viewed and brief notes were made to index them. The goal of this preliminary analysis was to identify representative parts of the videotapes indicative of teachers' approaches and strategies when performing specific tasks and of the ways in which use of the dynamic statistics software influenced their thinking. The selected occasions were transcribed and viewed several times. We carefully studied and analyzed teachers' talk and actions. The method of analysis involved inductively deriving the descriptions and explanations of how the teachers interacted with the software and approached selected ideas of statistics. We attempted to verify or refute the conjectures inherent in the design of Tinkerplots[®] by investigating the degree to which the features and structure of the software influenced teachers' approaches to statistical investigations, and how this, in turn, affected their individual schemes for key statistical concepts and their instruction. These interpretations were corroborated by the insights gained from examining the data collected from other sources. The

descriptions derived formed the study findings that are outlined in the next section.

4. RESULTS

The analysis of the data collected during the professional development seminars has provided us with rich insights into how teachers think and learn about statistics and how technology might impact their statistical reasoning and, subsequently, their teaching practices and their students' learning.

Initial observations in this setting confirmed earlier findings of the research literature indicating that teachers have a weak knowledge base in statistics. When they came to the professional development seminars, most of the teachers did not seem to have a global view of the features of a data distribution. They knew how to calculate measures like the mean and the median, but did not have a robust image of what these measures mean or how they are used (Hammerman & Rubin, 2003). When analyzing or interpreting data distributions, they tended to focus on measures of central tendency and avoided to take variation into account. Such an approach is not adequate since meaningful statistical analysis of a data distribution involves simultaneously attending to its center and the variation around that center. Comprehending what an average value or a distribution is about in relation to the variation around that value or distribution necessitates integration of the ideas of center and spread.

During the seminars, teachers' endeavors with Tinkerplots[®] brought about important changes in their ways of approaching statistical problems. The presence of the dynamic statistics software increased their interest in actively pursuing problems involving data. We also have some evidence for higher cognitive involvement, for improved overall comprehension of statistical concepts.

All the teachers, regardless of statistical background, became fully engaged in data explorations using the dynamic statistics software. They were enthusiastic about Tinkerplots[®] and the affordances it offers for delving deeply into the data to make sense of the situation at hand. They came to view technology as an indispensable tool in statistical endeavors and expressed eagerness to use it in their own classroom. At the same time, use of technology affected teachers' perceptions of data. Through their continuous participation in a variety of interesting computer activities that elicited conceptions of variability and difference rather than center and sameness, they "discovered the richness and complexity of data" (Hammerman & Rubin, 2003). We observed an improvement in teachers' intuitions about variation and its effects, accompanied by a parallel development of global perception of a data distribution as an entity with typical characteristics such as shape, center, and spread, and sample size (Ben-Zvi, 2003).

We present here an example indicative of the nature of the activities used during the professional development seminars and of the types of interaction teachers had with technology. This short example of statistical reasoning about a relatively simple dataset demonstrates the power of technology in supporting statistical reasoning — not just to calculate measures, but to generate visualizations that can reveal the structure of data. It illustrates how use of the dynamic statistics software could drastically change the culture of the mathematics classroom and support the development of data literacy skills by providing access to tools that allow one to see and manipulate data in ways that are impossible without technology (Hammerman & Rubin, 2003)

The following table was taken from your Geography textbook:

Country	Area (km ²)	Population (1990)	Population density	% Urban popul.	Per Capita (US\$)	Av. Life expectancy		Patients per doctor	Students per teacher	TV sets (% households)	Literacy rate
						Mal	Femal				
Great Britain	244000	5700000	234	89	16900	72	78	611	22	98	98
France	551000	5600000	102	74	21000	72	80	403	21	98	99
Germany	357000	7800000	221	84	18800	71	77	360	14	93	99
Denmark	43000	500000	119	87	25500	73	78	390	11	96	99
Switzerland	41000	670000	160	60	34000	74	80	360		88	99
Greece	132000	1000000	78	62	6600	74	78	300	23	97	94
Italy	301200	5770000	191	70	18500	72	79	233	16	94	97
Norway	324000	420000	13	75	25000	74	80	440	18	99	100
Russia	1700000	14000000	12	66		64	74				
Sweden	450000	8600000	19	84	26400	74	80	320	9	90	99
Finland	388000	5000000	15	60	27500	71	79	440	15	72	100
Cyprus	9251	710000	77	68	10000	73	78	510	22	95.4	95

Use the table to do the following:

- Find the mean life expectancy of the twelve European countries in the table, separately for men and for women. Compare this with the life expectancy for Cypriot men and women.
- Find the mean per capita income of the twelve European countries. Compare this with the per capita income for Cyprus.
- Find the mean for two other measures provided in the table. You can choose whatever interests you. Compare the overall mean with the corresponding value for Cyprus.

Figure 1: "Population in Europe task"

"Population in Europe" task

The "Population in Europe" task appears in the 5th grade mathematics textbook (ages 10-11). In this task, students are given a table representing information about the population in twelve

European countries, and are asked to calculate overall means for different variables and compare them to the corresponding values for Cyprus. This activity is indicative of the nature of the mathematical tasks through which 5th grade students in Cyprus develop their understanding of the notion of the mean of a dataset. Although the table in Figure 1 contains plentiful information and lends itself to rich data analysis, students are only asked to calculate and compare means. While students use algorithms to compute the mean, they do not get the opportunity to explore how this measure should be interpreted in the context of other characteristics of the data.

Teachers were asked to approach the task first using traditional paper-and-pencil means as investigation tools, and subsequently, the dynamic statistics package. Our goal in giving this task was to investigate the role of technology (specifically, the role of this dynamic statistics tool) in shaping teachers' approaches and strategies. We were interested in what effect the visualization affordances of the technology would have on their perceptions of center, spread, and distribution. Thus, our data analysis paid particular attention to the processes teachers used when they were actually solving this problem with and without the help of technology.

Teachers worked collaboratively on the task in groups of two or three. Next, we describe the way teachers approached the task, both as they began to analyze the data on paper and then as they moved to Tinkerplots[®]. Our analysis of the data revealed important differences in teachers' approaches to the problem – these differences will be the focus of our analysis and report.

Paper-and-pencil Stage: During the paper-and-pencil stage, teachers focused primarily on numeric strategies to complete the task. They perceived the provided table as a way to obtain the numbers needed to calculate means and respond to the task questions. None of the teachers attempted to approach the task using a visual strategy. Hence, none of the teachers took the initiative to investigate the problem situation visually by constructing a graph of the data values to gain a better perspective in solving the problem.

Here we share a description of the exploration of two female teachers, Anna and Sophia (these are pseudonyms), whose approach was typical of how most of the teachers approached the problem, first without and then with use of technology. Anna and Sophia spent only a few minutes on completing the task using paper-and-pencil. In the first question, they simply calculated the overall mean life expectancies of men and women and compared them to the corresponding life expectancies for Cyprus. They concluded that mean life expectancies for both genders (72 years for men, 78.4 years for women) were close to the corresponding values for Cyprus (73 years for men, 78 years for women). Also, they noted that women, both in Cyprus and throughout Europe, had a much higher life expectancy than men. In the second question, again all they did was to calculate the overall mean per capita income (\$20927) and compare it to the per capita income for Cyprus (\$10000). They concluded that the per capita income in Cyprus was much lower than the average per capita income of the European countries under consideration. In the last question, asking them to find the mean of any two other measures provided in the table and to compare them with the corresponding data values for Cyprus, they looked at two variables of particular interest to educators: “number of students per teacher”, and “literacy rate”. They did the calculations and found that the mean number of students per teacher in Cyprus was bigger than the corresponding overall mean (22 students vs. 17.1 students), whereas the literacy rate in Cyprus was lower than the mean literacy rate (95% vs. 98.1%).

Dynamic Statistics Stage. Teachers employed very different strategies during the dynamic statistics software assisted stage of instruction. Use of technology facilitated the use of advanced cognitive levels of statistical problem solving. It provided the means for teachers to focus on

statistical exploration and not on recipes and formal derivations, which became secondary in importance. At this stage, teachers did not limit their analysis to calculating means like they did during the paper-and-pencil stage. In order to get better understanding of the problem situation, they now explored the entire distribution of data values using a combination of visual and numerical strategies. These explorations went much beyond the task requirements and generated more questions for teachers to investigate. Teachers made conjectures about observed trends in the data, and actively searched for evidence to support their claims by creating, transforming, and interpreting graphical data representations. Using a variety of techniques afforded by Tinkerplots[®] like categorizing data into a small number of bins, imposing cut points, or clumping similar values together and declaring them the same, they were able to view and manipulate the data, to make comparisons, and to draw conclusions.

The two teachers discussed earlier, Anna and Sophia, next proceeded to using the dynamic statistics software Tinkerplots[®] as an aid in responding to the same task. This time the two teachers initiated the use of a graph and their approach to the first question was now much richer. Their comparisons were done visually, using a variety of graphical representations. First, they produced the graph in Figure 2-left, which is a dotplot of the life expectancy for men. They highlighted the data point corresponding to Cyprus and looked for the location of men's mean life expectancy. Once again, they concluded that life expectancy for Cypriot men (73 years) was close to the mean life expectancy (72 years). However, this time they looked for additional evidence (besides mean comparison) so they divided the values of life expectancy variable into four groups ("bins"). They got Figure 2-right. Looking at the location of the largest cluster of data points, Sophia noted: *"The fact that 9 of the 12 countries are in the 72-74 bin shows that Cyprus has a life expectancy similar to the rest of the European countries"*. This new view supported their previous argument.

The two teachers followed a similar procedure when computing the women's life expectancy and, based on their displays (Figure 3), concluded that Cypriot women had a life expectancy similar to that of the other European countries. Now, though, while looking at Figure 3-left, Anna observed that the life

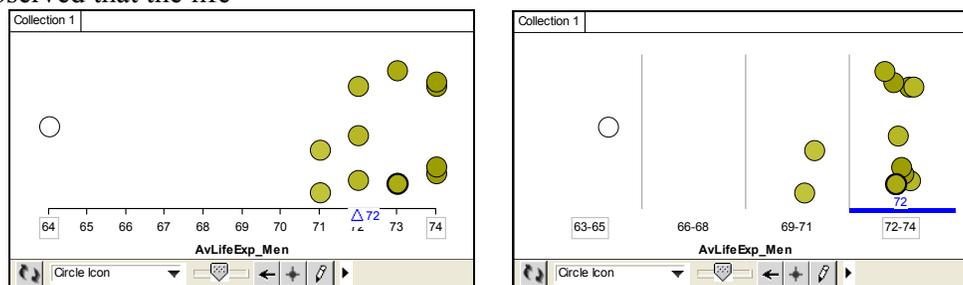


Figure 2: Dotplots of life expectancy for men

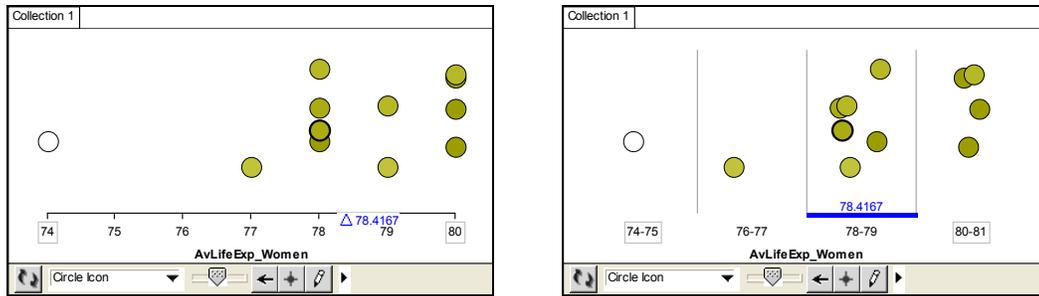


Figure 3: Dotplots of life expectancy for women

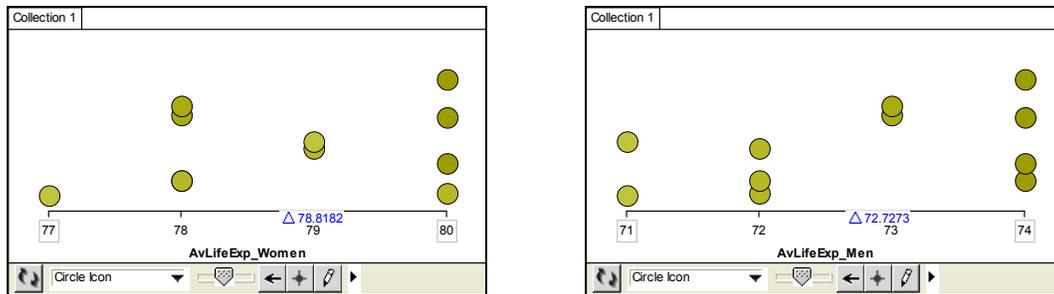


Figure 4: Life expectancy of women and men after deleting Russia

expectancy for one country (Russia, with a life expectancy of 74 years) was particularly low and that this might possibly lower the mean life expectancy for women. Sophia suggested to delete the data point corresponding to Russia and observe what happens. After doing so (Figure 4-left), they noticed that the mean life expectancy for women increased “*by almost half a year*” (from 78.4 to 78.8 years). They repeated their experiment with the men’s data (Figure 4-right), and, once again, they observed that *the mean increased more for men* (from 72 to 72.7 years) *because men’s life expectancy in Russia is only 64 and this lowered the mean a lot*”.

The two teachers also explored the relationship between life expectancy for men and life expectancy for women. Whereas during the paper-and-pencil stage the only observation they had made was that women tend to live longer than men, now access to technology allowed them to make more sophisticated comparisons. They drew a scatterplot of average age of men vs. average age of women (Figure 5-left), and looking at it concluded: “*Life expectancy for men tends to be higher in countries where life expectancy for women is also high*”. To further test their argument, they split the women’s life expectancy variable into four groups (74-75, 76-77, 78-79, 80-81) and compared the women’s mean life expectancy in each of these groups to the corresponding men’s mean life expectancy (Figure 5-right). Looking at the sequence of means of these distributions in the “sliced” scatterplot made it easier for

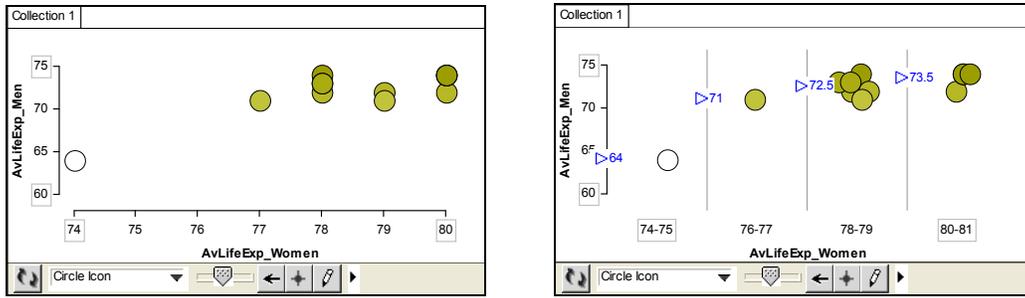


Figure 5: Life expectancy of men vs. life expectancy of women

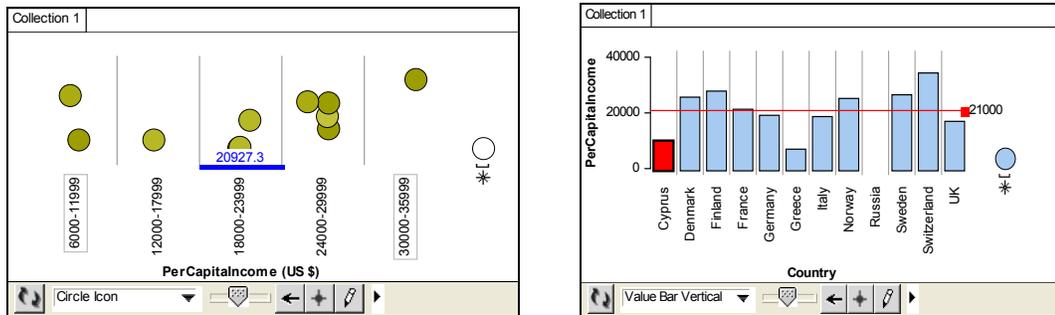


Figure 6: Dotplot and value bar graph of per capita income

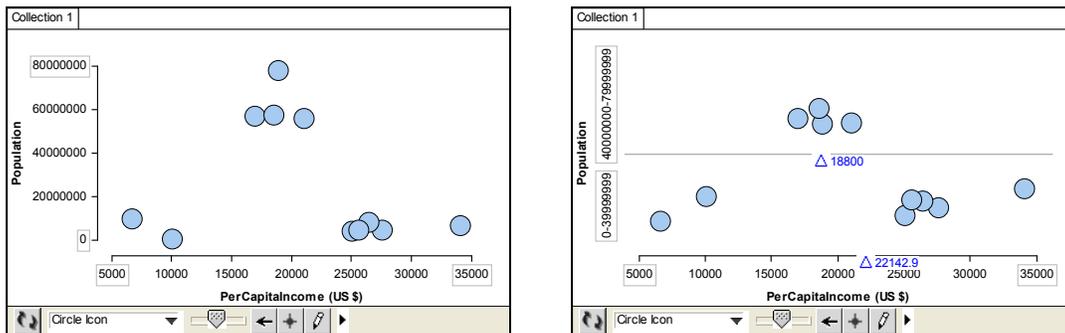


Figure 7: Per capita income vs. population size

teachers to explore the relation between the two variables (Konold, 2002; Noss, Pozzi, & Hoyles, 1999) and to further support their conjecture regarding the positive relationship between life expectancy for women and life expectancy for men.

Anna's and Sophia's approach to the second question, where they had to compare the overall mean per capita income to the per capita income for Cyprus was similar. Using various graphical displays, they concluded that not only the per capita income in Cyprus was much lower than in most of European countries, but they also observed that "per capita income clusters around 1800-24000...only three countries make less than 18000" (Figure 6-left). Similarly, by displaying the data as value bars (in which the length of each bar represents the per capita income for the corresponding country), and then introducing a cut point to divide the per capita income into two groups – one below the mean per capita income and one above it – they concluded that, "with the exception of Greece, the other countries are close to or well above the

overall mean” (see Figure 6-right).

Anna’s and Sophia’s exploration of the data gave rise to new questions in relation to per capita

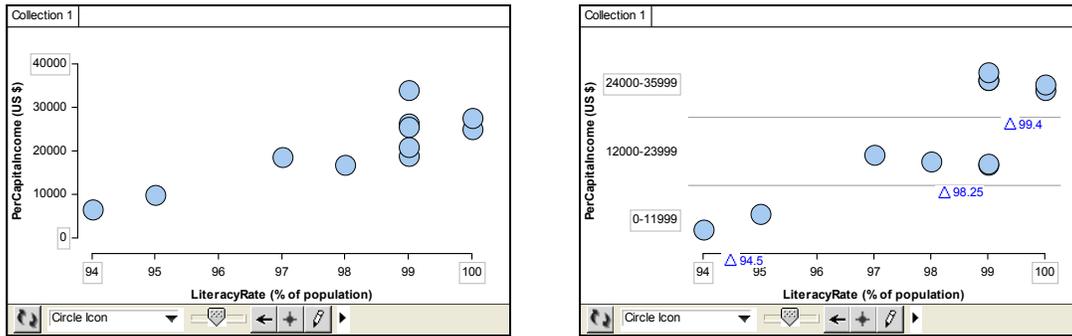


Figure 8: Literacy rate vs. per capita income

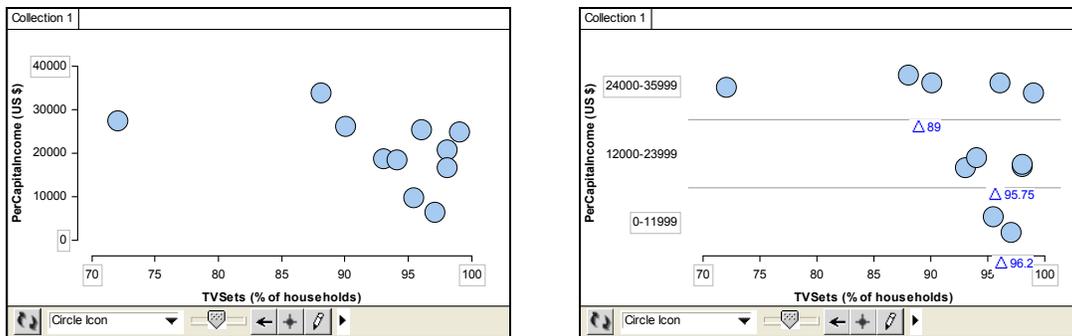


Figure 9: Percentage of households with TV sets vs. per capita income

income. For example, they investigated the relationship between per capita income and population size (see Figure 7). Looking at Figure 7-left, they concluded that, with the exception of Greece and Cyprus– the two countries on the lower far left of the plot – countries with a smaller population tended to have higher per capita income. They proceeded to investigate the correctness of their hypothesis by dividing the countries into two groups, those with population of at least 40 million people and those with population smaller than 40 million. The graph they got (Figure 7-right) supported their argument that per capita income tends to be smaller for countries with a larger population.

Anna and Sophia also explored the relation between per capita income and literacy level. Based on their graphs (Figure 8), they concluded that there seems to be a positive relationship between per capita income and literacy level.

Subsequently, they investigated whether there is any relationship between per capita income and percentage of households with a TV set. Before looking at the data, they conjectured that there would be a positive relationship between the two variables. However, their graphs (see Figure 9) did not support this conjecture. In particular, Anna made the important observation that Finland and Switzerland, the two countries with the smallest percentage of households with a TV set were at the same time the two countries with the highest per capita income.

The two teachers also explored the relationship of per capita income to other variables including life expectancy, number of patients per doctor, number of students per teacher, etc. They made important observations, which however will not be discussed here.

In the last question, asking teachers to compare the overall mean of two variables of their choice to the corresponding values for Cyprus, the two computer partners' explorations again went much beyond what the question required. One variable they investigated was total area. By plotting a dotplot of the area of the twelve countries (Figure 10-left) they noted that, with the exception of Russia, all the other European nations appear to have about the same size. *"This is very misleading"*, Sophia pointed out: *"Cyprus does not have the same area as England or France!"*

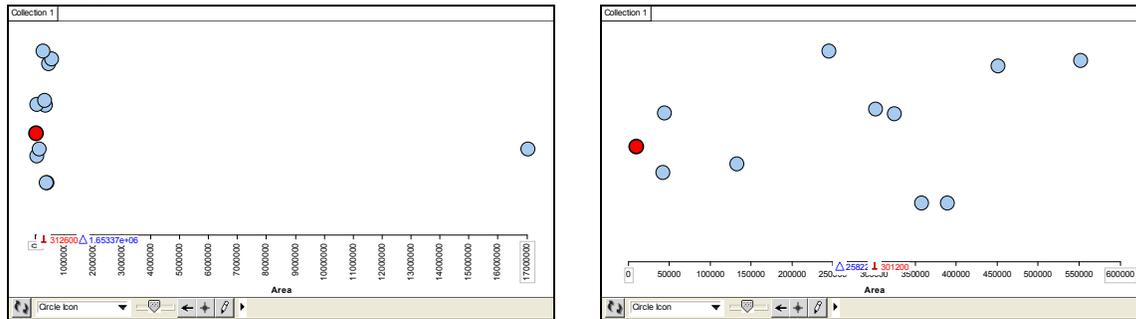


Figure 10: Dotplots of European countries' total area, before and after omitting Russia

Looking more closely, Anna and Sophia realized that the reason the dotplot looked misleading was Russia's huge area, which makes the rest of the countries appear close together on the graph and it also makes the mean area go up a lot. They noticed that the mean area (the point marked with a triangle in Figure 10-left) has a value exceeding the area of all countries in the data set other than Russia and concluded that, in this specific situation, the mean is not a very good summary of the "typical" area of a European country and that the median (the point marked with a line) is a better summary. They decided to delete Russia and see the effect of their action on the values of the mean and the median. Deleting the data point corresponding to Russia changed the scale and shape of the graph, while the mean value went down from 1 653 370 km² to 260 000 km² (Figure 10-right), and got close to the median, the value of which decreased only slightly (from 312 600 km² to 301 200 km²). The two teachers concluded that, in this context, deleting the case corresponding to Russia makes the mean a more informative measure of the center of a distribution.

Discussion on Pedagogy: The group activity was followed by a whole class discussion. During the discussion, teachers stressed the advantages of using Tinkerplots[®] to approach important statistical concepts like the mean. They noted that when teaching statistics using only traditional means of instruction, most students *"learn the different concepts as a set of techniques that they do not really understand and they cannot apply in real world settings"*. When for example being taught about the mean without using technology, most students can easily learn the procedure for calculating it, they do not however understand its meaning and how it can be used as a representative value of a set of data values. Dynamic statistics software, on the other hand, the teachers pointed out, offer tools that may aid even elementary school children, who have little statistical background, build understanding of some of the subtle aspects of the mean, such as its sensitivity to extreme values.

Teachers expressed once again their enthusiasm regarding Tinkerplots[®] and its capabilities compared to more conventional technological tools. One of the teachers, for example, said that he regularly uses calculators, and occasionally also uses the software Excel in his mathematics classes. When using Excel, he noted, his students can draw standard graphical representations

and can easily get numerical summaries of data like the mean and the median. However, he added, through use of Tinkerplots[®], students can do much more. They can make and test hypotheses; they can investigate relationships among different variables by easily constructing and manipulating their own representations of data:

Students can investigate numerous relationships. For example, they can easily provide answers to questions like: “Is there a relationship between life expectancy of men or women with the mean number of patients per doctor?” “What is the relationship between a country’s area, population, and population density?” “How are per capita income and percentage of households with TV sets related ?” I can’t imagine these relationships being explored as easily with other software. Use of Tinkerplots[®] allows us to ask questions, make and test conjectures, and discover relationships that we would not have even imagined without technology or when using some other type of software. This is one of only a handful of educational software I have worked with that has encouraged me to work using divergent reasoning.

The remaining teachers’ comments were in the same spirit:

Tinkerplots[®] allows students to quickly draw graphical representations to explore the relationships they are interested in. Without spending their time on meaningless procedures, they can focus on analyzing and interpreting data, and on drawing conclusions based on data.

Using this software, one can very easily and quickly make many comparisons among different variables and draw a lot of useful conclusions.

Several teachers pointed out that use of software like Tinkerplots[®] encourages students to generate their own questions, which can go much beyond what is required by the textbook problems. Additionally, when using technology students can easily have access to more recent data than what is available in the textbook. In particular, one teacher noted that the data which students are asked to analyse in the “Population in Europe” task is *“quite outdated since the textbooks were published several years ago”*. Other teachers agreed, pointing out that, for example, per capita income for Cyprus has doubled, life expectancy of Cypriot women has now reached 82 years and of Cypriot men 79 years, and the literacy rate has gone up to 98 percent. They stressed that, when using the software, the teacher could import from the Internet and provide his students with more recent statistics for both Cyprus and the other European countries. Additionally, students could have access to data for all European countries, not only the twelve in the table provided in the “Population in Europe task”. *“Increasing the number of countries would be extremely useful”*, a teacher noted, since *“here we had only twelve data values, and possibly some of the conclusions we drew might not hold for a larger number of countries.”*

During the discussion, we took advantage of the comment made by one of the teachers that a possible relation that the students could investigate is the relationship between a country’s area, population, and population density, to give teachers ideas as to how they could use Tinkerplots to help their students overcome one of the main difficulties in moving from elementary to secondary school – the transition from arithmetical to algebraic thinking. We stressed that the fact that software like Tinkerplots, which combine dynamic capabilities with the ability for the learner to enter formulas or commands, could engage elementary school students in constructing symbolic relationships, and thus help them build bridges to algebraic reasoning. When having

students work on the activity in Figure 1, the teacher could, through class discussion, help them see that population density describes the crowdedness of a country and that, to find out how crowded a country is, one needs to consider both area and population. Students could use the dynamic statistics software to figure out the relationship between population, area, and population density (i.e. $population\ density = area/population$). Subsequently, the teacher could encourage them to generalize their thinking by giving them the population and area of a country not in the dataset and asking them to find the population density of this country. This way, students would experience the advantages afforded by the power of generalization.

5. CONCLUSIONS

Data literacy has become a fundamental skill for living in an information era where important decisions are made based on available data. In order for students to develop a data-oriented mindset and robust data literacy skills, there ought to be significant changes to the instructional methods and tools typically employed in the classroom to teach statistical concepts. In particular, technology should assume a much more central role in statistics teaching and learning. Recognizing this need, the current study investigated the potential of the dynamic statistics software Tinkerplots[®], an educational package specifically designed to support statistics instruction in early grades. We explored the perceptions, actions and interactions that a group of Cypriot teachers had with this technological tool during professional development seminars introducing them to the software.

Findings of the study are very encouraging. They suggest that exposure, during the professional development seminars, to the dynamic statistics software Tinkerplots[®] brought about important changes in the participating teachers' approaches to statistics and its instruction. The presence of the dynamic software increased teachers' interest in statistical investigation, it gave them the opportunity to explore data in ways that had not been possible for them before (Hammerman & Rubin, 2003) The data analysis tools offered by the software provided the means for teachers to focus on statistical conceptual understanding and problem-solving, rather than on recipes and computational procedures. We witnessed the emergence of a community of highly motivated educators, enthusiastic about the affordances the software offers for delving deeply into the data. Being convinced that instructional use of Tinkerplots[®] could lead students to the construction of much more powerful understandings of statistical concepts, these teachers were eager to employ the software in their own classroom.

The qualitative methodology employed in this case study, the small scale of the study, and its limited geographical nature, means that generalizations to cases that are not very similar should be done cautiously as the specific group of teachers investigated might not be representative of all elementary school teachers. However, the study findings do suggest that use of dynamic statistics software does indeed have the potential to enhance statistics instruction. We do believe, and there are strong indications in this study to support our belief, that use of such software in the statistics classroom can promote active knowledge construction by encouraging students to build, refine, and reorganize their prior understandings and intuitions about statistics. Use of Tinkerplots[®] in particular – a software with a design based on the way young children learn statistics – can provide an inquiry-based learning environment through which genuine endeavors with data can start at a very young age. In combination with appropriate curricula and other supporting material, use of Tinkerplots[®] can help students develop a strong conceptual base on

which to later build a more formal study of statistics.

The research literature in the area of statistics education indicates very poor statistical intuitions among most college-level students and adults. Our firm belief is that people are capable of statistical reasoning and that the difficulties they face in reasoning about statistical phenomena are similar to other failures in mathematical understanding – they are primarily the result of deficient learning environments and of reliance on “brittle” formal methods (Wilensky, 1997). As researchers such as Pratt (1998), Paparistodemou et al. (2002), and Lehrer and Schauble (2004) have illustrated, when given the chance to participate in appropriate instructional settings that support active knowledge construction, even very young children can exhibit well-established intuitions for fundamental statistical concepts. Innovative educational software such as Tinkerplots[®], which are aligned with constructive views of learning, allow children to explore ideas in contexts that are both rich and meaningful to them. They afford young learners with tools they can use to construct their own conceptual understanding of statistical concepts – tools for not only data display and visualizations, but also thinking and problem solving. Use of such software, in combination with appropriate curricula and other supporting material, can help students develop a strong conceptual base on which to later build a more formal study of statistics.

REFERENCES

- Begg, A., & Edwards, R. (1999). *Teachers' Ideas about Teaching Statistics*. Paper presented at the Joint Conference of the AARE & NZARE, Melbourne.
- Bakker, A. (2002). Route-Type and Landscape-Type Software for Learning Statistical Data Analysis. *Proceedings of the Sixth International Conference on Teaching Statistics*. Durban, South Africa.
- Bakker, A. (2003). Reasoning about Shape as a Pattern in Variability. In C. Lee (Ed), *Reasoning about Variability: A Collection of Current Research Studies* [On CD]. Dordrecht, the Netherlands: Kluwer Academic Publisher.
- Ben-Zvi, D. (2000). Toward Understanding the Role of Technological Tools in Statistical Learning. *Mathematical Thinking and Learning*, 2, 127-155.
- Ben-Zvi, D. (2003). The Emergence of Reasoning about Variability in Comparing Distributions: A Case Study of Two Seventh Grade Students. In C. Lee (Ed), *Reasoning about Variability: A Collection of Current Research Studies* [On CD]. Dordrecht, the Netherlands: Kluwer Academic Publisher.
- Biehler, R. (1995). *Toward Requirements for More Adequate Software Tools that Support Both: Learning and Doing Statistics*. Revised version of paper presented at ICOTS 4. Occasional Paper 157. Bielefeld: University of Bielefeld.
- Carnel, L. J. (1997). Computers in Probability Education. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounters: Probability in education*, (pp. 169-211). Boston: Kluwer Academic Publishers.
- Fischbein, E. (1975). *The Intuitive Sources of Probabilistic Thinking in Children*. Dordrecht, The Netherlands: Reidel.
- Ghosh, J. K. (1997). Discussion. *International Statistical Review*, 65(2), 154-155.
- Hammerman, J., & Rubin, A. (2003). Reasoning in the Presence of Variability. In C. Lee (Ed), *Reasoning about Variability: A Collection of Current Research Studies* [On CD]. Dordrecht,

- the Netherlands: Kluwer Academic Publisher.
- Hirsch, L. & O' Donnell, A. (2001). Representativeness in Statistical Reasoning: Identifying and assessing misconceptions. *Journal of Statistics Education*, 9(2). (Online).
- Hoyles, C. & Noss, R. (1994). Dynamic Geometry Environments: What is the point? *Mathematics Teacher*, 87, 716-717.
- Konold, C. (2005). *Tinkerplots* [Software]. University of Massachusetts: Key Curriculum Press.
- Konold, C. (2002). Teaching Concepts Rather Than Conventions. *New England Journal of Mathematics*, 34(2), 69-81.
- Konold, C., & Khalimahtul Khalil, K. (2003). If U Can Graff These Numbers — 2, 15, 6 — Your Stat Literit. Paper presented at the Annual Meeting of the American Educational Research Association. Chicago, 2003.
- Lisbon Summit (2000, March). *Conclusions of the Lisbon Summit*. Online: [<http://ue.eu.int/summ.htm>].
- Lajoie, S., & Romberg, T. (1998). Identifying an Agenda for Statistics Instruction and Assessment in K-12. In S. Lajoie (Ed.), *Reflections on Statistics: Learning, Teaching, and Assessment in Grades K-12*, (pp. xi-xxi). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Lehrer, R. & Schauble, L. (2004). *Contrasting Emerging Conceptions of Distribution in Contexts of Error and Natural Variation*. Chapter prepared for the 33rd Carnegie Symposium on Cognition, "Thinking with Data," June 4-6, Pittsburg, PA.
- Makar, K., & Confrey, J. (2003). Chunks, Clumps, and Spread Out: Secondary Preservice Teachers' Informal Notions of Variation and Distribution. In C. Lee (Ed), *Reasoning about Variability: A Collection of Current Research Studies* [On CD]. Dordrecht, the Netherlands: Kluwer Academic Publisher.
- Mavrotheris, E., Meletiou-Mavrotheris, M., & Maouri, C. (2004). Towards a more Constructive Use of Technology in Mathematics and Science. *Proceedings of the 1st International Conference on Information & Communication Technologies*.
- Meletiou-Mavrotheris, M. (2003). Technological Tools in the Introductory Statistics Classroom: Effects on Student Understanding of Inferential Statistics. *International Journal of Computers for Mathematical Learning*, 8(3), 265-297.
- Meletiou-Mavrotheris, M., & Lee, C. (2002). Teaching Students the Stochastic Nature of Statistical Concepts in an Introductory Statistics Course. *Statistics Education Research Journal*, 1(2), 22-37.
- Meletiou-Mavrotheris, M., and Stylianou, D. (2003a). On the Formalist View of Mathematics: Impact on Statistics Instruction and Learning. *Proceedings of the Third Conference of the European Society for Research in Mathematics Education*, Online: [http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG5/TG5_list.html].
- Meletiou-Mavrotheris, M., & Stylianou, D. (2003b). A Cross-National Comparison of Teachers' Practices in the Third International Mathematics and Science Study: Instructional and Policy Implications. In A. Gagatsis, and S. Papastavridis (Eds.), *Proceedings of the Third Mediterranean Conference on Mathematical Education* (pp. 353-361). Athens: Hellenic Mathematical Society.
- National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.
- Nicholson, J. R. & Darnton, C. (2003). Mathematics Teachers Teaching Statistics: What are the Challenges for the Classroom Teacher? In *Proceedings of the 5th Session of the International Statistical Institute*, Voorburg, Netherlands: International Statistical Institute.

- Noss, R., Pozzi, S., & Hoyles, C. (1999). Touching Epistemologies: Meanings of Average and Variation in Nursing Practice. *Educational Studies in Mathematics*, 40, 25-51.
- Paparistodemou, E. & Noss, R. (2004). Designing for Local and Global Meanings of randomness *Proceedings of the Twentieth Eighth Annual Conference of the International Group for the Psychology of Mathematics Education Vol.3*, Bergen, Norway, p. 497-504
- Paparistodemou, E., Noss, R., & Pratt, D. (2002). Exploring in Sample Space: Developing Young Children's Knowledge of Randomness. In B. Philips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics*, Cape Town, South Africa. Voorburg, The Netherlands: International Statistics Institute.
- Pratt, D. C. (1998). *The Construction of Meanings In and For a Stochastic Domain of Abstraction*. Ph.D. Thesis, University of London.
- Rubin, A. (1999). *Technology Meets Math Education: Envisioning a Practical Future*. Paper Presented at the Forum on Technology in Education: Envisioning the Future.
- Rubin, A. (2002). Interactive Visualizations of Statistical Relationships: What Do We Gain? *Proceedings of the Sixth International Conference on Teaching Statistics*. Durban, South Africa.
- Rubin, A. (2005). Math that Matters. *Threshold*. 22-31.
- Shaughnessy, J. (1997). Missed opportunities in research on the teaching and learning of data and chance. In F. Bidduch & K. Carr (Eds.), *Proceedings of the 20th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 6-22). Rotorua, NZ: MERGA.
- Stigler, M. & Hiebert, J. (1999). *The Teaching Gap*. Free Press.
- Stake, R. (1995). *The art of case study research*. Thousand Oaks, CA: Sage Publications.
- U.S. Department of Education, National Center for Education Statistics (2001). Pursuing excellence: Comparisons of International Eighth-Grade Mathematics and Science Achievement from a U.S. Perspective, 1995 and 1999. (NCES 2001-028). Washington, DC: U.S. Government Printing Office.
- Watson, J.M. (2001). Profiling teachers' competence and confidence to teach particular mathematics topics: The case of chance and data. *Journal for Mathematics Teacher Education*, 4(4), 305-337.
- Wilensky, U. (1997). What is Normal Anyway? Therapy for Epistemological Anxiety. In R. Noss (Ed.), *Educational Studies in Mathematics. Special Issue on Computational Environments in Mathematics*, 33(2) (pp. 171-202).
- Yu, C. H., & Behrens, J. T. (1994). Identification of Misconceptions in Learning Statistical Power with Dynamic Graphics as Remedial Tool. *ASAProStEd*: 242-246.

TEACHING STATISTICS MUST BE ADAPTED TO CHANGING CIRCUMSTANCES: A Case Study from Hungarian Higher Education¹

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Abstract: Teaching statistics can bring up difficulties of various types for the teacher. Some of these are independent of the environment, i.e. they could occur at any place and time; some are specifically conditional on the surrounding circumstances. This paper presents an example for both of these kinds from the practice of two Hungarian teachers.

Keywords: contextual learning; Hungary; statistics education

Adjusting to the drastically changed environment

INTRODUCTION

Higher education in Hungary has been going through a turbulent transition period since many years. Traditionally Hungary had a well-established and successful higher education system³ but in the last decades it had to face new challenges coming from various directions. These challenges include the greatly increased number of students while the human and material capacity of educating institutions has not developed, the adjustment to the new needs of the labour market after the change of the political system in 1989 and adopting the two-cycle higher education system which is an obligation for Hungary by joining the European Union in May 2004. These issues have an over-all impact on the whole higher education system: we have to reconsider what and how we teach.

Probably the problems caused by these challenges are most apparent in business education since the deepest changes are to be observed in this field of life in a transition economy. The market needs a great number of well-qualified business professionals armed with usable practical knowledge but the philosophy of the traditional one-cycle system does not really fit to meet these

¹ *The views expressed in this paper – especially the sometimes sharp statements of the first part about the present situation of Hungarian higher education – are those of the author, and do not necessarily reflect in any sense the views of their institutions.*

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³ It can be certified, for example, by the fact that Hungary, although a small country with respect to its population, had over the course of the 20th century no less than twelve Nobel Laureates.

needs. The subject of statistics is an excellent example to illustrate the problem. As statistics is a very rigorous mathematical theory and at the same time a collection of practical analysing tools as well, it is always a matter of dispute how to weigh theory and practice in the curriculums.

Since I teach Statistics in a Hungarian business college⁴, here I will discuss the above mentioned issues through this particular subject. The paper is organized as follows. Section 1 briefly summarizes the current state of Hungarian higher education and enumerates the most important challenges it has to encounter. Section 2 discusses how these problems are related to the content and way of teaching Statistics, and Section 3 presents some suggestions based on my own experience and the results from a short survey which was carried out among second-year students of Statistics in December 2006. Section 4 summarizes the key findings.

1. CHALLENGES AND THE CHANGING ENVIRONMENT

Emerging mass education

The most characteristic feature of the Hungarian higher education was a dramatic increase in the number of students. Table 1 shows that between 1992 and 2003 the number of students more than tripled. At the same time the number of full time lecturers has just slightly increased after a significant fall in the middle 1990s. Comparing these two data it is apparent that the number of students per lecturer has increased from 7.3 to 21.9.⁵ It is also apparent that the budgetary support of higher education relative to GDP has not increased either, and it represents a quite low GDP proportion. This means that governmental support can be described with largely decreasing per capita value in real terms. One can summarize the last 15 years of the Hungarian higher education by *moving from a mostly elite-type education to a mass education*, which has far-reaching implications for the whole system.

Table 1: *Some characteristics of the Hungarian higher education, 1992-2003*

Year	Total number of students	Total number of full-time lecturers	Budgetary expenditures on higher education (per cent of GDP)
1992	117 460	16 157	1.12
1995	179 565	14 975	1.01
2000	295 040	15 649	1.15
2003	366 947	16 771	1.17

Source: CSO (2004)

Corresponding to market demand

Hungary has undergone fundamental structural transformation since the system-change which has also affected her labour market to a great extent. For example, traditionally we have had high proportion of students in engineering and in teacher's training. Owing to the emerging new economy, recently business and management studies are very popular: the market economy

⁴ The exact meaning of 'college' in the Hungarian higher education system will be given later in this paper.

⁵ It must be mentioned that part time lecturing has become more widespread that is many teachers have part-time jobs besides his full-time job.

needs professionals in the field of marketing, corporate finance, production management, human resource management, etc. Although the aim is obviously to *establish a course structure that will respond to the changing demands of the labour market*, very often it is mainly reflected only in the names of subjects. Management and business sciences is a quite new field of education in Hungary and sometimes the heritage of the past is more dominant in these programs than the orientation to *conveying practical knowledge which is required by the labour market*. The introduction of a two-cycle education system, which will be discussed in more details in the next paragraphs, may help to overcome some of these difficulties by clearly separating study programs qualifying to the labour market from those giving a more advanced and more theoretical education.

From one-cycle to two-cycle system

Traditionally Hungary had a binary higher education system provided by colleges (főiskola) and universities (egyetem), and both parts of this dual education system offered one-cycle programs. Traditional-style universities offered “long” four- to five-year degrees in arts and sciences, law, social sciences, economics and education. Colleges offered three- to four-year, professional-oriented programs in areas such as technology, business administration, health services and teacher training. Universities generally followed a one-tier system leading to an integrated master-level degree (Egyetemi Oklevél) that required a total of five years of study (six years for medicine). The system was one-cycle in the sense that colleges offered bachelor-level degrees (Főiskolai Oklevél) but hardly was it possible after graduation to continue on for a master’s at university nor was it a prerequisite to go to university and obtain a master-level degree. The conception behind this system was that one should decide whether she/he wants a bachelor or master-level degree at the beginning of their higher education and should enrol to the appropriate institution.

By joining the European Union, Hungary engaged herself that *the then existing dual education system would be gradually dissolved and a sequence of bachelor and master’s degrees built on each other would be created*. The shift to the two-cycle system took place in this academic year, so in September 2006 only the new, bachelor’s-level programs were launched at Hungarian institutions of higher education and the long, integrated programs were phased out. The shift, however, was not easy and painless, and over-many questions still remain open, for example about the curriculums. Hitherto both college and university curriculums were designed to make up a whole on their own, that is they knew what they could build on and there was no need to take into account what could be built on them. Evidently this autonomy no longer exists with the introduction of the two-cycle system, so *we should have redefined the place of each subject in the schedule*. This still ongoing process is very effortful mainly due to the institutional and personal inertia that can be experienced in higher education: no one likes changing what and how they teach.

2. TEACHING STATISTICS IN THE NEW ENVIRONMENT

Teaching the subject of Statistics had been always present in the modern, 20th century higher education in Hungary. Having a long tradition is definitely a valuable thing but *this honourable*

history also makes it more difficult to adjust to the new environment described shortly in the previous section.

When enumerating some of the troubles specifically related to teaching Statistics I will try to follow the same structure as before; thus I will categorize the arising problems in terms of whether it comes from mass education, from the changing demand of labour market or from the introduction of two-cycle higher education. Obviously these issues are not independent from each other, and this kind of grouping is arbitrary. My aim is only to give some framework of thinking, even for me myself.

Challenges to teachers of Statistics arising from the sudden shift to mass education

With moving from an elite-type higher education to mass education it is inevitable *that we face a descending average standard of students*. Ten or twenty years ago usually talented students from good secondary schools went to universities, so they had quite good basics which could be built on. Nowadays a much wider range of adolescents with very divergent backgrounds go on to higher education, so there is no firm common knowledge which can be taken for granted. It has far-reaching consequences on teaching Statistics. The most apparent is *the lack of ability to cope with formal mathematical arguments*, which would be essential to understand the theoretical side of the discipline.

Conventionally, the curricula of Statistics at universities and colleges used to lay special emphasis on the mathematical grounding. Students were not just provided with the appropriate formulas to use but they saw rigorous proofs and derivations resulting in those formulas. The typical student of now simply does not have the necessary preliminary training and, in my opinion, not even the intellectual capacity to accommodate such theoretical reasoning. The very important question of whether it is needful at all will be discussed later.

The multiplied number of students makes the teacher-student relationship much looser, too. In case of mass education practically there is no room for individual balancing, no room for handling personal problems. Uniform conditions and requirements are needed not only among students in the same study group but among lecturers of the same subject, as well. This means that *teachers' autonomy is away*. For me it causes the most inconvenience when compiling the tests. I think every teacher should have the right to weigh the parts of the curriculum to some extent, that is to emphasize stronger the methods he considers the most important and to talk a bit less about parts that does not seem to be of crucial importance. Obviously it is impossible if all the students have to take the same exam. In that case *I have to teach them what they will be asked in the centrally compiled test and not what I think they should know*.

Challenges to teachers of Statistics when trying to meet the needs of the labour market

In the past the main objective of the Hungarian higher education was not to give directly usable practical knowledge, but rather to provide the students with extensive general knowledge – or to be more elevated, a ‘view of the world’ – on their widely interpreted field which they could use as a basis and they could develop themselves building on that ground. With the rapid change in the structure of society and economy *this role cannot be sustained any longer*. *The labour market*

does not need ‘little scientists’ who can learn anything if they have enough time. It *longs for ‘professional specialists’* who are armed with all the practical knowledge of their specific narrow field, everything they need to start work (and make profit) immediately.

I would illustrate this point with the following example from the field of Statistics. A company does not want its new marketing assistant to understand how two sample t-test actually works. It wants him to know what it is good for and to be able to perform a t-test on a computer using an adequate program. It has the consequence, again, that *our emphasis should shift from theory towards practice*. It is more important for graduates to be able to run a regression, for instance, with MS-Excel than to be aware of the theoretical stuff with all those uncorrelated, standard normally distributed error terms.

In my view, *teaching Statistics in Hungary has gotten stuck somewhere in the middle of the way from the old concept to the new, market-oriented one*. Most institutions realized that due to the increasing number of students and falling standard they cannot expect as much theory as before, so they reduced the requirements. But, concurrently, almost nothing happened in order to rationalize the curricula and to adjust them to market demand; that is to increase the number of real-life examples and computer-aided seminars or to skip parts that are not really important from a practical point of view.

Challenges to teachers of Statistics arising from the new two-cycle higher education

Perhaps some of the dilemmas mentioned earlier will be solved by the introduction of the two-cycle higher education system. Many of the current problems arise from the fact that we are in trouble when defining the role of higher education: Is it mainly a ‘scientific workshop’ giving theoretical education no matter what practical skills the labour market requires (close to the old view) or is it a ‘conveyor belt’ producing good professionals with immediately usable knowledge but not really wide-ranging thinking. The system of the two-cycle higher education may reconcile these two somewhat opposite views. *The bachelor’s programs, as the stage of mass education, could provide the market with the labour force it needs and the master’s programs could place much more emphasis on the theoretical grounding*.

In my esteem, this division is also inevitable when talking about Statistics. To describe the present situation I would use the proverb ‘too much is as bad as none at all’. *At the moment we try to teach a lot*, from the basic concepts through hypothesis testing and multiple regressions to time series analysis, and all these rather deeply.⁶ Given the available time and the preliminary training of students it is just too much to cover. I am convinced that *some of the topics should be allocated into the master’s program*, and those remaining should be backed up with more real-life and practical examples.

Another important question is *the place of probability theoretical grounding of Statistics*. As I have already mentioned, traditionally Hungarian institutions of business and economic higher education gave a very serious mathematical grounding. They offered at least one semester of Calculus, one semester of Linear Algebra and Linear Programming and one semester of

⁶ The list is quite similar at universities too, the difference is that later they offer more advanced courses based on Basic Statistics such as Econometrics, Multivariable Statistics or Time Series Analysis.

Probability Theory.⁷ The main point is that Probability Theory was a standalone course, so the students were familiar with the basic mathematical concepts (for example random variable, density function, central limit theorem, etc.) used in Statistics. Probably in a short and practice-oriented bachelor's program Probability Theory cannot claim an own semester for itself, so *the introduction of basic concepts will devolve upon Statistics*. I guess it would also have some advantages. For instance, probability theoretical grounding could be more goal-oriented, namely we could avoid such topics which are irrelevant for our purposes.

3. SOME PROPOSALS WHICH MAY HELP TO OVERCOME THE DIFFICULTIES

In this section I will try to draw some conclusions and phrase some short suggestions concerning teaching Statistics in Hungary in the future. These directions of development are crucial if we want to fulfil the requirements placed on us by the mass education, the labour market and the transformation process of Hungarian higher education. In doing so, I will rely on two sources of thought. As being a teacher of Statistics myself at a college, *I have my own ideas about what should we do in a different way*, indeed. These ideas have emerged during the years of teaching and are based on direct experience. But *I was also interested in the opinion of the other side of the classroom, namely that of the students*. Therefore I carried out a 'little survey research' with 72 participating second-year college students studying Statistics II⁸ in the first semester of the 2006/2007 academic year. I asked them to fill out *a short questionnaire consisting of four open-ended questions*. Answering was fully voluntary and anonymous. The questions were the following:

- According to your opinion, will you make any use of statistical knowledge in the future? It is important that the question is not how bad or good the present education is, but that if it makes any sense, in general, to teach Statistics at the collage.
- Which parts of the curriculum do you think should be discussed more thoroughly, and on which parts should be placed less emphasis?
- What alterations would you carry out concerning the methods of education?
- Any other opinions, advices, experiences that can be useful for us to make teaching Statistics better.

Since the answers to these questions cannot be analyzed numerically, I will not present any descriptive statistics or figures from this survey. Instead, in the following reasoning I will refer to reactions and judgements that steadily emerged from the answers. It is important to note that all my statements and suggestions apply to the bachelor's level programs, i.e. the level on which I teach.

1. Minimizing the amount of mathematical proofs and derivations, concentrating on the methods: what they are good for, when they can be applied.

⁷ I must emphasize that these really represent a minimum; a lot of universities have more semesters of the listed courses, and also Operational Research in the first two years.

⁸ Statistics II is the second semester of Basic Statistics covering methods of statistical induction (inferential statistics).

It is very difficult for me, as well, to talk about e.g. linear regression without correctly deriving the general formulas by ordinary least squares, but I had to accept that most students are just not enough into probability theory and matrix algebra so that they could make any use of my eager manipulations at the blackboard. Students' answers confirmed my impression, as almost 70 per cent of them considered Statistics a useful subject, but said that they saw no sense in dealing with the theory and neither did they understand it.

2. Much more emphasis on computer-aided seminars. Using at least MS-Excel, but rather SPSS.⁹

I emphasized several times how strong is the pressure from the market that our students be familiar with the most important software packages, and how much we fall behind to meet this requirement. The students are also aware of this deficiency, since about half of them indicated explicitly that they wanted more software-aided demonstrations. This would contribute to improve their routine more than doing the calculations 'by hand'.

3. Real-life data and problems to persuade students that Statistics is useful and really can answer practical questions.

This point is closely related to the previous one. If we would like to motivate our students, we have to persuade them that Statistics is actually relevant to real-life, i.e. it can help in answering actual questions and solving true problems. To this end we have to show them real-size problems with real data and analyze them on a computer. The results of the questionnaire show how important this issue is: almost two-thirds of the students declared that they did not see the connection between the exercises at the seminars and the challenges they will face in their jobs. For me it means that we failed to give them statistical methods as 'tools' to use.

4. Confine ourselves to teaching less on the one hand, but much more carefully and in a practice-oriented way on the other hand.

It is necessary to reconsider the content of the curriculum, and to let some topics into the master's program in order to give narrower but more usable knowledge. Of course, there may be a lot of dispute around the exact breakdown of the topics. As for myself, I would consider leaving the following methods to the master's level: estimation from stratified sample, analysis of variance (ANOVA) and goodness-of-fit test. This would not be a significant loss, as usually hardly anyone understands the essence of these methods on an ordinary course at the college.

4. CONCLUSION

In this paper I summarized the most important changes in the environment of Hungarian higher education which have far-reaching consequences on the role of universities and colleges in the new market economy. I demonstrated how these challenges affect the way Statistics is taught in these institutions. Finally, based on my own experience and on a short research among second-year students at a Hungarian college I submitted some suggestions and directions of possible further development. One thing can be stated for sure: no matter how Hungarian higher

⁹ It is important to note that the students already have a course where they study such software but it is not integrated into Statistics. The point is that computer applications should be parallel to learning the methods.

education will change in the future, this evident mismatch between the current practice and the circumstances cannot be sustained for long.

REFERENCES

- Rappai, G. (2005): A Bologna-folyamat kihívásai a statisztika felsőfokú oktatása számára (Challenges of the Bologna Process for Teaching Statistics in the Higher Education), *Statisztikai Szemle*, 83(6), available only in Hungarian
- CSO (2004): Tertiary Education and R&D, *Hungarian Central Statistical Office*, 2003, (<http://portal.ksh.hu/pls/ksh/docs/hun/xftp/idoszaki/oktkut/oktkut03.pdf>)
- Temesi, J. (2004): Challenges and reforms in the Hungarian management education, paper presented at the *Budapest Polytechnic Jubilee Conference: Science in Engineering, Economics and Education*, September 4, 2004 (<http://www.bmf.hu/conferences/jubilee/Temesi.pdf>)
- Clark, N. (2004): Bologna Country Updates: Central Europe, *World Education News and Reviews*, 17(1), (<http://www.wes.org/eWENR/04jan/Hungary.htm>)
- Ahn, Thomas von (2005): Higher education in Hungary, *EAIE Forum*, 2005 winter, 50-51., (<http://www.eaie.org/pdf/F73art8.pdf>)

STATISTICS TEACHING IN AN AGRICULTURAL UNIVERSITY: A Motivation Problem

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Abstract: There are many teaching methods and there are various teaching materials even in one university, not to mention different universities specialising in different disciplines. So I cannot talk about Hungarian method in general, but about my experience in teaching statistics. I teach statistics on several levels (BSc, MSc, PhD) and in different faculties (Agriculture and environmental Sciences, Economic and Social Sciences) and in different forms. I find different problems according to the faculties and forms. In this paper I focus on only one of them, which is the most important for me.

Keywords: agricultural programs; teaching and learning statistics; reflective practice

1. TEACHING STUDENTS FROM THE AGRICULTURAL FACULTY

To increase the efficiency of my teaching method is the most important task for me. Knowledge of statistics would be essential for the students especially when they are working on their final theses and they need to evaluate their data. Problems involving data need statistics to solve them. Nowadays the scientific paper needs some kind of statistical analysis, which helps to verify the subjective statements in an objective way. But students in third semester do not understand why they have to learn statistics and how it could help them later, so they do not spend enough time on it.

The statistics is taught only in one semester with a 1 hour lecture and 2 hours of seminars per week. It is a very short time to learn and practice the statistics methods. It is only enough to give a very short summary of what the subject is about, and to remember later that there are methods to solve problems. I have met several students from the fifth year in my class who sit in because they realise that they need statistical analysis for their final theses.

Summarizing the main problem, the subject is taught when the student cannot catch the sense of it and the student is not interested in it. What can I do?

2. BACKGROUND

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In 1996 I was lucky to spend a full semester in Beloit College at the Department of Biology taking part in the Genetics course of Professor Jungck. In that time I taught plant breeding, genetics and biometrics. I was there to learn more about BioQuest Software Library and new teaching methods connected with it. During this semester I gained a closer view in the teaching methods and the way of motivating students. It was quite different from the method we used in our university.

I entered the classroom. There were several little groups speaking inquisitively about something. Every group stood near a funny poster from which a slouching tortoise or a laughing clock or another strange image looked at me. Where am I? And what are these young people doing around me? And what do these many amusing and colourful posters mean? These questions intrigued me. I went nearer to one group to find out what was going on. And then I realized that the playful posters contained very serious topics. They spoke about one of the most interesting areas of genetics, the inheritance of quantitative traits. Every poster presented a different special question and its answer too. In front of posters, students discussed different topics. They spoke easily about the estimation of the number of genes, components of variability, and heritability. They spoke about it just like about what was on TV in the last evening. It was fantastic: not only how much they knew but also how sure of themselves they were in dealing with the topic. I enjoyed the contradiction of the playful form and the serious scientific content.

The first lecture was the second big surprise. Having seen the posters before, I supposed I would hear a very hard, serious lecture. Instead of this I heard a kind, friendly conversation. The students asked a lot of questions. They were inquisitive and they had purposeful questions. (I am accustomed to our students who hardly ask any questions and who are usually quite inactive during the lesson.)

I had to think about it. What is the reason for this difference between the students of Beloit and in our university? Our students are intelligent and they can work hard, too. But there is a big difference between the behaviour and attitudes. I had to recognize the difference is not between the students but much more between the two education systems. We usually do not have enough time. When we meet with the students we want to tell them everything. We want to give more concrete knowledge to them, but we don't give them time to think, to think about different problems, to pose questions and, what is most important, we don't really give them an opportunity to solve a problem alone. So they don't have a daily experience of being successful. They have no time to "live together" with the subject and they have no time to become fond of it.

Arriving home I changed my teaching style. I did not want to teach everything but let them freely choose a topic and speak about it. In that way I successfully made them active during class and increased their interest in my classes. In that time I taught facultative subjects, such as biometrics and population genetics, that students chose because they were interested in those subjects and wanted to learn them. Teaching statistics is a completely different situation. For 3 years I have been responsible for teaching statistics to students of the agriculture faculty. It means about 120-140 students in a semester. I give lectures to all students and I lead seminars to one group. Leading seminars is the best way to keep contact with students and discover what they understood from the lecture and learn what is their opinion.

The first two years I was quite disappointed because the results of the final exam were worse than I expected. What was the reason for it? Maybe I expected too much from students? Did I ask wrong questions? Did I give the wrong lecture? Did they not learn enough? After asking them I got the following answer: “We do not understand the material and we do not need it. We only want to pass, not more.”

3. ADAPT THE A NEW STYLE

I remembered my experience of Beloit and decided to try to adopt it in my statistics teaching as well. My aims were 1, to make the students active and to force them to practise the methods learned; 2, students should obtain experience in using statistics in solving a problem; 3, students should recognize that the statistics could be useful to demonstrate and solve different problems; 4, do not be afraid of statistics, get closer to the subject, try to enjoy a little.

It was not difficult to motivate them. I offered that year that one of their written tests (from two) they can replace with a “presentation”. There was not a big risk. Knowing the results of statistics exams of the last few years, both the students and I thought that we could take advantage from it. The possibility of taking a presentation was chosen freely and it took place in the spare time of students. Approximately 75% of students made the best of the opportunity.

The conditions were:

- 1, two or three students can work together on one topic
- 2, the topic can be chosen freely (crazy, funny, special topics were preferred)
- 3, students have to declare the aim of presentation (analysis), source of data, type of analysis used, have to show the results (table or figure), have to give a conclusion connected to the aim
- 4, the results can be shown by poster or slide or computer and so on
- 5, time of presentation 5-10 minutes

4. EXPERIENCE OF PRESENTATION

Generally it was successful. Most of the students worked well. They found suitable problems for analyses, they chose and applied the statistical method in a correct way and drew the right conclusion. And all this they did with pleasure. They were proud of their results and I was proud as well as I reached my aims.

Furthermore it was very interesting to learn more about the students through the topics they chose. Some of the presentations were based on fictive data but there were some based on real data from experiments or a web site connected to agriculture science.

Students certainly learned more during the preparation of the presentation than they learned for an exam. And they listened to each other and discussed the results. What was good, and what has to be done better or in another way? So they saw more examples of how we could use statistics to solve different problems.

5. NEGATIVE EXPERIENCE

What I cannot solve yet is how to give grades to the students. Listening to all presentation I corrected the biggest mistakes and I could evaluate the presentation "itself". Sometimes I felt that only one student worked in the group and the other members got the results without doing anything.

6. CONCLUSION

In summary I can say that giving more freedom to students to show what they know, we can get better results. I think this experiment was successful because many of the students who closed their eyes and said "I am stupid for statistics - I don't want deal with it" now got results from their own work and got closer to the subject.

REFERENCE

Toth, K.L. (1997): Can Less Be More? BioQUESTnotes 7(2) March issue.

CALCULATING DEPENDENT PROBABILITIES

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Key Words: conditional probability, independent events, bookmaker's odds.

In the 2004 European soccer competition France were one of the favourites to win the World Cup and Thierry Henry, their star forward, was one of the favourites to be top goal scorer. Bookkeepers were offering odds of 4 : 1 on France winning the competition and odds of 8: 1 on Thierry Henry being the top scorer. A large number of punters went into betting shops in the United Kingdom and made a single bet that France would win the competition **and** that Thierry Henry would be top scorer. The counter clerks in the betting shops accepted the bets and punters making the bets believed that a £1 stake would bring a return of £42. (*A £1 stake on France winning the competition at odds of 4 : 1 gives £5 (=£4 plus return of the £1 stake). The £5 then being bet on Thierry Henry being top scorer at odds of 8 : 1 gives £45(= £40 plus £5.) In general if a bet is made on two outcomes and the odds of each outcome are $m : 1$ and $n : 1$ then the return on a £1 stake is $£(m + 1)(n + 1)$.*)

This calculation is only valid, however, if the two events are independent. In this case the events are clearly not independent since if France do win the competition they will have played more games and are likely to have scored more goals. Since Thierry Henry is their most likely goal scorer it follows that he is more likely to be the top goal scorer overall. The example below shows how the probabilities should be worked out.

In November 2004 England played Spain in a friendly soccer match. The tables below show some of the odds being offered by the bookmakers William Hill.

Spain to win 2 – 0	16:1
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First player to score	
Raul (Spain)	2:1
W Rooney(England)	3:1
Morientes (Spain)	7:2
M Owen	9:2
Another Spanish player	6:1
Another English player	7:1

Spain to win 2 – 0 and Raul to score first	25:1
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The bookmaker has acknowledged that Spain winning 2 – 0 and Raul scoring first are not independent events since a £1 bet would receive only £26 and not £(16 + 1)(2 + 1) = £51! Clearly the events are not independent since if Spain win 2 – 0 a Spanish player must have scored first!

Are the odds of 25 :1 consistent with the other odds offered?

Consider the odds offered against the player to score first. We first change these odds to probabilities(see ‘Odds that don’t add up’ Teaching Mathematics and its Applications 1994)

First player to score			
	Bookmaker’s odds	‘Adjusted’ probabilities	True Probabilities
Raul	2:1	0.3333	
W Rooney	3:1	0.2500	
Morientes	7:2	0.2222	
M Owen	9:2	0.1818	
Another Spanish	6:1	0.1429	
Another English	7:1	0.1250	
		Total 1.2552	1.000

(Note that in this context the ‘True Probabilities’ merely reflect the amount of money staked by the punters on each player. They do not measure the real probability a player will score first – if indeed such a probability exists!)

If Spain win 2 – 0 an English player could not have scored first. The conditional probabilities of each of the Spanish players scoring first are shown below

First Spanish player to score	Conditional probabilities given that a Spanish player scores first
	Total = 1.000

In order to make the same percentage profit as before the bookmaker adjusts these conditional probabilities by multiplying by 1.2552. The table below shows the adjusted probabilities and the associated odds.

First Spanish player to score	Probabilities	Adjusted probabilities	Bookmaker’s Odds
		Total = 1.213	

(for explanation of changing probabilities to odds see ‘Odds that don’t add up’ Teaching Statistics 1994)

The bookmaker should therefore, to be consistent, be offering the following odds.

Luis Figo to score first given that Portugal won 2 - 0	3.202:1
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A punter who bets £1 should, therefore, receive $£(20 + 1)(3.202 + 1) = £88.24$ if Portugal win 2 – 0 and Luis Figo scores first. In practice he or she would receive only £41.

Readers are invited to submit their answer to the following.

On the same match the bookmaker also quoted the following odds.

England to win 3 – 1	16:1
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England to win 3 – 1 and Emile Heskey to score the first goal	66:1
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To be consistent what odds should be offered on England winning 3 – 1 and Emile Heskey scoring the first goal?

(Assume that the odds of 16:1 against England winning 3 – 1 and 9:2 against Heskey being first player to score a goal are sensible odds. i.e. they reflect the amount of money bet by punters.)

Fletcher

FOR THE REST OF YOUR LIFE

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Keywords: game show mathematics; pay-offs

'For the Rest of Your Life' is a new TV game show. Contestants play to win money every month. This can be for as little as one month or, if every one of their guesses is correct, for the rest of their lives. The rules are shown in table 1.

First half of the programme

Contestants are faced with 11 tubes. Eight of these tubes have a white light inside and three have a red light. The contestant chooses a tube at random. Picking a white light increases their prize by £150, picking a red lowers it by the same amount. Once they are four steps up the money ladder they can stick with the prize they have. i.e. once they have £600 they can stop.

As an example, consider if the tubes chosen were white, white, white, red, white, red white, white. In this case the contestant's possible prizes would have been £150, £300, £450, £300, £450, £300, £450, £600. At this point the contestant is allowed to stop guessing and take the £600 since this is the fourth step up the ladder. In this case it may well be worthwhile to do so because once all three reds have been picked the contestant wins nothing.

Second half of the programme

Contestants are faced with 15 tubes. Eleven of these tubes have a white light inside and four have a red light. The lights now count for months for which the money won in the first half of the programme is paid. The possibilities are 1 month, 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 10 years, 15 years, 25 years and 'the rest of your life'. (The 'rest of your life' is taken as 40 years.)

If a contestant drew all 11 white lights without drawing a red light then he/she would win an amount of money (won in the first half of the programme) every month 'for the rest of their life'.

Table 1

The first problem that will be analysed is: **In the first part of the programme, if the contestant stops as soon as he/she has £600, how likely is it that he/she will win £600?**

The scenarios that will win £600 are the tubes being drawn in the following orders.

1) WWWW Probability = $\frac{8}{11} \times \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{7}{33}$

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- 2) WWWRWW Probability = $4 \times (8/11 \times 7/10 \times 6/9 \times 3/8 \times 5/7 \times 4/6) = 8/33$
3) WWRWWW
4) WRWWWW
5) RWWWWW

- 6) WWWRRWWW Probability = $14 \times (8/11 \times 7/10 \times 6/9 \times 3/8 \times 2/7 \times 5/6 \times 4/5 \times 3/4)$
7) WWWRWRWW = $14/55$
8) WWRWWRWW
9) WWRWRWWW
10) WWRRWWWW
11) WRWWWRWW
12) WRWWRWWW
13) WRWRWWWW
14) WRRWWWWW
15) RRWWWWWW
16) RWRWWWWW
17) RWWRWWWW
18) RWWWRWWW
19) RWWWRWWW

Probability £600 is won = $7/33 + 8/33 + 14/55 = 117/165$

The contestant has a good chance of winning £600 - approximately $\frac{3}{4}$
Suppose, however, he/she decides to try for £750.

The second problem that will be analysed is: **In the first part of the programme, if the contestant stops as soon as he/she has £750, how likely is it that £750 will be won and is it worth trying for £750?**

The scenarios that will win £750 are the tubes being drawn in the following orders.

- 1) WWWWW Probability = $8/11 \times 7/10 \times 6/9 \times 5/8 \times 4/7 = 4/33$
2) WWWWRWW Probability = $5 \times (8/11 \times 7/10 \times 6/9 \times 5/8 \times 3/7 \times 4/6 \times 3/5)$
3) WWWRWWW = $2/11$
4) WWRWWWW
5) WRWWWWW
6) RWWWWWW
7) WWWRRWWW Probability = $20 \times (8/11 \times 7/10 \times 6/9 \times 5/8 \times 3/7 \times 2/6 \times 4/5 \times \frac{3}{4} \times 2/3)$
8) WWWRWRWW = $8/33$
9) WWRRWWWW
10) WWRWRWWW
11) WWWRWRWW
12) WRRWWWWW
13) WWRWRWWW
14) WWRWRWWW

- 15) WWRWWWRWW
- 16) WRRWWWWW
- 17) WRWRWWWWW
- 18) WRWWRWWWWW
- 19) WRWWRWWWW
- 20) WRWWRWWWW
- 21) RRWWWWW
- 22) RWRWWWWW
- 23) RWRWWWWW
- 24) RWRWWWWW
- 25) RWRWWWWW
- 26) RWRWWWWW

Probability of winning £750 = $\frac{4}{33} + \frac{2}{11} + \frac{8}{33} = \frac{6}{11}$

Interestingly, the contestant has a reasonable chance of winning £750

If we look at the expected winnings, however, we see that the contestant is better off trying for £600

Expected winnings, given that he/she is trying for £600, = $\text{£}600 \times \frac{117}{165} = \text{£}425$

Expected winnings, given that he/she is trying for £750, = $\text{£}750 \times \frac{6}{11} = \text{£}409$

Let us suppose that the contestant takes £600 into the second part of the programme.

What strategy should the contestant use to maximise their expected winnings?

Suppose the contestant tried to win £600 a month for the rest of his/her life. The probability of pulling out 11 white lights in succession is

$\frac{11}{15} \times \frac{10}{14} \times \frac{9}{13} \times \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} = \frac{11!}{15!} = \frac{1}{1365}$

clearly it is not in his/her interest to try and win the money for the rest of his/her life!

In fact most contestants adopt a strategy of playing until only one red light remains.

This strategy will be analysed.

What are the expected winnings of a contestant who plays until three red lights have been revealed?

(In fact it may be the case that the contestant who adopts this strategy never actually sees three red lights – all the white lights may show before three red lights are revealed.)

The outcomes, their associated probabilities and winnings are shown in table 2.

Outcome	Probability (=p)	Winnings (=W)	pW
11 white	1/1365	£600x480	£600x480/1365
11 white, 1 red	11/1365	£600x300	£600x300x11/1365
11 white, 2 red	66/1365	£600x180	£600x180x66/1365
11 white, 3 red	0	0	0
10 white, 3 red	132/1365	£600x60	£600x60x132/1365
9 white, 3 red	165/1365	£600x36	£600x36x165/1365
8 white, 3 red	180/1365	£600x24	£600x24x180/1365
7 white, 3 red	180/1365	£600x12	£600x12x180/1365
6 white, 3 red	168/1365	£600x6	£600x6x168/1365

5 white, 3 red	147/ 1365	£600x3	£600x3x147/1365
4 white, 3 red	120/1365	£600x1	£600x1x120/1365
3 white, 3 red	90/1365	0	0
2 white, 3 red	60/1365	0	0
1 white, 3 red	33/1365	0	0
3 red	12/1365	0	0

Table 2

(Note that it is not possible to pull 11 white lights and 3 red lights using this strategy.

Since the last light pulled is red the 11th white light will have been pulled previously.

The contestant stops pulling once the 11th white light has been pulled and hence it is not possible to pull 11 white lights and 3 red lights.)

(To see how these probabilities are calculated consider the probability of 5 white and 3 red

$$= \underbrace{(11/15 \times 10/14 \times 9/13 \times 8/12 \times 7/11)}_{5 \text{ white}} \times \underbrace{(4/10 \times 3/9)}_{2 \text{ red}} \times \underbrace{(2/8)}_{\text{last one red}} \times \underbrace{(7!/(5!2!))}_{\text{number of combinations of 5 white and 2 red}}$$

$$= \frac{11! 4! 7! 7!}{6! 15! 5! 2!}$$

(Note that the last tube picked has to be red because the contestant stops pulling once 3 red lights show)

In general the probability of X white and 3 red is $\frac{11! 4! (15 - (X+3))! (X + 2)! / (X! 2!)}{15! (11 - X)!}$

where X < 11)

The expected winnings are $\Sigma pW = \text{£}16513$.

Not bad winnings for pulling lights out of a tube at random!!

LEARNING, PARTICIPATION AND LOCAL SCHOOL MATHEMATICS PRACTICE*

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Abstract: This paper reports on a study whose aim was to examine students' learning in terms of participation in collective mathematical discussions. Our basic theoretical assumptions are based on a combination of situated learning perspectives and a framework that links social and psychological approaches of mathematical activity and learning. The research was carried out in a Year-8 classroom (students were aged 12 to 13), and the mathematical subject under investigation was area measurement. Data are presented to illustrate possible correspondences between 'signs' of learning and 'local' changes of participation. We conclude by discussing some pedagogical implications resulting from the study.

Keywords: classroom discourse; communities of practice; local practices; mathematical learning; psychology of learning; sociological approaches;

Introduction

In the context of mathematics education participation during the classroom interactions has been examined by distinct approaches and foci. These have strongly indicated that factors like the affective domain, the other participants (especially their power relation to the person), the means of communication (especially language), the artefacts involved and the physical surroundings influence the process of participation. For example, Tatsis and Rowland (2006) argue that the participants are engaged in an interpretive process during their interactions; while they may wish to fulfil the purpose of the interaction (e.g. to solve a problem), at the same time they are interested in maintaining their face. Back and Pratt (2007) examine a student's participation in an online discussion board; their work demonstrates the significance of the medium of communication (in their case written speech) in participation and identity formation. McVittie (2004) has used discourse analysis, particularly Wegerif and Mercer's (1997) categorisation, to describe the regularities found in students' talk. According to this scheme, students use three different kinds of

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* This paper is a development of two papers presented at PME-31, Seoul, Korea (8-13 July 2007) and 2nd ScTIG Conference, Manchester, UK (10-11 September 2007).

** Supported by FAPEMIG and CNPq.

talk – depending on the task and the discourse community they are involved – disputational, cumulative and exploratory talk. These different kinds of talk are used to signify different participation patterns or, in other words, different selves. Williams and Clarke (2003) focus on patterns of participation in dyadic and triple collaborative interaction aiming at solving mathematical activities. The authors suggest that in both interactions the communication among the students is characterised not only by the sharing of the meanings of the mathematical terms they exchange during the interaction, but also by some pre-existing (but not always stated) established modes of particular participation of each dyad/triple. Jaworski (2007) extends the concept of participation into engagement, which denotes active participation and mental inclusion; she uses this concept – together with the concept of community of inquiry – to examine the ways in which teachers engage in their school community, align with its practice and exercise imagination to achieve their own professional goals. Frade, Winbourne and Braga (2006) examine students' participation in terms of crossing boundaries between different disciplinary school practices. From interdisciplinary work carried out by secondary mathematics and science teachers, the authors conclude that it was mainly the activity of these teachers that enabled the crossing of the boundaries between their disciplines: they have translated for each other their specific discipline codes, worked together to prepare and organise their collaborative work and shared their goals and purposes with the students.

Cobb, Stephan, McClain and Gravemeijer (2001) offer a framework that links social and psychological approaches of mathematical activity and learning. In doing so, they attempt to see participation as a coordination between the establishment of common mathematical practices (social perspective) and the individuals' reorganisation of mathematical reasoning during the evolution of these practices (psychological perspective). This attempt to address to any coordination between the social and the individual in studies of participation is shared, in some way, with other researchers. Indeed, from a situated perspective of learning, Wenger (2007) says that as long as we enter, engage with and leave communities of practice, learning – in these communities – is a social journey as well as a cognitive process. In developing a *discursive participationist* view of learning, Sfard (2006) emphasises the interrelationship between what she calls 'collective and individual editions': developmental transformations are the result of two complementary processes, that of *individualization of the collective* and that of *collectivization of the individual*. For her, these two processes are in a constant dialectical flux between both individual and collective forms of doing.

Lave and Wenger (1991) describe participation by following a movement from the 'peripheral' to the 'central', i.e. the process of becoming a member of a particular community of practice. This movement, adapted accordingly for the classroom context, can be used to analyse the students' participation and identity formation during an extended period of time. However, teachers also need to evaluate students in smaller periods of time, e.g. during a single lesson or even an activity. Motivated by this and based on Lave's (1993) discussion of practice, Winbourne and Watson (1998) have introduced the notion of *local community of practice* (LCoP) for everyday school mathematics. Grounded on a combination of this situated theoretical construct and that of Cobb et al. above-mentioned, our study aims at providing a possible way of analysing students' learning in terms of participation (something we could also discuss in terms of identity formation, though that is beyond the scope of this paper) during a specific classroom activity: a collective mathematical discussion. In doing so we will also look for the evolution of sociomathematical norms, as Yackel and Cobb (1996) put it and, particularly, how these norms influence participation and the

establishment of a LCoP. Our basic theoretical assumptions will be presented firstly, followed by our methodology and analysis of the data.

Theoretical Background

Situated learning perspectives are fundamentally characterised by two main epistemological premises: i) learning means changing participation and formation of identities within communities of practice (e.g. Lave 1988; Lave and Wenger, 1991); ii) cognition is seen as a process situated in practices, and so always changing or transforming individuals – including teachers and students, activities and practices (e.g. Frade, Winbourne, & Braga, 2006; Lerman, 2001;). These very features are thoughtfully expressed by Lave and Lerman as follows:

(Quotation 1 here)

(Quotation 2 here)

Concerning school practices these mean a shifting of teacher's focus on individual differences, and an abandonment of comparative notions, for instance, 'good' or 'bad', 'more' or 'less' learning, among students groups. This is challenging or, at least, unusual, for it demands other way of thinking from the teacher's part. Learning now should be seen as occurring socially; collectively in activities which the students develop in specific, situated practices. Student and learning environment are closely connected, and the student's performance is strictly linked to his/her participation and identity formation in learning practices.

The most expressive elaboration of the concept of participation is offered by Wenger (1998). For him, participation is a process related to social experiences "in terms of membership in social communities and active involvement in social enterprises" (p. 55). In elaborating this concept Wenger (1998) differentiates participation from mere engagement as the former has the potential of mutual recognition. In doing so, he explicitly takes into account people, interaction, community, identity, and so on. Participation includes talking, doing, feeling and belonging; it is treated as learning in terms of distinctions between kinds of enterprises rather than distinctions in qualities of human experience and knowledge.

As indicated in the introduction, the movement from the 'peripheral' to the 'central' to describe the process of becoming a member of a particular community of practice (Lave & Wenger, 1991) can be used to examine the students' participation during an extended period of time. This can be done by adapting the concept of 'legitimate peripheral participation' (Lave & Wenger, 1991) to school mathematics practice as proposed by the first author of this paper in her doctoral thesis; it can be thought of as the ways the students fit their experiences in order to engage in such a practice, or their intention to preserve a collective fruitful environment for learning. According to Frade (2003) it does not make sense to say that peripherality has to do with a necessary distance from full participation aiming at the mastery of a profession. Peripherality in classroom practices is a mode of participation, which is associated with the students' commitment (more or less intensive) to their learning. In other words, the movement from the 'peripheral' to the 'central' in school mathematics context should be associated with motivation and predisposition for learning. Interpreting this movement as such, the aspects 'non-voluntarism' and 'not aiming at being a mathematics teacher or a mathematician' that make difficult the direct translation of Lave's and Wenger's ideas to classrooms cannot be obstacles to regard a classroom as a particular community of practice. In fact, these aspects, says Frade, do not necessarily imply that students will construct

an identity of non-participants in school mathematics practice, or yet, that they will not wish to develop possible trajectories or to invest in themselves.

Having said this, let us return to our exploration of students' learning in terms of participation, considering smaller periods of time, e.g. a single lesson or a specific activity. Based on Lave's (1993) discussion about local practices, Winbourne and Watson (1998) have introduced the notion of *local community of practice* (LCoP) for everyday school mathematics:

(Quotation 3 here)

This notion of LCoP it is clearly compatible with both Wenger's concept of participation and Frade's (2005) adaptation of the movement from the 'peripheral' to the 'central' to classroom contexts. Also, it is very important for our research purpose, for it provides a situated background for us to talk about learning and 'local' changes of participation (as well as on identity, as we said before) during a single lesson, in particular where the students were involved in a whole-class discussion.

From this notion, Winbourne and Watson have proposed a helpful tool consisting of six characteristics, which allow us to identify whether a local community of practice is constituted in classroom:

(Quotation 4 here)

Taking a closer look to these characteristics, one could see that they refer to two aspects of the activities; on the one hand we have the social aspect (2, 4 and 5) and, on the other hand, the personal aspect (1, 3 and 6), though the one cannot be thought without the presence of the other. Or yet, these characteristics allude to something that we could call 'collective cognition' to emphasise the dynamic character and the indissolubility between the social and individual facets of an interaction that emerges from a certain practice. This has led us to look for a theory that could somehow combine these aspects – or explain a collective cognition event – in order to provide a full account of the classroom practices. Cobb et al.'s (2001) framework offers this possibility¹:

(Quotation 5 here)

The idea of participation above can be viewed as a 'local mode' of talking, doing, feeling and belonging when restricted to a classroom microculture, or else, to a LCoP as Winbourne and Watson put it. Another idea that proved useful in our analysis was that of the norm, particularly the social and the sociomathematical norm. The social norms refer to regularities in classroom activity that are jointly established by the teacher and the students (Cobb et al., 2001), while sociomathematical norms are "normative aspects of mathematics discussions specific to students' mathematical activity" (Yackel & Cobb, 1996, p. 461). These norms regulate the classroom's practice and play a part in shaping the participants' acts. Based on these ideas we analysed the interactions that took place in a whole-class discussion, taking participation in such a practice as involving sharing of purpose, ways of interpreting and arguing, and forms of mathematical reasoning and argumentation, in relation to the suggested classroom norms. At the same time we have looked to this whole-class discussion as a LCoP.

The basic elements of the context of our study, together with our methodology are presented in the following section.

The study

Research context and data collection

The study was carried out in a Brazilian urban secondary school and was part of a wider research project whose general aim was to investigate the development of area measurement knowledge of 28 students (11 girls and 17 boys) of a Year 8 mathematics class (ages approximately 12 to 13). This class was not a multiethnic class, though we can say that it was characterized by a cultural diversity concerning the children's socio-economical position. The context of the study refers to two sequential lessons (50 minutes each) dedicated to the collective correction of a diagnostic questionnaire the students had answered in order for the teacher-researcher – the first author of this paper – to start working on the subject. She was an experienced teacher who had been teaching in this school for eighteen years. She had also been teaching in this class since Year 7 when her students were introduced to area measurement by prioritising their daily-life and school previous knowledge, the basic concept and some informal procedures for calculating area. During the research she worked professionally both as a regular teacher fully involved in the routine of the classroom work, and as a researcher, having that classroom as her research setting. At this time two undergraduate students who had done their teaching practice in her class for the period of one month prior to the beginning of the research were asked to help the teacher-researcher in the data collection. They promptly agreed and stayed in the class during all data collection under the teacher-researcher supervision. Before starting the data collection the teacher-researcher gave them instructions on how to collect data and to help the students. If requested by them the two undergraduate students could offer explanations about the activities proposed in the questionnaire, working with the students in a similar way they both had done previously during their teaching practice.

The day in which the diagnostic questionnaire was applied the students sat individually. The day after the students had answered the questionnaire the teacher-researcher allowed them to sit in small groups to discuss it collectively. On this day the teacher-researcher walked around the classroom all the time, picking out students who were contributing to the discussion. Sometimes the teacher-researcher addressed to some students asking them to respond to a specific question; other times she addressed to the whole class asking volunteers to talk about their answer. Such practices had been typical of the culture of this classroom from the year before. With the help of the two undergraduate students data were collected by video, audio and students' questionnaires.

The questionnaire consisted of seven questions. For our research purposes we opted to mention just one question from it. This question presented ten alternative situations to the students and asked them to mark with an 'x' those which contained the concept of area measurement. These situations were:

- a) to calculate the quantity of paint needed to paint a wall
- b) to compare the quantity of water of two reservoirs
- c) to decide about the size of a carpet to be put in a living room
- d) to measure the distance from your house to the nearest bakery
- e) to decide in which of two wardrobes you can put more clothes
- f) to calculate the quantity of wood needed to cover the floor of a house
- g) to measure the height of a building

- h) to decide which umbrella protects more from rain
- i) to calculate the quantity of wire needed to surround a terrain
- j) to decide which of two gardens is the biggest.

Due to the rich collective discussion that occurred during the correction of this question we restrict our analysis to the first situation: to calculate the quantity of paint needed to paint a wall. It was expected that all students had marked this alternative with an 'x'. The particular or 'local' discussion lasted seven minutes and was chosen because it explicitly demonstrates some of the students' doubts concerning volume and area measurement and how the discussion has evolved to reach a common understanding.

Methodology

For our analysis we adapted Cobb et al.'s (2001) methodology to fit our research purposes; we have tried to locate the forms of participation that were legitimate in this discussion. This led us to develop conjectures "both about the ways of reasoning and communicating that might be normative at a particular point in time and about the nature of selected individual students' mathematical reasoning" (p. 128). By focusing on students' utterances we were able to view them as constituting the classroom practices and at the same time their own forms of participation. These two aspects of the same activity were then tested against Winbourne and Watson's characteristics to see if a local community of practice has been established. In relation to these characteristics (C1, C2, ..., C6), the criteria we have used to produced evidence of them are shown in table 1. We note that we have rephrased C3 and C6 into one characteristic, namely C3,6; for the latter looks like to us the same as the former plus the teacher's participation. The reliability of the criteria used was achieved by the two authors' agreement.

Characteristics	Criteria
C1. pupils see themselves as functioning mathematically and, for these pupils, it makes sense for them to see their 'being mathematical' as an essential part of who they are within the lesson	Self-reflexive statements related to mathematical processes
C2. through the activities and roles assumed there is public recognition of developing competence within the lesson	Utterances of appraisal or expressions of satisfaction towards one's own work
C3,6. participants see themselves as engaged in the same activity, working purposefully together towards the achievement of a common understanding.	Procedural utterances (used to assist the regular flow of the discussion) or prompts for actions needed for common understanding (as it is perceived by the speaker)
C4. there are shared ways of behaving, language, habits, values, and tool-use	Regularities in the discussion and common assumptions revealed in the discussion; the social and sociomathematical norms are the core theoretical constructs related to this characteristic
C5. the lesson is essentially constituted by the	Requires an holistic approach to the activity and

active participation of the students	can be identified by observing the significance of each student's contribution and the number of students that participated
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Table 1. Local Community of Practice characteristics and identification criteria

Having done this, we decided on how to refer to these characteristics and the respective criteria in our analysis in such a way that they would not blur our main analytical purpose: to build possible correspondences between ‘signs’ of learning and ‘local’ changes of participation. So, utterances in the discussion were coded when they indicated to us evidence of the above characteristics; this coding was used additionally to support or to withdraw our claims for the existence of a local community of practice. We note that this evidence was produced by analysing the discussions related to the whole questionnaire, during the two sequential lessons.

Analysis

The discussion that was transcribed took place between the teacher and some students, and among students. The sign (...) indicates that some utterances or parts of an utterance were omitted. Our notes are in brackets. Next to each turn there is a code that refers to the characteristics mentioned before; this is done for all codes except C4 and C5 which can be identified in a sequence of verbal exchanges and not in single utterances as we show in the analysis that follows. The duration of the discussion was seven minutes and all students’ names are pseudonyms.

- 01 Teacher: Children, pay attention now (...) we are going to discuss [an exercise] and if needed you should make it over again, okay? (...) Among the alternatives below, mark with an ‘x’ the situations in which the mathematical concept of area measurement is involved (...) the first alternative is ‘to calculate the quantity of paint needed to paint a wall’. Hands up those who marked this (...). [Most students raise their hands. After observing the raised hands the teacher invites the students who did not mark this alternative to speak]
- 02 Marcelle: I don’t know.
- 03 Paula: Why don’t you know?! [C3,6], [C4]
- 04 Teacher: Don’t you? What did you think here? [addressing Marcelle] (...)
- 05 Marcelle: I thought it was...oh, I thought it was to measure the paint litres.
- 06 Teacher: Then you thought that the litre has nothing to do with area measurement.
- 07 Stephanie: I also thought this... The same thing she [Marcelle] thought.
- 08 Teacher: Felipe, why didn’t you mark this? Do you remember?
- 09 Felipe: I forgot, Miss; I don’t know why I didn’t mark it.
- 10 Teacher: You don’t know why. Didn’t anybody else mark? [some students raise their hands] Amanda, why?
- 11 Amanda: Likely, liquid has nothing to do with area. How do I measure...
- 12 Teacher: Okay, you thought that litre, the quantity corresponding to a litre has nothing to do with area measurement. Now let’s see who has marked this alternative (...)

- 13 Paula: Let's suppose that I am going to buy, to paint my house. Let's suppose that I am going to buy ten cans of paint, but I don't know how many square metres my house is. How would I know how many litres of paint I would need? (...) [C1]
- 14 Calvin: I did.
- 15 Teacher: And does your justification coincide with [that of Paula]?
- 16 Calvin: The same thing. (...)
- 17 Lucas: Oh, I have marked it like this. Let's suppose that he needs paint to paint, to paint five square metres like this. So he has to measure how many square metres there are in order to know what to paint. (...)
- 18 Livia: See, I haven't thought in this way. You gave me the, the example of the room. But I haven't thought about the room. I have thought, for example, about a football field. [The teacher hears the student carefully] I would know the length, I mean, the width if it had been a square. But it is not a square, it is a rectangle. Then I have to know the length and the width of it. I would know the area of the field for me to cover it with grass. [C1]
- 19 Teacher: Hum, but I am thinking about the first exercise, to calculate the quantity of paint.
- 20 Livia: Then, this classroom is an example of this. I have to know the area. For example, I am going to paint just one wall in one colour. I have to know the area of the wall for me to paint it; for me to see how much paint I am going to buy. [C1]
- 21 Teacher: Okay. (...) Felipe [who had already said to the teacher that he didn't know why he didn't mark it] (...) what do you think now, after? [this discussion]
- 22 Felipe: I think I should have marked it.
- 23 Teacher: Sorry?
- 24 Felipe: I should have marked it.
- 25 Teacher: Why?
- 26 Felipe: Because knowing the size of the, knowing how much paint I am going to need to paint the wall I don't need to buy a lot of paint. [C1] [C4]
- 27 Teacher: So, it was enough for you to know what about the wall?
- 28 Felipe: The square metres.
- 29 Teacher: The size of the wall. Okay, that's fine. Did you mark, Mateus? [C2]
- 30 Mateus: I did (inaudible) like this, you, I am going to paint a wall, don't I? So I calculate the quantity of paint I have, so I paint just with was not enough (inaudible) So a bit of the area of the wall rests without painting. This is because I didn't calculate. I have to calculate how much I am going to spend to paint.
- 31 Teacher: Okay. That's fine kids (...) Wait! Yes, Amanda? [Amanda calls the teacher] [C2]
- 32 Amanda: Let me ask a question. Which one is correct?

- 33 Teacher: Kids, see! [The teacher asks for silence to hear Amanda]
- 34 Amanda: Because like this. I don't know how many, how many like this, one paint can I can paint?
- 35 Teacher: You do.
- 36 Amanda: How?
- 37 Teacher: It [the quantity] is written on the paint can.
- 38 Amanda: Is it?
- 39 Teacher: Yes, it is.
- 40 Herbert [and some others]: It is always written. [C3,6]
- 41 Teacher: The quantity of paint is written, but the painter or even you, you can buy one litre of paint [referring to a paint can]. Even if it were not written on the container, you could go there and hold a small can of paint. So, you calculate more or less. Say in this way: observe that that little can was enough to paint this area. Then how many cans am I going to need to paint the whole area? Then you have to know the whole area of the wall. This means, the concept of area measurement is involved for you to know how many litres of paint you need. Otherwise, you won't know, okay? [C3,6]
- 42 Barbara: So, whoever marked [an 'x'] is right. Whoever marked it is right?
- 43 Teacher: So, whoever marked it is right [some students exclaim happily 'Yeah!'].

Valuing the practice of sharing and comparing the students' responses the teacher asks the students who have marked the 'x' to put their hands up aiming at an evaluation of the consonances and dissonances, and decides firstly to involve the students who have not marked the 'x', i.e. the 'wrong' responses. When Marcelle says that she does not know why she did not put the 'x' Paula intervenes immediately suggesting that Marcelle should know. Encouraged by Paula the teacher asks Marcelle what she had thought and she immediately explains her interpretation of the question. In utterance 06 the teacher conjectures about Marcelle's reasoning and completes her thought; by doing that she scaffolds her students' thinking as she formulates in a clear and comprehensible way the idea that volume ("litres") does not seem to be related to area. Moreover, she encourages three more students to express their opinion. Felipe says that he does not remember, whereas Stephanie and Amanda confirm the teacher's conjecture. Amanda goes a bit further in her participation when she gives clues about their mathematical reasoning and interpretation of the question (11). Note that the teacher does not intervene in the students' responses concerning their inappropriateness.

In utterance 12 the teacher repeats the proposition that "the litre has nothing to do with area measurement" and decides to listen to the responses of the students who marked the alternative. This time she encourages six students to talk about their responses (note that in a classroom with 28 students, eleven have demonstrated public participation in the task by this time) despite their difficulties in making their utterances intelligible or clear. In particular, Livia participates with an analogical reasoning between the situation under discussion and a 'football field' situation. The teacher did not intervene in the students' responses because she wanted to promote a learning practice in which the students may uncover for themselves the dissonances or inconsistencies in

their responses. The students' utterances up to this point do seem to agree with the reason they have marked the 'x'. On the other hand, these utterances may serve to mark disagreement, and to raise the discovery and exploration of dissonances between responses of the participants, teacher and students.

Utterances 21, 23, 25, 27 and 29 reflect the teacher's intention to check if meanings were negotiated and some knowledge was co-constructed during the conversation. Utterances 22, 24, 26 and 28 are evidence that Felipe has negotiated meanings and has constructed a 'new' knowledge for him (perhaps for others as well, but not in the same way) since he assumes that he should have marked the 'x'. Moreover, he is aware now that if he does not know how much paint he needs he can run the risk of buying more paint than necessary. Up to this point the teacher did not make any straight suggestion about which students are right: those who marked the 'x' or those who did not. The way in which the teacher guides the discussion was intentional, for she valued the negotiation of meanings and co-construction of knowledge between the students-participants. Indeed, she supports the establishment of the sociomathematical norm of clarity of expression (utterances 27-29) by assisting Felipe to express his thoughts. Until now, twelve students have participated publicly in the activity.

The utterances 32-40 demonstrate Amanda's attitude in wanting to confirm/test after all which responses are correct. This is related to a sociomathematical norm, according to which a question is validated by its answer. Utterance 33 shows that the teacher asks the other students to pay attention to their classmate's doubt. This action may configure a social classroom norm, according to which a query is expected to be discussed and evaluated by the whole class and not solely by the teacher. However, Amanda's doubt is related to a convention that paint manufacturers should provide clear information for their clients; she wants to be sure about the quantity of paint there is in a can. Utterances 37, 39 and 40 show that both the teacher and Herbert were able to allay her doubt. At this stage, the number of students who have participated publicly is fourteen, so we can claim that the lesson is essentially constituted by the active participation of the students (characteristic C5 in Table 1). The teacher has not yet said anything about which responses are right or wrong.

Finally, the teacher decides to summarise the discussion trying to reach a collective agreement concerning the meanings constructed by the participants. In doing so, she extends the discussion of the alternative in question to a situation of how many paint cans one needs in order to paint a specific surface, and concludes that this involves the concept of area measurement. In her final response, when she agrees about "who was right", Barbara shows some awareness of what she has learned. Finally, the teacher agrees with Barbara and makes an 'indirect' synthesis of the situation: "... whoever marked it is right".

Throughout the excerpt one can see the sociomathematical norm of justification, i.e. that one is expected to justify his/her opinion. It is evident from the beginning in Paula's question (03) and then it is re-expressed by the teacher (04, 08). Students' participation structure is based on this norm, since we see them during the whole episode trying to explain their choice, and when their explanation is not accepted (18) elaborating it (20). Another interesting norm revealed is that mathematics seems to be closely related to everyday practice; contrary to other research findings (Tatsis and Koleza, 2008). Paula and Livia enrich the initial question with 'everyday' examples in order to make the problem more comprehensible, thus more easily solved.

Conclusions

Regarding our research objective – to build possible correspondences between ‘signs’ of learning and ‘local’ changes of participation – we take Felipe and Amanda’s participation to claim that there was a correspondence between leaning and changing participation during the collective discussion. Indeed, we have established which forms of participation were expected in such a practice. In doing so, we could identify Felipe and Amanda’s forms of participation, according to the classroom norms suggested, as well as how their forms of participation changed during the event (Felipe: 09, 22-29; Amanda: 11, 34-38). From a situated perspective this participation has constituted some learning (something we could also discuss in terms of identity formation, though that is beyond the scope of this paper). In fact, it is very reasonable to say that ‘bits’ of Felipe and Amanda are not the same as before, for the practice clearly afforded transformations in their way of thinking. On the other hand, Felipe and Amanda’s participation contributed to the regeneration of the practice as they afforded the participation of both the teacher and their classmates. Amanda’s unexpected doubt “...I don’t know how many, how many like this, one paint can I can paint?” revealed that such kind of practices include emergent phenomena that overlap already-established/expected ways of reasoning and communicating into which students are suppose to be inducted (see Cobb et al., 2001).

From the video recording of the two sequential lessons (50 minutes each) dedicated to the collective correction of the diagnostic questionnaire and the analysis of these lessons as a whole, we can say that a LCoP was constituted during the particular activity. The degree of the students’ participation – as we took participation above – was very significant. We have identified an expressive number of students exposing their interpretation of the questions, their reasoning and argumentation, based on the relevant norms established. These norms and especially the justification norm proved very important in the structuring of the participation, by helping students establish some shared ways of understanding each other and of evaluating each other’s contribution (the most characteristic case is Paula’s question towards Marcelle in 03, asking her to justify her opinion). Another common assumption established was that the problems posed can be integrated into everyday situations (coming from the students’ everyday experience) in order to be more effectively dealt with. This assisted their purposeful participation towards the achievement of a common understanding. The discussion took place not only between the teacher and the students, but also among the students: many of them addressed their classmates to question or comment about their responses, which reveals the social norm of collaboration towards a common end. Also, the intensity of the students’ participation was so high in certain moments that the teacher-researcher had to do interventions likely “Children, please! I think it’s great that all of you are excited to participate, but we don’t need to do this by shouting so much and at the same time! I still have to teach until 5:30 p.m.!”.

We suggest that the main pedagogical implication of this study points to the teachers’ role in guiding a collective discussion and scaffold the students’ thinking. Also, teachers should be clear and transparent with students about which ways of participation they expect and value in classroom practice in order to support and evaluate their learning. This does not mean that students’ participation should be constrained; it means that the teacher should be able to frame the students’ actions according to the established norms and at the same time be flexible in the establishment of new norms and practices. Moreover, if the teacher is able to identify the ‘small’ changes in each student’s participation, s/he will be able not only to better monitor the evolution of the activity but also to perform his/her interventions in the most effective way.

Quotations

1. ...learning is an aspect of changing participation in changing “communities of practice” everywhere. Wherever people engage for substantial periods of time, day-by-day, in doing things in which their ongoing activities are interdependent, learning is part of their changing participation in changing practices. (Lave, 1996, p.150)
2. As a person steps into a new practice, in social situations, in schooling, in the workplace, or other practices, the regulating effects of that practice begin, positioning the person in that practice... Even if a person withdraws from a practice after a short time, she or he has been changed by that participation. (Lerman, 2001, p. 98)
3. Such communities are local in terms of time as well as space: they are local in terms of people’s lives; in terms of the normal practices of the school and classrooms; in terms of the membership of the practice; they might ‘appear’ in a classroom only for a lesson and much time might elapse before they are reconstituted... (p. 94)
4. C1. pupils see themselves as functioning mathematically and, for these pupils, it makes sense for them to see their ‘being mathematical’ as an essential part of who they are within the lesson;
C2. through the activities and roles assumed there is public [from the participants] recognition of developing competence within the lesson;
C3. learners see themselves as working purposefully together towards the achievement of a common understanding;
C4. there are shared ways of behaving, language, habits, values, and tool-use;
C5. the lesson is essentially constituted by the active participation of the students;
C6. learners and teachers could, for a while, see themselves as engaged in the same activity. (p. 103)
5. ... normative activities of the classroom community (social perspective) emerge and are continually regenerated by the teacher and students as they interpret and respond to each other’s actions (psychological perspective). Conversely, the teacher’s and students’ interpretations and actions in the classroom (psychological perspective) do not exist except as acts of participation in [and constitutive of] communal classroom practices. When we take a social perspective, we therefore locate a student’s reasoning within an evolving classroom microculture, and when we take a psychological perspective, we treat that microculture as an emergent phenomenon that is continually regenerated by the teacher and students in the course of their ongoing interactions. (p. 122)

References

- Back, J., & Pratt, N. (2007). Spaces to discuss mathematics: Communities of practice on an online discussion board. *2nd Socio-cultural Theory in Educational Research and Practice Conference, University of Manchester, U.K.* Retrieved March, 24, 2008 from:
<http://www.lta.education.manchester.ac.uk/ScTIG/papers/Jenni%20Back.pdf>
- Boaler, J. (2002). The development of Disciplinary Relationships: Knowledge, Practice and Identity in Mathematics Classrooms. *For the Learning of Mathematics*, 22, 1, 42-47.
- Cobb, P. (1994). Where Is the Mind? Constructivist and Sociocultural Perspectives on Mathematical Development. *Educational Researcher*, 23 (7), 13-20.

- Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences, 10*, 113-164.
- Frade, C. (2003). *Componentes Tácitos e Explícitos do Conhecimento Matemático de Áreas e Medidas* (Tacit and Explicit Componenets of Area Measurement Knowledge). Belo Horizonte: Faculdade de Educação-UFMG. Tese de Doutorado não publicada (Unpublished Doctoral Thesis).
- Frade, C., Winbourne, P., & Braga, S. (2006). Aline's and Julia's stories: Reconceptualizing transfer from a situated point of view. In J. Novotná, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*. Prague, Czech Republic: PME, vol. 3, 97-103.
- Jaworski, B. (2007). Theory in developmental research in mathematics teaching and learning: Social practice theory and community of inquiry as analytical tools. In *2nd Socio-cultural Theory in Educational Research and Practice Conference, University of Manchester, U.K.* Retrieved March, 24, 2008 from: <http://www.lta.education.manchester.ac.uk/ScTIG/papers/Barbara%20Jaworski.pdf>
- Lave, J., & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. New York: Cambridge University Press.
- Lave, J. (1993). The Practice of Learning. In S. Chaiklin & J. Lave (Eds.), *Understanding Practice: Perspectives on Activity and Context*. New York: Cambridge University Press, 3-32.
- Lerman, S. (2001). Cultural, Discursive Psychology: A Sociocultural Approach to Studying the Teaching and Learning of Mathematics, *Educational Studies in Mathematics, 46*, 87-113.
- McVittie, J. (2004). Discourse Communities, Student Selves and Learning. *Language and Education, 18* (6), 488-503.
- Sfard, A. (2006). Participationist discourse on mathematics learning. In J. Maaß & W. Schölglmann (Eds.), *New mathematics education research and practice*. Rotterdam, The Netherlands: Sense Publishers B.V., 153-170.
- Tatsis, K., & Rowland, T. (2006). Vague language in Greek and English mathematical talk: A variation study in face-work. In J. Novotná, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*. Prague, Czech Republic: PME, vol. 5, 257-264.
- Tatsis, K., & Koleza, E. (2008). Social and Sociomathematical Norms in Collaborative Problem Solving. *European Journal of Teacher Education, 31* (1), 89-100.
- Wegerif, R., & Mercer, N. (1997). A dialogical framework for researching peer talk. In R. Wegerif & P. Scrimshaw (Eds.), *Computers and Talk in the Primary Classroom*. Clevedon: Multilingual Matters, 49-61.
- Wenger, E. (1998). *Communities of Practice: Learning, Meaning and Identity*. Cambridge, UK: Cambridge University Press.

- Wenger, E. (2007). Learning for a small planet: Agency and structure in the constitution of identity. In *2nd Socio-cultural Theory in Educational Research and Practice Conference, University of Manchester, U.K.* Retrieved March, 24, 2008 from: <http://www.lta.education.manchester.ac.uk/ScTIG/2007conferencekeynotespeakerabstracts.html>
- Williams, G., & Clarke, D. (2003). Dyadic Patterns of Participation and Collaborative Concept Creation: 'Looking In' as a Stimulus to Complex Mathematical Thinking. Paper presented as part of the symposium "*Patterns of Participation in the Classroom*" at the Annual Meeting of the American Educational Research Association, Chicago, April 21-25, 2003.
- Winbourne, P., & Watson, A. (1998). Participating in Learning Mathematics through Shared Local Practices in Classrooms. In A. Watson (Ed.), *Situated Cognition and the Learning of Mathematics*. Oxford: Centre for Mathematics Education Research, University of Oxford. Chapter 7, 93-104.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27 (4), 390-408.

¹ Sfard's (2006) discursive participationist view of learning could be another possibility. However, given that we are using the notion of LCoP as our background, and the discursive interactions are, in our opinion, contemplated in it (characteristic number 4) as well as in Cobb et al.'s framework (part of normative activities in classroom practices), we believe that the latter allows us to talk about learning as local changes of participation within a LCoP more flexibly.

If $A \cdot B = 0$ then $A = 0$ or $B = 0$?

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Dedicamos este trabajo a la memoria de nuestro querido amigo, el Maestro Carlos García

Abstract: We present a study carried out in Uruguay, with secondary school students and tertiary level mathematics students, concerning the zero-product property. In our research we observed that when early secondary and late secondary school students have to solve equations of the form $(ax + b)(cx + d) = 0$, they do not always apply the property, even when it is the only available tool and have received specific instruction on its application to the resolution of equations of this type. We also detected an error that students make when they have to verify the solutions of this type of equations. The error consists of the assignment of two different values to the unknown simultaneously. Our study also revealed that late secondary and tertiary level students show a certain tendency to generalize the zero-product property to other algebraic structures where it is not always valid.

Keywords: error analysis; false generalizations; linear equation products; secondary school students; Uruguay; zero-product property

Introduction

According to Bednarz, et al. (1996, p. 3) the introduction of school algebra can take many different directions: “the rules for transforming and solving equations”, “the solving of specific problems or classes of problems”, “the generalization of laws governing numbers”, “introduction

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of the concepts of variable and function” and “the study of algebraic structures”. Each one of these options has its conceptual difficulties associated with it and brings about didactical problems to be solved. On the other hand one cannot deny their importance and we may wonder if it is possible to have a curricular approach that reaches a balance among all these possibilities. One way to accomplish this would be to search mathematical topics that can lend themselves to the design of mathematical situations encompassing different facets of algebra. Our research is a step in this direction.

The content of our study touches upon all the approaches mentioned in the above paragraph. On the one hand this makes its mathematical focus a valuable resource for didactical purposes, and on the other hand we get a glimpse into student difficulties concerning the related notions.

Background and research questions

In the title of this article the well-known zero-product property appears in the form of a question. What is the answer to it? And what would happen if we asked it of the students?

Of course the answer depends on the characteristics of the structure to which A and B belong, and it allows us to distinguish those structures that contain divisors of zero from those that do not. This question and the related property have been fundamental in the development of structural algebra.

Texts about the historical development of mathematics (see for example Corry, 1996) show how, little by little, the study of the algebraic structures becomes the main task of Algebra at the beginning of the 20th century. The property on which we focus our attention and the distinction between structures (their classification as having or not having divisors of zero), were particularly important in the development of Abstract Algebra. The zero-product property is the defining characteristic of a type of commutative ring called an integral domain (Wikipedia).

In Uruguay, this property in the context of the real numbers is known as the Hankelian property. Although some textbooks assert that this is due to the name of the mathematician Hankel who discovered it, we could not find any evidence or reference about this claim.

This denomination was, apparently, introduced in Uruguay by a mathematics teacher in a textbook he published in 1958². There the author points out the “brief but deep exposition of Hankel about the theory of numbers and their operations” but he does not make any specific reference that links Hankel with this property³.

The zero-product property appears along the curriculum in different ways. To illustrate them we will give some examples taking into account the directions stated by Bednarz, et al. (1996, p. 3) that we have mentioned above.

The second degree equation is a common topic of the secondary school curriculum. Usually, teachers present to their students the incomplete forms of the second degree equation before teaching the quadratic formula to solve them. For instance, equations of the form $ax^2 + bx = 0$, can be transformed into $x(ax + b) = 0$. Hankelian property is a useful tool to solve an equation of this type and in general, to solve any second degree equation of the form $(ax + b)(cx + d) = 0$. When students study the quadratic function and they want to find, for example,

the x -intercepts (if they exist), the Hankelian property may be a tool to find the roots of the function if its analytic expression is given in an appropriate form.

During early secondary school students study different sets of numbers, the operations defined on each of them and their properties. The Hankelian property is observed in the context of multiplication working with concrete numbers and then it is generalized. Textbooks give students activities such as the following⁴:

Complete:

x 5 = 0

17 x = 0

x = 0

x x 7 x 3 = 0

Observe that for the product to be zero at least one of the factors must be zero.
In general: If $a \times b = 0 \Rightarrow a = 0$ and/or $b = 0$.

Many times in the late secondary school or at the university, students have the opportunity to study other subjects where they can analyze the validity of the zero-product property in different contexts. For example, in the context of the matrices an Uruguayan textbook –universitary level– gives students the following activity⁵:

Find examples of real 2x2 matrices where:

$$B^2 = O, B \neq O$$

This activity can lead the students implicitly to realize that the zero-product property is not valid in the context of the matrices.

In our study we wanted to identify student difficulties related to the zero-product property through mathematical situations that address its different aspects. We also wanted to offer didactical strategies that might contribute to a better understanding of this important topic by students at different educational levels.

Three phenomena

In this article we present three didactical phenomena related to this property as we discuss below. First we describe these phenomena shortly and then we elaborate on them, presenting evidence from our research. Since we are exploring different phenomena and what unites them is a mathematical property, we use different viewpoints to provide interpretations for each of them.

First Phenomenon. In our research we observed that when early secondary (14-15 years old) and late secondary school students (17-18 years old) are required to solve equations of the type $(2x - 6)(18 - 2x) = 0$, they do not immediately apply this property, even when it is the only available tool to them and when they have received specific instruction on its application to the resolution of equations where factored polynomial expressions appear equal to zero.

Second Phenomenon. We also detected an error that students make when they have to verify the solutions of an equation such as the one mentioned above. The error consists of the assignment of two different values to the unknown simultaneously (which we will refer to as double assignment), as it is illustrated in the following task carried out by a 17 year-old late secondary level student. The task can be translated as follows:

- 1) i) Solve the equation $(2x - 6)(18 - 2x) = 0$. Explain how you do it.
- ii) How many solutions did you obtain? _____ What are they? _____
- iii) Verify the solution(s) that you obtained.

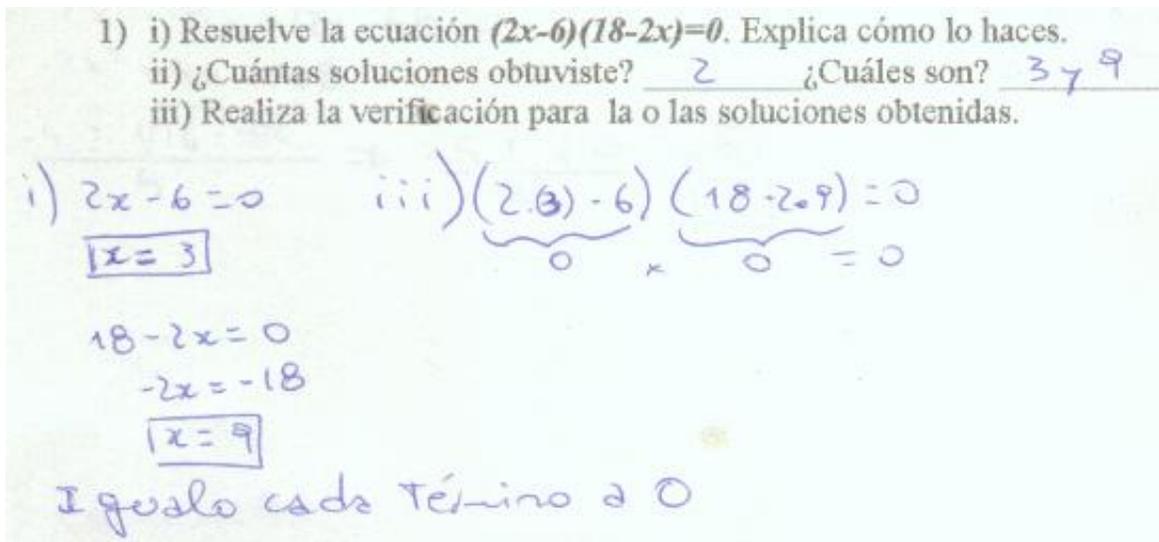


Fig. 1

As can be seen from his work, after finding the solutions as 3 and 9, this student substitutes 3 in the factor $(2x - 6)$, and 9 in the factor $(18 - 2x)$ simultaneously to verify them.

Vaiyavutjamai, et al. (2005) studied the extent to which students from three different nations (coming from Thailand, Brunei Darussalam and the USA) correctly solved equations in the form $x^2 = K$ ($K > 0$) and $(x - a)(x - b) = 0$ (where a and b can be any real numbers). They report that

some students in their research study “tended not to know what their solutions represented in relation to the original equation” (Vaiyavutjamai, et al., 2005). Furthermore, “many students did not realize that if a variable appeared twice in an equation, then it had the same value in the different “places” in which it appeared” (Vaiyavutjamai, et al., 2005). They conclude that the thinking of many students in this task is guided by a misconception related to variables. The authors point out the need for creating a research agenda centered on the topic of quadratic equations as this is an unexplored field in terms of student understanding and difficulties.

A similar phenomenon that corresponds to thinking that the same letter does not necessarily stand for the same value in a given mathematical expression was reported in other studies, as well. Filloy & Rojano (1984) observe that when solving first degree equations such as $x + 5 = x + x$, some students think that the x on the left side of the equation can be any number, but the second x on the right side has to be 5. Fujii (2003) uses expressions such as $x + x + x + x = x$ and $x + x + x = 12$ in a study to illustrate this misconception. In the first case the students are asked whether the expression is correct, and in the second case they are to choose possible correct answers from among three choices provided to them. When students who think that the expression $x + x + x + x = x$ can be correct were questioned about whether “ x does not have to be the same number”, a student answered by saying “It doesn’t have to be the same thing. It’s a variable” (Fujii, 2003). The same student who chose (2,5,5) and (10,1,1) as acceptable solutions for the equation $x + x + x = 12$ was questioned whether $x + x + x$ would be replaced by $3x$, and he replied:

It can, but it can also be wrong. It depends on what x equals, which, because x can equal 10, the first x , and then second x can equal 2. (Fujii, 2003).

According to Fujii (2003, referring to Van Engen, 1961a, b), this misconception stems from the fact that some students consider only the *unspecified* aspect of the concept of variable, and the *definite* aspect, which is in tension with the former, tends to be missing.

In this paper we add another interpretation to this phenomenon, in the context of our research.

Third Phenomenon. We also observed that late secondary as well as tertiary level mathematics students⁶ (older than 21 years with various ages) show a certain tendency to extend this property to other algebraic structures where this property is not always valid, as in the context of matrices or real functions.

Erroneous generalization of rules or properties to other contexts where they do not hold true can have its roots in the prior learning experiences of the students with the topic in question, and the intuition that they develop in relation with it (Fischbein, 1987, p.198). For example Aguilar & Oktaç (2004) found that teachers involved in their study tried to solve equations in modular arithmetic structures as if the elements were real numbers and the operations were the usual ones.

Method

In the study that we conducted in order to research the understanding and the use of the zero-product property, we applied a written questionnaire consisting of eighteen questions to two groups (corresponding to 14 early secondary and 14 late secondary level students) and another

one consisting of six questions to two groups of students (corresponding to 10 late secondary and 23 tertiary level students). We interviewed three students from the lower secondary level, seven from the upper secondary level and three from the tertiary level as well as one teacher. The written questionnaires differed slightly depending on the level of the students and the interview probed on those aspects that we considered revealing for the purposes of our research.

The complete questionnaires are given in the appendix. Here we consider a few of the questionnaire items in detail, in order to look into the three phenomena described above. When there are remarkable differences as to the way different groups answer a certain question, we note that, as well.

First Phenomenon. How do students go about solving equations of the form $(ax + b)(cx + d) = 0$?

Kieran (1996, p. 22) distinguishes between three types of activities of school algebra: generational, transformational and global/meta-level. The first one emphasizes the forming of algebraic objects such as expressions and equations, possibly within the frame of a mathematical situation. The transformational type refers to equation solving and manipulation of expressions to get equivalent expressions, among others. The global/meta-level activities are the ones for which algebra is used as a tool such as modeling, noticing patterns and problem solving. As Kieran notes “[a]lgebra textbooks have traditionally emphasized the transformational aspects of algebraic activity, with more attention paid to the rules to be followed in manipulating symbolic expressions and equations than to conceptual notions that support these rules or to the structural underpinnings of the expressions or equations being manipulated” (Kieran, 1996, p. 24).

In our research we observed that when early secondary students try to solve $(ax + b)(cx + d) = 0$ type equations, they usually apply the distributive law and/or try to use some well-known technique for solving first degree equations, as they do not yet know the quadratic formula. The procedure they apply is usually erroneous; below we present examples of students’ work illustrating some of these strategies:

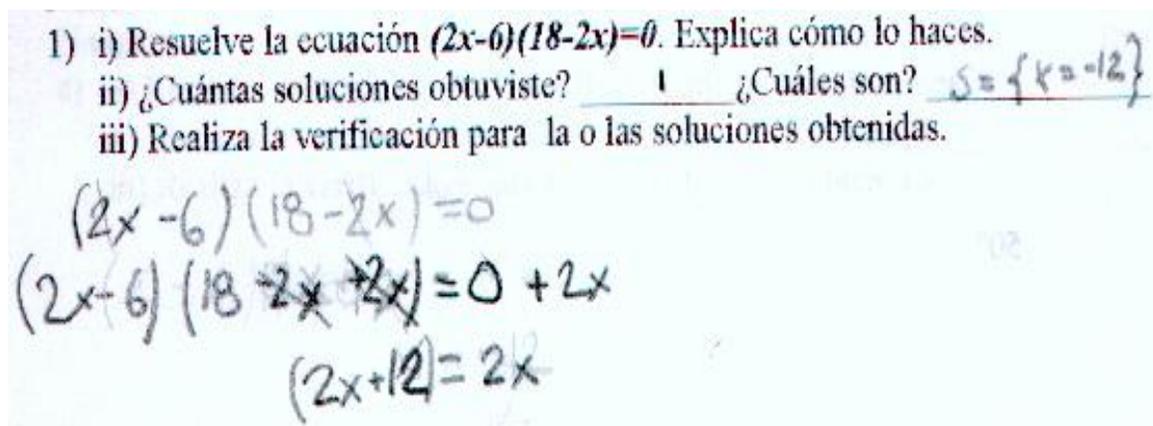


Fig. 2

In Fig. 2 the student introduces $2x$ on both sides of the equation, however on the left side it is added within one of the factors of the product.

1) i) Resuelve la ecuación $(2x-6)(18-2x)=0$. Explica cómo lo haces.
 ii) ¿Cuántas soluciones obtuviste? _____ ¿Cuáles son? _____
 iii) Realiza la verificación para la o las soluciones obtenidas.

i) $36x - 4x^2 - 108 + 12x = 0$
 $48x - 4x^2 - 108 = 0$
 $48x - 4x^2 = 108$
 $48x - 4x^2 = 108 + 48x$
 ~~$48x - 4x^2 - 48x = 108$~~
 $-4x^2 = 108$
 $x^2 = \frac{108}{-4}$
 $x^2 = -27$
 $x = -\sqrt{27}$

Fig. 3

Fig. 3 shows the work of a student who introduces the term $48x$ to the right side of the equation, without doing the same thing on the left side. This allows him to get rid of the term involving x and making it possible to arrive at an “answer”.

About five months before they completed this questionnaire, these students had been instructed to solve this type of equations by applying the zero-product property. However, they do not seem to recognize its applicability even though they apparently know about it, as we see in the following type of answers:

6) i) Se sabe que $b \cdot d = 0$, ¿qué puedes deducir sobre b y d a partir de esta información?
 ii) ¿Qué representan para ti b y d ?

que b o d es 0
 Numeros

Fig. 4

The question in Fig. 4 reads:

6) i) We know that $b \cdot d = 0$. From this information, what can you conclude about b and d ?

ii) What do b and d represent for you?

For (i), the student writes: “that b or d is 0”. For (ii), her answer is: “numbers”.

Most of the students are not successful in the task of solving $(2x - 6)(18 - 2x) = 0$ due to the complexity of the equation that they obtain after applying the distributive law, as they do not have resources like the quadratic formula to solve it (they have not studied it yet). As an example, in the following figure we can see the work of a student whom we will call Clarise, an early secondary student:

1) i) Resuelve la ecuación $(2x-6)(18-2x)=0$. Explica cómo lo haces.
ii) ¿Cuántas soluciones obtuviste? _____ ¿Cuáles son? _____
iii) Realiza la verificación para la o las soluciones obtenidas.

$$(2x-6)(18-2x)=0$$
$$36x - 4x^2 - 108 + 12x = 0$$
$$48x - 4x^2 - 108 = 0 + 48x$$

Fig. 5

Although in some cases students said that they could give the solutions of the equation mentally without carrying out any operation, it seems that this does not satisfy them because they have not applied an algorithmic procedure to solve it. We can see this in the following translation of Clarise’s interview, which revealed that actually she knew the answer:

[...]

Interviewer: So, you, according to what you answered in question number 6 (see Fig. 4), you knew that if the product of two factors is zero, then one or the other must be zero.

Clarise: Yes.

I: Nevertheless you didn’t apply it to solve the equation (see Fig. 5). Can you explain why?

C: Because then I started thinking about it, right? It was something that I was deducing without performing any operation. Let’s say I was able to find it, what I couldn’t do was by seeing, I mean, solving for the x .

I: So, the idea that you just explained to me, that x must be 3 or 9, you knew it well...

C: Of course, what I didn’t know was how to arrive at the result using a formula, how can I tell you? I couldn’t follow an operation, I couldn’t find according to all the procedures that x was equal to 3 or x was equal to 9.

In some cases this could be due to the fact that mathematics is taught as a set of rules or procedures to apply, giving more emphasis to the transformational type of activities (Kieran, 1996, p. 24). When students face a situation that is not familiar to them, they will try to apply certain algorithms known to them, since they believe that this way they will surely reach a solution – as they do, on occasions.

When Lima and Tall (2006) asked 15-16 year old students to solve equations of the type $(y - 2)(y - 3) = 0$ among others, not only no one mentioned or used the zero-product rule, but “students did not seem to believe it. The only met-before seem to be numeric ‘guess and test’ to seek solutions, or an attempt to use the quadratic formula. The students therefore are at a procedural level relying on a single procedure, without the appreciation of several procedures to give alternative approaches” (Lima & Tall, 2006).

In our study only a small percentage of students was able to solve the equation in question, equating each factor to zero in order to find the roots. These students could also formulate an explanation for what they were doing as we see in the work shown in Fig. 6:

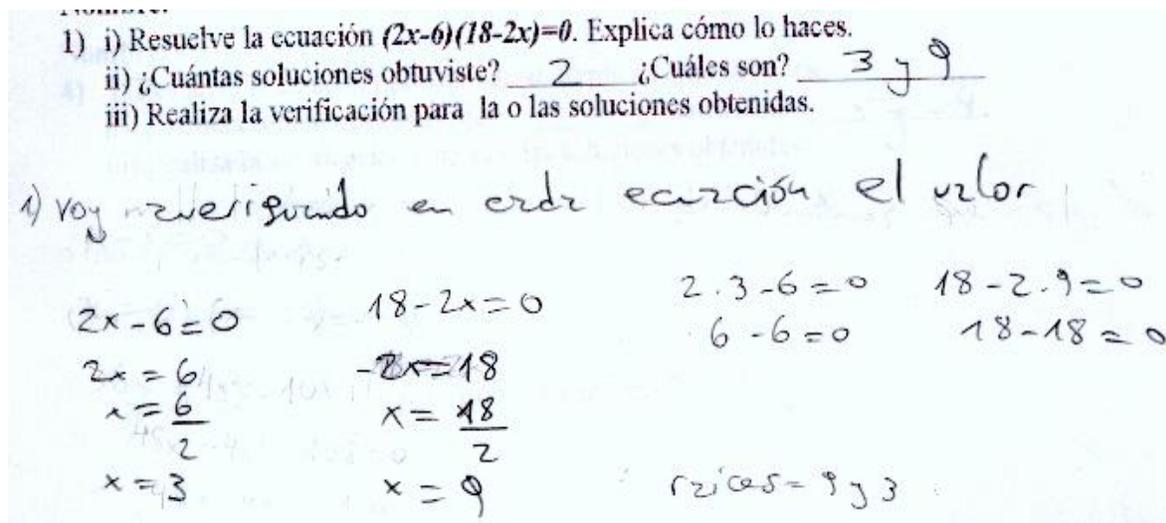


Fig. 6

Some late secondary students that apply this property to solve the equation do so because their teachers have taught them that “it is done this way” or because they were “taught to do it this way” (as they explained themselves) and they cannot always offer an explanation with mathematical arguments for what they do. But in general, the strategy preferred by students at this level consists of developing the polynomial expression and applying the quadratic formula as we can see in the following figure. Even when the student knows the two procedures (application of the property and the use of the quadratic formula), he/she does not choose the application of the property as a simpler procedure.

1) i) Resuelve la ecuación $(2x-6)(18-2x)=0$. Explica cómo lo haces.
 ii) ¿Cuántas soluciones obtuviste? 2 ¿Cuáles son? 3 y 9
 iii) Realiza la verificación para la o las soluciones obtenidas.

i) $(2x-6)(18-2x)=0$ ~~36x - 4x^2 - 108 + 12x~~
 $36x - 4x^2 - 108 + 12x$
 $-4x^2 + 48x - 108 = 0$
 $x = \frac{-48 \pm \sqrt{(48)^2 - 4(-4)(-108)}}{2(-4)}$
 $\frac{-48 \pm \sqrt{2304 - 1728}}{-8} = \frac{-48 \pm \sqrt{576}}{-8}$
 $\frac{-48 \pm 24}{-8}$ $\begin{cases} \frac{-24}{-8} = 3 \\ \frac{-72}{-8} = 9 \end{cases}$

Fig.7

On the other hand the tertiary level mathematics students have generated enough autonomy to decide and to choose what tool to use according to the situation they face. In our study they always applied the property whenever the equation allowed it.

Second Phenomenon. Verification of the solutions of an equation of the form $(ax + b)(cx + d) = 0$

Although verification and validation of solutions is an important part of the problem solving process, students usually don't feel the need to check their answers. Furthermore, even though in a problem-solving context validating an answer might make sense to the students, in an algebraic setting there is little meaning given to this type of activity.

In Uruguay when early secondary students begin to study first degree equations, teachers usually use the verification as a way to explain them what it means for a specific number to be a solution of an equation. Students' textbooks also use the verification in the same sense. In the case of the verification of the solutions of an equation of the form $(ax + b)(cx + d) = 0$, textbooks give students exercises such as the following⁷:

¿True or false?
 In the equation $(x - 1)(x + 2) = 0$ the numbers -2 y 1 are roots.

Nevertheless, none of the students' textbooks show how to do the verification of an equation of this form. In the classroom, it very much depends on the course instructor whether students would engage in an activity of verification.

When asked, students frequently verify the solutions of this type of equations replacing the x of the first factor by $-b/a$ and that of the second factor by $-d/c$ simultaneously. We detected this strategy at all educational levels at which we applied our instrument, that is at early and late secondary, and tertiary levels. A priori we had supposed that it was closely related to the application of the property: We think that the student could believe that the second degree equation is fragmented into two first degree equations and therefore he/she does the verification this way. In the interviews the students manifested certain confusion in relation to whether the equation $(2x - 6)(18 - 2x) = 0$ was one equation or two equations. As the student ends up solving two first degree equations, it is possible that he/she treats the whole equation as two linear equations even when the substitution has been made in an equation of second degree, without the student being aware of it. In Fig. 8 we see an example illustrating this phenomenon.

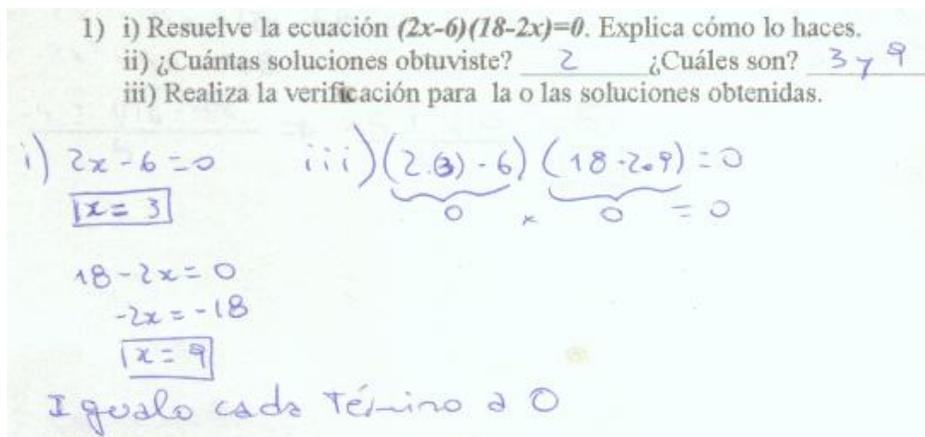


Fig. 8

We saw that this error is also present when other strategies such as the application of the quadratic formula are used to solve the equation. We can see this in the following work of a late secondary level student:

1) i) Resuelve la ecuación $(2x-6)(18-2x)=0$. Explica cómo lo haces.
 ii) ¿Cuántas soluciones obtuviste? 2 ¿Cuáles son? 3 y 9
 iii) Realiza la verificación para la o las soluciones obtenidas.

i) $(2x-6)(18-2x)=0$ $36x - 4x^2 - 108 + 12x$
 ~~$36x - 4x^2 - 108 + 12x$~~
 $-4x^2 + 48x - 108 = 0$
 $x = \frac{-48 \pm \sqrt{(48)^2 - 4(-4)(-108)}}{2(-4)}$

iii) $(2(3)-6)(18-2(9))=0$ $\frac{-48 \pm \sqrt{2304 - 1728}}{-8} = -48 \pm$
 $\begin{matrix} \text{"} & \text{"} \\ 0 & 0 \end{matrix}$ $x = \frac{-48 \pm 24}{-8}$ $\begin{cases} \frac{-24}{-8} = 3 \\ \frac{-72}{-8} = 9 \end{cases}$

Fig. 9

One might think that the students may be using this strategy as a shortcut, however we observe that although the students claim that if a product is zero then one of the factors should be zero, a strong belief seems to exist that both factors should be simultaneously zero. This situation that we consider of an intuitive nature might condition how the student thinks that the concept of variable functions, leading him/her to make the double assignment. This belief was evidenced through the following activity where most of the early and late secondary level students replied that the numbers were 6 and 19. The activity reads:

13) The papers are hiding numbers, can you find them? Explain your reasoning.

13) Los papelitos tapan números, ¿puedes averiguarlos? Explica tu razonamiento.

$$(\boxed{6} - 6) \times (\boxed{19} - 19) = 0$$

pero que la ecuación sea 0, los dos términos deben ser = 0

Fig.10

This student wrote: "In order for the equation to be 0, both terms should be = 0".

The error was not exclusive to the equation that is given in factored form but it also appeared when verifying the roots of an equation whose polynomial expression was developed. The following activity can be translated as:

- 17) Are 3 and 4 roots of the equation $x^2 - 7x + 12 = 0$?
 Explain your answer, clarifying whatever you think is appropriate.

17) ¿Son 3 y 4 raíces de la ecuación $x^2 - 7x + 12 = 0$?
 Explica tu respuesta. Realiza los planteos que sean necesarios.

$$\begin{array}{l}
 3^2 - 7 \cdot 4 + 12 = 0 \\
 9 - 28 + 12 = 0 \\
 + 21 - 28 = -7 \\
 -7 \neq 0
 \end{array}$$

No xq al sustituir los valores en la ecuación no da 0

Fig. 11

The student's answer is: "No, when the values are substituted the equation doesn't give 0". We observe that here 3 was substituted in the x^2 term of the equation, and 4 was substituted in the $-7x$ term. It seems that this student wants to make sure that 3 **and** 4, being two roots of the given equation, appear in the verifying process simultaneously. This might cause a conflict with the solving process when we get " $x = 4$ **or** $x = 3$ " as a result.

If the students find the roots of the equation themselves using a procedure and if they use the developed expression to verify them by substituting two different values simultaneously, they are more likely to detect that there is something that is not working since they do not obtain an expression of the type $0 = 0$. On the other hand if they use the factored expression they will obtain $0 \cdot 0 = 0$, in which case they are not likely to realize the error.

Vaiyavutjamai and Clements (2006) report about Thai students who thought that the two x 's in the equation $(x - 3)(x - 5) = 0$ represented different numbers (writing $(3 - 3)(5 - 5) = 0$ to check their answers) and even after an instructional treatment continued with this belief. They observed the same type of phenomenon when they used the equation $x^2 - x = 12$. In this case students were wondering why they were not getting an equality when they substituted different numbers for the two x 's.

In order to observe if the students would detect their error, we prepared a contextualized situation where it was impossible that the variable (in this case the hour of the day) could take two

different values at the same time. For that, we worked in the context of real functions and the calculation of their images.

To evaluate the need for traffic lights at an intersection, a device that counts cars is placed. The results of the study are given by the expression:

$$A = (2x - 6)(18 - 2x)$$

where A represents the number of cars that go through the intersection at the hour x .

- a) How many cars crossed the intersection at 5 o'clock? Show your calculations.*
- b) Calculate how many cars crossed at 3 o'clock and at 9 o'clock. Show your calculations.*
- c) At what time did no car cross? Why?*

Let's see an extract from Martín's interview, a late secondary student. After Martín claims that to verify if 3 and 9 are roots of the equation $(2x - 6)(18 - 2x) = 0$ he should substitute 3 in the first factor and 9 in the second, he was presented with the traffic lights problem. After he solves this problem correctly, the interviewer probes more:

[...]

Interviewer: If I ask you how many cars cross the intersection at 3 in the morning and at 9 in the morning, what would you do to solve this Martin?

Martín: Substitute both x by the same number, that is, by the same root, both by 3 or both by 9.

T: Right. So, between this attempt and this one⁸, which one you consider appropriate to answer the problem of the cars and traffic lights?

M: (Points out the correct one)

I: And now if we leave the context of this problem and forget that we are talking about cars and traffic lights and we just want to know if 3 and 9 are roots of the equation...

M: Yes.

I: There, you can do...

M: I can do any of the procedures.

I: And in the context of the problem, why did you think that x could not value at the same time 3 in the morning and 9 in the morning?

M: In the context of the cars?

I: Yes.

M: Because we are talking about hours. It can't be, It wouldn't be a possible situation.

The student pointed out that in the case of this problem x should take the same value because it could not be 3 in the morning and 9 in the morning at the same time. However in verifying the roots of an equation it could, since according to them x was not representing the hour of the day

in this case. In this way the student isolated the case of the verification of the roots of the equation as a special situation, separating it from the calculation of image values of a function. In this way he was maintaining the coherence of their mental schemes. We could say that in this case, the compartmentalization phenomenon (Vinner, 1990) appears as a resource of the mind that allows the student to avoid contradictions and therefore to maintain internal coherence, isolating the two situations and recognizing them as different things that are not connected.

The evidence in our work shows that the verification process is not obvious, and neither is the idea that the unknown cannot be replaced at the same time by two different values.

The assignment of different values to the same variable was a resource used by the youngest students when they were asked to build an equation with two given roots. We see it in the following task that reads:

11) Give an equation with 4 and 3 as its roots. How do you do it?

11) Escribe una ecuación que tenga por raíces a 4 y 3. ¿Cómo lo haces?

$$\begin{array}{l}
 x + 1 + x + 2 = 19 \\
 \textcircled{4} + 1 + \textcircled{3} + 2 = 19 \\
 5 + 12 + 2 = 19 \\
 19 = 19
 \end{array}$$

Fig.12

This procedure and the one that considers an equation in two variables, such as $(x - 4)(y - 3) = 0$, assigning to x the value 3 and to y the value 4, was more natural or more spontaneous for them than conceiving a second degree equation in one variable that had the two given roots. Only three students out of fourteen wrote a second degree equation with one variable to answer this task. Maybe conceiving a second degree equation with one variable with the possibility of the existence of two roots is much more complex for the students than educators might believe. With reference to this, Trigueros and Ursini (2003) point out that the great majority of the students with whom they worked thought that the unknown involved in a quadratic equation could take only one value.

Third Phenomenon. Finally... does $A \cdot B = 0$ imply that $A = 0$ or $B = 0$?

Another aspect on which we centered our attention was whether the students extended the zero-product property to structures where this is not valid. We also wanted to shed light on the reasons for doing so, even in the case when they had received specific instruction on this subject.

To go about finding this out, we asked the students, among others, three questions that we consider key:

- ✓ *It is known that $b \cdot c = 0$. Based on this information, what can you conclude about b and c ?
What do b and c represent for you?*
- ✓ *f and g are two functions whose domain is R . It is known that $f \cdot g = 0$, that is to say that the product of f and g is the zero function. Based on this information, what can you conclude about f and g ?*
- ✓ *D and B are two matrices. It is known that $D \cdot B = 0$, that is to say that the product of the two matrices is the zero matrix. Based on this information, what can you conclude about D and B ?*

The first question refers to two factors a and b whose product is zero, where the nature of a and b is not specified. The second question is about two functions whose product is the zero function, and the third one involves two matrices whose product is the zero matrix. In each case the students were asked what they could conclude about the factors.

When they took the questionnaire, the late secondary students in our study had already taken a course in Analysis and another in Algebra and Analytic Geometry. The tertiary level students had also studied those subjects at the university level. In the case of the functions, we do not know whether the students had seen examples of non-zero functions whose product was the zero function. On the other hand, in the case of matrices all students who took the test had seen that the property was not valid. In the three cases that we present here, a high percentage of the students answered that one of the factors was zero in the first question, was the zero function in the second question and the zero matrix in the third question.

Several late secondary school students reached their conclusions thinking about the property of absorption, that is, if one of the two factors is zero, then the product will also be zero. They did not realize that the inverse property does not necessarily hold in all the structures.

In the case of the tertiary level students, the validity of the zero-product property in the context of real numbers greatly influenced their answers. In the interviews they stated explicitly that when they gave their answer (that one of the factors must be zero) they thought of this property in the set of the real numbers. For these students this property appears extremely linked to experience, in the sense that they try to apply it whenever possible, factoring in a convenient way, as they themselves pointed out.

Besides the experience in applying the property in familiar situations and contexts, we consider that its incorrect extension is favored by the textbooks that give the rule as “if $a \cdot b = 0$ then $a = 0$ or $b = 0$ ”, without specifying what a and b represent, and without warning that its validity is not

universal. For example, in an Uruguayan textbook⁹ for early secondary students we find the following statement about the multiplicative property of zero:

$a \times 0 = 0 \times a = a$ $\forall a \in N$	PROPIEDAD DE ABSORCIÓN
---	------------------------

Later in the same page, the following statement appears:

$Si a \times b = 0 \Rightarrow a = 0 \ y/o \ b = 0$	PROPIEDAD HANKELIANA
---	----------------------

We can see that in the case of the multiplicative property of zero (known in Uruguay as the property of absorption) what “*a*” represents is clearly specified (in this case a natural number). However the same thing does not happen in the zero-product property that is frequently known in Uruguay, as we have already said, as the Hankelian property.

This could lead the students to consolidate a thought model that does not include the nature of the objects *a* and *b*, fixing the attention only in the syntax of the writing. According to English and Halford (1995, p. 230), students frequently generate ‘malrules’ by constructing prototype rules whose surface structure corresponds to the writing of a property. We think that the syntactic features of the writing would favor the application of the property by students, in contexts where it is not valid. While the mathematical objects change, the visual syntactic features remain practically unchanged, giving rise to a mental image such as:

@ . € = 0 ⇒ @ = 0 or € = 0, where the symbols @ and € can be replaced with anything.

We think that the rule in question could have the characteristics of an implicit model of thought which is based on the visual syntactic features of the expression involved. Since in their experience students reinforce constantly the validity of that rule in the context of real numbers, they find it difficult to incorporate further and new information that goes contrary to this experience (Fischbein, 1987, p. 194-195), as in the case of the multiplication of matrices.

Another possible interpretation can be made if we refer to the work of Tirosh and Stavy (1999) about intuitive rules: Students could be applying a rule of the type “Same hypothesis (null product of two factors) - Same conclusion (one of the factors is null)”, without paying attention to the semantic aspects of the mathematical objects involved. In particular, the generalization of this property to the case of the first question above, where the nature of the factors was not specified, seems to support this hypothesis.

Later, in order to observe the reactions of early and late secondary level students before a structure that admits divisors of zero, they were given a sequence of activities on residual classes modulo 6. None of the two groups had previous experience with this topic. Here we present part of the sequence:

(I) We will work with the elements of set A . The set A is the following:

$$A = \{0, 1, 2, 3, 4, 5\}$$

We will define an operation that we will call multiplication and we will represent it with the symbol: \bullet

This operation works according to the following table:

\bullet	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

For example, to compute $2 \bullet 5$ you have to look for the intersection of the line and the column as shown in the table above, and you obtain 4 as a result.

a) Using the multiplication table that appears above, calculate:

$$4 \bullet 2 =$$

$$3 \bullet 3 =$$

$$5 \bullet 4 =$$

b) Using the table, find the values of x that satisfy the equation $3 \bullet x = 0$

Write here the value or the values you have found for x

Explain what you did.

c) a and b are elements of the set A as above. It is known that $a \bullet b = 0$; based on this information, what can you say about a and b ? Explain your answer.

In this activity students were asked what they could conclude about two factors whose product is zero ((Ic) of the sequence). While the youngest students took into account the existence of divisors of zero giving correct responses, most of the late secondary students gave the answer that one of the factors had to be zero. Hence the youngest students demonstrated a more versatile thought than that of late secondary level students; this might be because the experience with structures without divisors of zero had still not greatly influenced the consolidation of stable schemes of thought.

From the point of view of the didactical strategies that are commonly used in classrooms, the arithmetic operations and their properties constitute the entrance to the understanding of the algebraic operations and their properties. From a cognitive point of view this first interpretation could block later and more abstract generalizations as Fischbein (1987, p.198) points out. Let us add to this the epistemological element that we find in the work of the mathematician Peacock (1791-1858) when he built the axioms of the symbolic algebra starting from the fundamental laws of arithmetic. According to Boyer (1992, p. 711), Peacock provoked, unintentionally, a stagnation in the evolution of algebra when he institutionalized the universal validity of these laws because he suggested that they remain the same and do not depend on the mathematical object. The laws he wrote did not include, for example, the existence of operations that were not commutative.

Didactical suggestions

In relation with the first phenomenon (not applying the zero-product property) and on the evidences obtained in our study, we can suggest that educators make more emphasis on how an equation is solved rather than on why it can be solved more efficiently in one way or another. As a pedagogical strategy, we think that before early secondary students are taught to apply the zero-product property to solve equations, they can be faced explicitly with appropriate second degree equations, given both in expanded and factored forms. In this way, they can realize that the *sui generis* procedures that they use to discover the unknown, are in most of the cases ineffective. This way, on the one hand they may value the tool that is to be taught, and on the other hand they can see that it is not always necessary to carry out a long sequence of operations to be able to solve an equation, which is what many of them believe. Of course this strategy has to be coupled with methods to allow the students to see whether a particular answer is correct or not.

About the second phenomenon (making a double assignment in verifying) we can say that the students who participated in this study seemed to have it clear that the variable can take one value at a time when calculating the image values of a function. We thus suggest that a possible didactic alternative to avoid this error could be to teach the resolution of equations in the context of functions; that is, ask the students to find the roots of a real function f given in the form $f(x) = (ax + b)(cx + d)$. This way the students could solve the equation $(ax + b)(cx + d) = 0$ in order to find the roots of the function f . To verify that the real numbers they have found are, in fact, the roots of the function the students could analyze if each one has image zero under f . In this way they would be calculating images, so we think that perhaps the double assignment would not occur since they would be focusing on a single root at a time. However more research is needed to find out if this approach would result in a different outcome.

As a didactic suggestion in relation with the third phenomenon (generalizing the zero-product property to other structures) we propose the possibility that the study of Algebra not only begins starting from the arithmetic operations and their properties, but also puts the students in contact with other structures that are within their reach and that offer them a wider vision of the algebraic properties. For example, after students work with natural numbers and whole number division, we can introduce the concept of residual classes to early secondary students, by means of activities such the following (Ochoviet, 1999):

*The company that supplies mineral water “Coolish”, divided the city of Montevideo in 164 zones in order to make the punctual weekly delivery of this bottled water in each home.
Mrs. Mary Jo, who organizes the delivery, established the following chronogram taking into account the number of the zone.*

<i>Monday</i>	<i>Tuesday</i>	<i>Wednesday</i>	<i>Thursday</i>	<i>Friday</i>
<i>1</i>		<i>2</i>		<i>3</i>
	<i>5</i>		<i>4</i>	
<i>6</i>		<i>7</i>		<i>8</i>
	<i>10</i>		<i>9</i>	
<i>11</i>		<i>12</i>		<i>13</i>
	
.....	

Find out which day of the week the company delivers the mineral water to zone 164.

After students work with activities that lead them to the concept of residual classes we can offer them others such as the one we used in the present research, where students worked with residual classes modulo 6 and experienced the existence of zero divisors.

This way we believe that the obstacles can be minimized so that in future the students can conceptualize more abstract or general structures.

Final comments

Understanding of the zero-product property which has been the subject of this article is very important for students at all levels, starting with the secondary level. Furthermore it is a topic that encompasses different facets of algebraic activity and can serve as a source for design of

suitable mathematical situations for different age levels. Difficulties associated with it can be related to the understanding of the concept of variable, structural thinking in Algebra and the understanding of procedures that are linked to mathematical properties. Further research can point out the nature of these relationships and focus on the design of appropriate didactical strategies to overcome possible obstacles.

References

- Aguilar, P. & Oktaç, A. (2004) Generación del conflicto cognitivo a través de una actividad de criptografía que involucra operaciones binarias. *Revista Latinoamericana de Investigación en Matemática Educativa* 7(2), 117-144.
- Bednarz, N., Kieran, C. & Lee, L. (1996). Approaches to Algebra: Perspectives for Research and Teaching. In N. Bednarz, C. Kieran & L. Lee (Eds), *Approaches to Algebra: Perspectives for Research and Teaching* (pp. 3 – 12). The Netherlands: Kluwer Academic Publishers.
- Boyer, Carl. (1992). *Historia de la matemática*. Madrid: Alianza Universidad Textos.
- Corry, L. (1996). *Modern Algebra and the Rise of Mathematical Structures*. Basel and Boston: Birkhäuser Verlag.
- English, L. & Halford, G. (1995). *Mathematics Education. Models and Processes*. Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.
- Fischbein, E. (1987). *Intuition in Science and Mathematics. An Educational Approach*. Dordrecht: D. Reidel Publishing Company.
- Filloy, E. & Rojano, T. (1984) From an arithmetical to an algebraic thought: A clinical study with 12-13 years old. *Proceedings of the Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 51-56.
- Fujii, T. (2003) Probing Students' Understanding of Variables Through Cognitive Conflict Problems: Is the Concept of a Variable so Difficult for Students to Understand? *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education*, vol. 1, 49-65.
- Kieran, C. (2004). The Core of Algebra: Reflections on its Main Activities. In K. Stacey, H. Chick & M. Kendal (Eds.) *The Future of the Teaching and Learning of Algebra. The 12th ICMI Study*. (pp. 21 – 33). Boston: Kluwer Academic Publishers.
- Lima, R. N. & Tall, D. (2006). The concept of equation: what have students met before? *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, Prague, Czech Republic, vol. 4, 233–241.
- Ochoviet, C. (1999). $2 \times 3 = 0?$. *Revista do Professor de Matemática*, 41, 30-33. San Pablo: Sociedade Brasileira de Matemática.

Tirosh, D., & Stavy, R. (1999). Intuitive rules: A way to explain and predict students' reasoning. *Educational Studies in Mathematics*, **38** (1-3), 51-66.

Trigueros, M. & Ursini, S. (2003). First-year Undergraduates' Difficulties in Working with Different Uses of Variable, *Research in Collegiate Mathematics Education*, Vol. 12, 1-29.

Vaiyavutjamai, P. & Clements, M.A. (2006). Effects of Classroom Instruction on Students' Understanding of Quadratic Equations. *Mathematics Education Research Journal*, **18**(1), 47-77.

Vaiyavutjamai, P., Ellerton, N.F., & Clements, M.A. (2005). Students' attempts to solve two quadratic equations: A study in three nations. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practice* (proceedings of MERGA 28, Vol. 2, pp. 735-742). Melbourne, Australia: MERGA Program Committee.

Van Engen, H. (1961a) A Note on "Variable". *The Mathematics Teacher*, March, 172-173.

Van Engen, H. (1961b) On "Variable" _a rebuttal. *The Mathematics Teacher*, March, 175-177.

Vinner, S. (1990). *Inconsistencies: Their Causes and Function in Learning Mathematics*. Focus on Learning Problems in Mathematics, **12** (3 & 4), 85-98.

Wikipedia. Consulted on January 23, 2007.

http://en.wikipedia.org/wiki/Zero-product_property

Endnotes

This research study has been funded by Project Conacyt 2002-C01-41726S.

[2] Infantozzi, C. (1958). *Número Natural*. Montevideo: La casa del estudiante. Pp. 22.

[3] The same denomination appears in: Infantozzi, C. (1975). *Número Real*. Montevideo: Librería Selecta Editorial. Pp. 39.

[4] Borbonet, M., Burgos, B., Martínez, A. & Ravaioli, N. (1995). *Matemática 1*. Montevideo: Editorial Fin de Siglo. Pp. 23.

[5] Instituto de Matemática y Estadística "Prof. Ing. Rafael Laguardia". (2005). *Geometría y Álgebra Lineal 1*. Montevideo: Facultad de Ingeniería. Universidad de la República. Pp. 185.

[6] Specifically, they are preservice mathematics teachers.

[7] Belcredi, L. y Zambra, M. (1998). *Gauss 3*. Montevideo: La Flor del Itapebí. Pp. 38.

[8] The teacher refers to the correct attempt (one assignment at a time) and the wrong one (double assignment), as both have been done by the student at different moments of the interview.

[9] Borbonet, M., Burgos, B., Martínez, A. & Ravaioli, N. (1995). *Matemática 1*. Montevideo: Editorial Fin de Siglo. Pp.23. See also Apostol, T. (1972). *Calculus*. Volumen 1. (2ª Ed). España: Editorial Reverté S. A. Pp. 23.

APPENDIX

Questionnaire 1

- 1) i) Solve the equation $(2x - 6)(18 - 2x) = 0$. Explain how you do it.
 ii) How many solutions did you obtain? _____ What are they? _____
 iii) Verify the solution(s) that you obtained.
- 2) i) Solve the equation $(x + 6)(2x - 8) = 0$. Explain how you do it.
 ii) How many solutions did you obtain? _____ What are they? _____
 iii) Verify the solution(s) that you obtained.
- 3) i) Solve the equation $(3x - 6)(x - 7) = 0$. Explain how you do it.
 ii) How many solutions did you obtain? _____ What are they? _____
 iii) Verify the solution(s) that you obtained.
- 4) i) Solve the equation $(x - 5)(x + 4) = 0$. Explain how you do it.
 ii) How many solutions did you obtain? _____ What are they? _____
 iii) Verify the solution(s) that you obtained.
- 5) i) Solve the equation $(x - 9)(x - 6) = 0$. Explain how you do it.
 ii) How many solutions did you obtain? _____ What are they? _____
 iii) Verify the solution(s) that you obtained.
- 6) i) Solve the equation $x(2x - 10) = 0$. Explain how you do it.
 ii) How many solutions did you obtain? _____ What are they? _____
 iii) Verify the solution(s) that you obtained.
- 7) i) Solve the equation $x(x - 8) = 0$. Explain how you do it.
 ii) How many solutions did you obtain? _____ What are they? _____
 iii) Verify the solution(s) that you obtained.
- 8) i) Solve the equation $x^2 = 6x$. Explain how you do it.
 ii) How many solutions did you obtain? _____ What are they? _____
 iii) Verify the solution(s) that you obtained.
- 9) i) Solve the equation $5x = 0$. Explain how you do it.
 ii) How many solutions did you obtain? _____ What are they? _____
 iii) Verify the solution(s) that you obtained.
- 10) Give an equation with 8 as a root.
- 11) Give an equation with 4 and 3 as its roots. How do you do it?
- 12) In the equation $(2x - 4)(\dots) = 0$ we do not know the second factor.
 Is 2 a root of the equation? Why?
 Is 3 a root of the equation? Why?
- 13) The papers are hiding numbers, can you find them? Explain your reasoning.

$$(\square - 6)(\square - 19) = 0$$
- 14) Is 7 a root of the equation $(3x - 21)(x - 3) = 0$? Explain your answer.
- 15) Are 6 and 2 roots of the equation $(2x - 12)(5x - 10) = 0$? Explain your answer.
- 16) Are 5 and 4 roots of the equation $(2x - 10)(3x - 8) = 0$? Explain your answer.
- 17) Are 3 and 4 roots of the equation $x^2 - 7x + 12 = 0$?
 Explain your answer, clarifying whatever you think is appropriate.
- 18) i) We know that $b \cdot d = 0$. From this information, what can you conclude about b and d ?
 ii) What do b and d represent for you?

Questionnaire 2

1) Solve in \mathbb{R} the equation $(2x - 6)(5x + 10) = 0$.

Verify the solution(s) that you obtained.

2) It is known that $b.c = 0$. Based on this information, what can you conclude about b and c ? What do b and c represent for you?

3) f and g are two functions whose domain is \mathbb{R} . It is known that $f.g = 0$, that is to say that the product of f and g is the zero function. Based on this information, what can you conclude about f and g ?

4) D and B are two matrices. It is known that $D.B = 0$, that is to say that the product of the two matrices is the zero matrix. Based on this information, what can you conclude about D and B ?

5) p and q are two polynomials. It is known that $p.q = 0$, that is to say that the product of the two polynomials is the null polynomial. Based on this information, what can you conclude about p and q ?

6) It is known that $b.c = 0$. Based on this information, what can you conclude about b and c ? What do b and c represent for you?

THE ORIGINS OF THE GENUS CONCEPT IN QUADRATIC FORMS

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ABSTRACT: We present an elementary exposition of genus theory for integral binary quadratic forms, placed in a historical context.

KEY WORDS: Quadratic Forms, Genus, Characters
AMS Subject Classification: 01A50, 01A55 and 11E16.

INTRODUCTION: Gauss once famously remarked that “mathematics is the queen of the sciences and the theory of numbers is the queen of mathematics”. Published in 1801, Gauss’ *Disquisitiones Arithmeticae* stands as one of the crowning achievements of number theory. The theory of binary quadratic forms occupies a large swath of the *Disquisitiones*; one of the unifying ideas in Gauss’ development of quadratic forms is the concept of genus. The generations following Gauss generalized the concepts of genus and class group far beyond what Gauss had done, and students approaching the subject today can easily lose sight of the basic idea.

Our goal is to give a heuristic description of the concept of genus – accessible to those with limited background in number theory – and place it in a historical context. We do not pretend to give the most general treatment of the topic, but rather to show how the idea originally developed and how Gauss’ original definition implies the more common definition found in today’s texts.

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BASIC DEFINITIONS: An integral binary quadratic form is a polynomial of the type $f(x, y) = ax^2 + bxy + cy^2$, where a , b , and c are integers. A form is *primitive* if the integers a , b , and c are relatively prime. Note that any form is an integer multiple of a primitive form. Throughout, we will assume that all forms are primitive. We say that a form f represents an integer n if $f(x, y) = n$ has an integer solution; the representation is *proper* if the integers x, y are relatively prime. A form is *positive definite* if it represents only positive integers; we will restrict our discussion to positive definite forms.

The *discriminant* of $f = ax^2 + bxy + cy^2$ is defined as $\Delta = b^2 - 4ac$. Observe that $4af(x, y) = (2ax + by)^2 - \Delta y^2$. Thus, if $\Delta < 0$, the form represents only positive integers or only negative integers, depending on the sign of a . In particular, if $\Delta < 0$ and $a > 0$ then $f(x, y)$ is positive definite. Moreover, $\Delta = b^2 - 4ac$ implies that $\Delta \equiv b^2 \pmod{4}$. Thus we have $\Delta \equiv 0 \pmod{4}$ or $\Delta \equiv 1 \pmod{4}$, depending on whether b is even or odd. Moreover, we will write $(\mathbb{Z} / \Delta\mathbb{Z})^*$ to denote the multiplicative group of congruence classes which are relatively prime to Δ .

We say that an integer a is a quadratic residue of p if $x^2 \equiv a \pmod{p}$ has a solution. When discussing quadratic residues, it is convenient to use Legendre symbols. If p is an odd prime and a an integer relatively prime to p , then $\left(\frac{a}{p}\right)$ is defined as follows:

DEFINITION:
$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a solution} \\ -1 & \text{otherwise} \end{cases}$$

This notation allows us to concisely state some well-known facts about quadratic residues; here p, q are distinct odd primes:

$$\text{i) } \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \qquad \text{ii) } \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$$

$$\text{iii) } \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} \quad \text{iv) } \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right).$$

Item (iii) is called the Quadratic Reciprocity Law; discovered independently by Euler and Legendre, the first correct proof appeared in Gauss' *Disquisitiones*. Items (i) and (ii) are known as the First and Second Supplements to Quadratic Reciprocity and were proved by Euler (1749) and Legendre (1785) respectively.

More generally, let $m = p_1 p_2 \cdots p_k$, and let a be any positive integer. The Jacobi symbol is defined as $\left(\frac{a}{m}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right) \cdots \left(\frac{a}{p_k}\right)$. Observe that if a is a quadratic residue

modulo m , then $\left(\frac{a}{m}\right) = 1$, but the converse is not true. The Jacobi symbol has many of the same basic properties as the Legendre symbol; in particular the four results above are valid when p and q are replaced by arbitrary odd integers. The Jacobi symbol also satisfies $\left(\frac{a}{m}\right)\left(\frac{a}{n}\right) = \left(\frac{a}{mn}\right)$. The reciprocity law for Jacobi symbols was also proved by

Gauss [7, Art 133], and can be stated as follows: If m and n are odd integers, then $\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right)$ if either of $m, n \equiv 1 \pmod{4}$ and $\left(\frac{m}{n}\right) = -\left(\frac{n}{m}\right)$ if $m \equiv n \equiv 3 \pmod{4}$.

HISTORICAL BACKGROUND: The earliest investigations concerning the representation of integers by binary quadratic forms were due to Fermat. In correspondence to Pascal and Marseenne, he claimed to have proved the following:

THEOREM

1:

1. Every prime number of the form $4k + 1$ can be represented by $x^2 + y^2$.
2. Every prime number of the form $3k + 1$ can be represented by $x^2 + 3y^2$.
3. Every prime number of the form $8k + 1$ or $8k + 3$ can be represented by $x^2 + 2y^2$.

These results motivated much later research on arithmetic quadratic forms by Euler and Lagrange. Beginning in 1730, Euler set out to prove Fermat's results; he succeeded in proving (1) in 1749 (as well as the more general Two-Square Theorem), and made significant progress on the other two [1]. In a 1744 paper titled *Theoremata circa*

divisors numerorum in hac forma paa ± qbb contentorum, Euler recorded many examples and formulated many similar conjectures (presented as theorems). It was in this paper that he also established many basic facts about quadratic residues. His most general result along these lines was the following:

THEOREM 2: Let n be a nonzero integer, and let p be an odd prime relatively prime to n . Then $p \mid x^2 + ny^2$, $\gcd(x, y) = 1 \Leftrightarrow \left(\frac{-n}{p}\right) = 1$.

In 1773, Lagrange published the landmark paper “*Recherches d’arithmetique*”, in which he succeeded in proving Fermat’s conjectures concerning primes represented by the forms $x^2 + 2y^2$ and $x^2 + 3y^2$. The same paper contains a general development of the theory of binary quadratic forms, treating forms of the type $f = ax^2 + bxy + cy^2$. Lagrange’s development of the theory is systematic and rigorous – it is here that he introduces the crucial concepts of discriminant, equivalence, and reduction. One of the first results is a connection between quadratic residues and the representation problem for general quadratic forms:

THEOREM 3: Let m be a natural number that is represented by the form $ax^2 + bxy + cy^2$. Then $\Delta = b^2 - 4ac$ is a quadratic residue modulo m .

One of Lagrange’s primary innovations was the concept of equivalence of forms (although the terminology is due to Gauss). We say that two forms are *equivalent* if one can be transformed into the other by an invertible integral linear substitution of variables. That is, f and g are equivalent if there are integers p, q, r , and s such that $f(x, y) = g(px + qy, rx + sy)$ and $ps - qr = \pm 1$. It can be shown (e.g. see [6] or [11]) that equivalence of forms is indeed an equivalence relation. Moreover, equivalent forms have the same discriminant and represent the same integers (the same is true for proper representation). Gauss later refined this idea by introducing the notion of proper equivalence. An equivalence is a proper equivalence if $ps - qr = 1$, and it is an improper equivalence if $ps - qr = -1$. Following Gauss, we will say that two forms are in the same class if they are properly equivalent. Using these ideas, we obtain the following

partial converse of Theorem 3:

THEOREM 4: Let p be an odd prime. Then p is represented by a form of discriminant Δ

if and only if $\left(\frac{\Delta}{p}\right) = 1$.

Proof: Let $f = ax^2 + bxy + cy^2$ represent p , say $p = ar^2 + brs + cs^2$. Because p is prime, we must have $\gcd(r, s) = 1$. Hence, we can write $1 = ru - st$ for integers t, u . If $g(x, y) = f(rx + ty, sx + uy)$, then g is properly equivalent to f and thus has discriminant Δ . Moreover, by direct calculation we have $g = px^2 + b'xy + c'y^2$. Thus, $\Delta = b'^2 - 4pc'$ and so $b'^2 \equiv \Delta \pmod{p}$.

Next, suppose that $m^2 \equiv \Delta \pmod{p}$. We can assume that m has the same parity as Δ (replacing m by $m + p$ if necessary). Writing $m^2 - \Delta = kp$, and recalling that $\Delta \equiv 0$ or $1 \pmod{4}$, we have $kp \equiv 0 \pmod{4}$. Thus the form $px^2 + mxy + (k/4)y^2$ has integer coefficients and represents p

Once we have partitioned the set of binary quadratic forms into equivalence classes, the next logical step is to choose an appropriate representative for each class. This naturally leads another of Lagrange's innovations, the concept of reduction. A primitive positive definite form $ax^2 + bxy + cy^2$ is said to be *reduced* if $|b| \leq a \leq c$ and $b \geq 0$ if either $|b| = a$ or $a = c$. Lagrange showed that every primitive positive definite form is properly equivalent to a unique reduced form, and that there are only finitely many positive definite forms with a given determinant Δ [6, 11]. We write $h(\Delta)$ for the number of classes of primitive positive definite forms of discriminant Δ . Thus, $h(\Delta)$ is the number of reduced forms of discriminant Δ .

In the special case where $h(-4n) = 1$, the only reduced form of discriminant $-4n$ will be the form $x^2 + ny^2$. In this case, $p = x^2 + ny^2 \Leftrightarrow \left(\frac{-n}{p}\right) = 1$. This situation is in fact quite rare – Gauss conjectured that the only values of n for which $h(-4n) = 1$ are $n = 1, 2, 3, 4,$ and 7 . The conjecture was proved by Landau in 1903. More generally,

we call Δ a *fundamental discriminant* if it cannot be written as $\Delta = k^2\Delta_0$, where $k > 1$ and $\Delta_0 \equiv 0$ or $1 \pmod{4}$. Gauss conjectured that if $\Delta < 0$ is a fundamental discriminant then $h(\Delta) = 1$ only for $\Delta = -3, -4, -7, -8, -11, -19, -43, -67, -163$. This was proved in 1952 by Heegner [12].

GENUS THEORY: We say that two primitive positive definite forms of discriminant Δ are in the same *genus* if they represent the same values in $(\mathbb{Z}/\Delta\mathbb{Z})^*$. Recall that equivalent forms represent the same integers and so must be in the same genus. Thus, the concept of genus provides a method of separating reduced forms of the same discriminant according to congruence classes represented by the forms. In his table of reduced forms, Lagrange showed forms grouped according to the congruence classes represented by the forms. For this reason, many authors credit the original idea of genus to Lagrange. Some authors have even attributed the idea to Euler [10]. However, Gauss is the first to explicitly discuss the concept of genus. More importantly, he is the first to put it to use.

Before presenting Gauss' definition of genus, a few remarks concerning notation and terminology are in order. Throughout most of the *Disquisitiones Arithmetica*, Gauss assumes forms have even middle coefficient – that is, he mostly considers forms of type $ax^2 + 2bxy + cy^2$. (Forms with odd middle coefficient are called “improperly primitive”, and are treated separately.) Instead of discriminants, he uses the *determinant* of the form, defined as $D = b^2 - ac$. Note that the discriminant Δ satisfies $\Delta = 4D$.

The following result, found in Article 229 of *Disquisitiones Arithmetica*, is the foundation of genus theory. The proof is paraphrased slightly from the original text.

THEOREM 5: Let F be a primitive form with determinant D and p a prime number dividing D : then the numbers not divisible by p which can be represented by the form F agree in that they are either all quadratic residues of p , or they are all nonresidues.

Proof: Let $m = ag^2 + 2bgh + ch^2$ and $m' = ag'^2 + 2bg'h' + ch'^2$. Then

$$mm' = [agg' + b(gh' + hg') + chh']^2 - D(gh' - hg')^2.$$

Thus mm' is a quadratic residue mod D , and hence is also a quadratic residue mod p for any p dividing D . It follows that m, m' are either both residues, or both are non-residues mod p . That is, if m and m' are both represented by F , then $\left(\frac{m}{p}\right) = \left(\frac{m'}{p}\right) \square$

From the relation $\Delta = 4D$ we get two important observations: First, any odd prime that divides D also divides Δ . Moreover, if p is an odd prime, then Δ is a residue mod p if and only if D is. Thus Theorem 5 still holds if the word determinant is replaced by discriminant. Henceforth, we will revert to the more common practice of using discriminants.

The argument used to prove Theorem 5 also shows that if $8 \mid D$ or $4 \mid D$, then the product of two numbers represented by F will be a quadratic residue mod 8 or a quadratic residue mod 4, respectively. Hence if $8 \mid D$, then exactly one of the following is true: all numbers represented by F are $\equiv 1 \pmod{8}$, or all are $\equiv 3 \pmod{8}$, or all are $\equiv 5 \pmod{8}$, or all are $\equiv 7 \pmod{8}$. Likewise, if $4 \mid D$, but $8 \nmid D$, then all numbers represented by F are $\equiv 1 \pmod{4}$, or all are $\equiv 3 \pmod{4}$.

These observations are then used to classify forms according to *characters*. Let p_1, p_2, \dots, p_k be the odd prime divisors of D . Define $\chi_i = Rp_i$ if the numbers represented by F are quadratic residues of p_i , and $\chi_i = Np_i$ if the numbers represented by F are quadratic non-residues of p_i . We define one additional character, χ_0 , which will be an ordered pair a, b chosen from the list $\{(1,4), (3,4), (1,8), (3,8), (5,8), (7,8)\}$, where all numbers m represented by the form f satisfy $m \equiv a \pmod{b}$. For example, we write $\chi_0 = 1,4$ to indicate that all numbers represented by the form are congruent to 1 mod 4. Finally, the *complete character* for a form is then defined as: $\chi_0; \chi_1, \chi_2, \dots, \chi_k$. Two forms then said to be in the same *genus* if they have the same complete character.

In Article 231, Gauss discusses the possibilities for χ_0 based on the prime factorization of the determinant, as well as the number of potential complete characters in each case. In each case, the number of potential complete characters is a power of 2.

Let $p_1 p_2 \cdots p_k$ be all of the odd primes dividing Δ . We summarize the results in the table below:

Δ	Possible χ_0	Number of potential complete characters
$\Delta = 8 \cdot 2^r \cdot p_1 p_2 \cdots p_k$ ($r \geq 0$)	1,8 3,8 5,8 7,8	2^{k+2}
$\Delta = 4 \cdot p_1 p_2 \cdots p_k$	1,4 3,4	2^{k+1}
$\Delta = p_1 p_2 \cdots p_k \equiv 1 \pmod{4}$	1,4	2^k

Table

1

EXAMPLE: Let $\Delta = -55$; then $\chi_0 = 1,4$ and there are four reduced forms:

$$f_1 = x^2 + xy + 14y^2, \quad f_2 = 2x^2 + xy + 7y^2$$

$$f_3 = 2x^2 - xy + 7y^2, \quad f_4 = 4x^2 + 3xy + 4y^2$$

f_1 represents 1, and 1 is a residue for any prime p , so the complete character for f_1 is 1,4; $R5, R11$. f_2 and f_3 each represent 2, which is a non-residue mod 5 and mod 11, so the complete character for each of these forms is 1,4; $N5, N11$. Finally, f_4 represents 4, which is a residue modulo any odd prime p . Thus the complete character for f_4 is $R5, R11$.

It follows that there are two genera, each with two proper equivalence classes:

Complete Character

Reduced Forms

1,4; $R5, R11$

$$f_1 = x^2 + xy + 14y^2, \quad f_4 = 4x^2 + 3xy + 4y^2$$

1,4; $N5, N11$

$$f_2 = 2x^2 + xy + 7y^2, \quad f_3 = 2x^2 - xy + 7y^2$$

Note that f_2, f_3 are equivalent, so they must be in the same genus. However, they are *not* properly equivalent since $f_3 = f_2(-x, y)$. Thus they represent two distinct elements within the genus.

Observe also that in the example above, there were four possible complete characters, but only two actually defined a genus. In Articles 261 and 287, Gauss

shows that the number of genera is always exactly half the number of possible complete characters and must always be a power of 2. For odd, non-square discriminants, this is easy to see: Let m be an odd integer represented by a form f of odd discriminant Δ , and let p be an odd prime dividing Δ . If R_p is a character, then $\left(\frac{m}{p}\right) = 1$, whereas if N_p is a character, then $\left(\frac{m}{p}\right) = -1$. Replacing the characters by their respective Legendre symbols and multiplying, we get $\left(\frac{m}{p_1}\right)\left(\frac{m}{p_2}\right)\cdots\left(\frac{m}{p_k}\right) = \left(\frac{m}{\Delta}\right)$, where $\left(\frac{m}{\Delta}\right)$ is the Jacobi symbol and $\Delta = p_1 p_2 \cdots p_k$. By reciprocity we have $\left(\frac{m}{\Delta}\right) = (-1)^{(m-1)(\Delta-1)/4} \left(\frac{\Delta}{m}\right)$. Since m is odd and $\Delta \equiv 1 \pmod{4}$, we have $\left(\frac{m}{\Delta}\right) = \left(\frac{\Delta}{m}\right)$. Finally, since m is represented by f , we have $\left(\frac{\Delta}{m}\right) = 1$ by Theorem 3. Thus, for m represented by f , the product of the characters is always 1; if $k-1$ of the characters are known, the k -th is also determined. It follows that there must be 2^{k-1} complete characters.

Reciprocity plays a critical role in the argument above, and this is no accident. In Article 261, Gauss shows that at least half the possible complete characters cannot belong to a genus – this fact serves as the basis of his second proof of the Quadratic Reciprocity [7, Art 262].

The argument above (or Theorem 3) shows that if m is represented by a form of odd discriminant Δ , then $\left(\frac{\Delta}{m}\right) = 1$. Gauss' Theorem 5 then allows us to extend this relationship to elements of $(Z/\Delta Z)^*$. That is, $\chi(\bar{m}) = \left(\frac{\Delta}{m}\right)$ is a well-defined map from $(Z/\Delta Z)^*$ to $\{\pm 1\}$. This is a homomorphism since $\left(\frac{\Delta}{m}\right)\left(\frac{\Delta}{n}\right) = \left(\frac{\Delta}{mn}\right)$. Moreover, this is the *unique* homomorphism $\chi : (Z/\Delta Z)^* \rightarrow \{\pm 1\}$ such that $q \in \ker(\chi)$ if and only if q is represented by a form of discriminant Δ . A famous result of Dirichlet guarantees that there are infinitely many primes in an arithmetic progression, provided the first term and

common difference are relatively prime. Thus, each element of $(Z/\Delta Z)^*$ can be represented as \bar{q} , for some odd prime q not dividing Δ . From this, it follows that the condition $\chi(\bar{q}) = \left(\frac{\Delta}{q}\right)$ for odd primes q determines χ uniquely.

Let $\Delta \equiv 0, 1 \pmod{4}$ be a discriminant. The *principal form* is defined by

$$x^2 - \frac{\Delta}{4}y^2 \quad \text{if } \Delta \equiv 0 \pmod{4}$$

$$x^2 + xy + \frac{1-\Delta}{4}y^2 \quad \text{if } \Delta \equiv 1 \pmod{4}$$

The class and genus containing the principal form are called the *principal class* and *principal genus*, respectively. Note that the principal form has discriminant Δ and is reduced. When $\Delta = -4n$, the principal form is $x^2 + ny^2$. Many fundamental properties of genus can be described in terms of the homomorphism χ and the principal form:

THEOREM 6: Given a negative integer $\Delta \equiv 0, 1 \pmod{4}$, let χ be the homomorphism of Theorem 4, and let f be a form of discriminant Δ .

- i) For an odd prime not dividing Δ , $\bar{p} \in \ker(\chi)$ if and only if p is represented by one of the $h(\Delta)$ forms of discriminant Δ .
- ii) $\ker(\chi)$ is a subgroup of index 2 in $(Z/\Delta Z)^*$
- iii) The values in $(Z/\Delta Z)^*$ represented by the principal form of discriminant Δ form a subgroup $H \subset \ker(\chi)$
- iv) The values in $(Z/\Delta Z)^*$ represented by $f(x, y)$ form a coset of H in $\ker(\chi)$.
- v) For odd Δ , $H = \{x^2 \mid x \in (Z/\Delta Z)^*\}$

Part (i) of the theorem is a restatement of Theorem 3: $\chi(\bar{p}) = \left(\frac{\Delta}{p}\right) = 1$ if and only if p is represented by some form of discriminant Δ . Part (ii) states that exactly half the congruence classes in $(Z/\Delta Z)^*$ are represented by some form of discriminant Δ ; for odd Δ , this follows from our argument that exactly half of all possible complete characters actually result in a genus. Parts (iii) and (iv) get to the heart of genus theory; since distinct cosets are disjoint, different genera represent disjoint classes in $(Z/\Delta Z)^*$. That is, we can now describe genera in terms of cosets kH of H in $\text{Ker}(\chi)$. We could then

define a genus to consist of all forms of discriminant Δ that represent the values of $kH \pmod{\Delta}$. Note that this definition could be used to show that each genus contains the same number of classes [9, Art. 252].

EXAMPLE: Recall that there were four reduced forms of discriminant $\Delta = -55$:

$$\begin{aligned} f_1 &= x^2 + xy + 14y^2, & f_2 &= 2x^2 + xy + 7y^2 \\ f_3 &= 2x^2 - xy + 7y^2, & f_4 &= 4x^2 + 3xy + 4y^2 \end{aligned}$$

There are $\Phi(55) = 55(1 - \frac{1}{5})(1 - \frac{1}{11}) = 40$ elements in $(Z/55Z)^*$. Of these 40 elements, exactly 20 are represented by a form of discriminant -55 .

Since $f_1(x, 0) = x^2$, the principal form $f_1 = x^2 + xy + 14y^2$ represents all of the squares:

$$H = \{1, 4, 9, 14, 16, 26, 31, 34, 36, 49\}$$

Thus the set of classes in $(Z/55Z)^*$ represented by f_1, f_4 is H , which is easily verified to be a subgroup of $(Z/55Z)^*$. Also note that $f_2(0, y) = 7y^2$, so the set of classes represented by f_2, f_3 can be written as $7H = \{2, 7, 8, 13, 17, 18, 28, 32, 43, 52\}$.

Of special interest are those discriminants Δ such that each genus contains exactly one class; in this situation, the primes that are represented by a form of discriminant Δ are determined by congruence conditions mod Δ . (See [2] for details.)

COMPOSITION OF FORMS: The theory of composition is intricately linked to that of genus. Composition of forms was first investigated by Legendre and Lagrange, but the theory was brought to fruition by Gauss, who discovered a remarkable group structure. Gauss' exposition is long and technical, and is one of the most difficult parts of the *Disquisitiones*. However, the main result – that classes of binary quadratic forms of fixed discriminant form an abelian group under the operation of composition – is justly celebrated as one of the milestones of 19th century mathematics. Mathematicians following Gauss were able to streamline the theory considerably.

Gauss showed that any two forms of the same discriminant can be composed in such a way that composition is a well-defined operation on (proper) equivalence classes of forms. For simplicity, we present a version of the operation developed by Dirichlet [2,

3] which is based on a case singled out by Gauss for special consideration [7, Art 242].

We say that $f_1 = a_1x^2 + b_1xy + c_1y^2$ and $f_2 = a_2x^2 + b_2xy + c_2y^2$ are *concordant* (the terminology is due to Dedekind [3]) if the following conditions hold:

$$\text{i) } a_1a_2 \neq 0 \quad \text{ii) } b_1 = b_2 \quad \text{iii) } a_1 \mid c_2 \quad \text{and} \quad a_2 \mid c_1$$

If two concordant forms have the same discriminant, say $b^2 - 4a_1c_1 = b^2 - 4a_2c_2$, then $a_1c_1 = a_2c_2$, and so $c_1/a_2 = c_2/a_1$. We then define the composition of two concordant forms f_1, f_2 of discriminant Δ as $f_1 * f_2 = a_1a_2x^2 + bxy + cy^2$, where $b = b_1 = b_2$ and $c = c_1/a_2 = c_2/a_1$. Dirichlet showed that given two equivalence classes of forms C_1, C_2 , it is always possible to find concordant forms f_1, f_2 with $f_1 \in C_1$ and $f_2 \in C_2$.

Suppose that $f_1 = a_1x_1^2 + bx_1y_1 + a_2cy_1^2$ and $f_2 = a_2x_2^2 + bx_2y_2 + a_1cy_2^2$ are concordant forms. Then setting $X = x_1x_2 - cy_1y_2$ and $Y = a_1x_1y_2 + a_2y_1x_2 + by_1y_2$, we have $(a_1x_1^2 + bx_1y_1 + a_2cy_1^2)(a_2x_2^2 + bx_2y_2 + a_1cy_2^2) = a_1a_2X^2 + bXY + cY^2$ (by direct calculation). Using this identity and the definition of composition given above, we quickly deduce that $f_1 * f_2$ represents m_1m_2 whenever f_1 represents m_1 and f_2 represents m_2 . The following theorem summarizes the main properties of composition [7, Art 242]:

THEOREM 8 [Gauss]: For a fixed discriminant Δ , the set of equivalence classes of primitive positive definite forms comprise an abelian group under the operation of composition. The identity of this group is the class containing the principal form. The class containing the form $ax^2 + bxy + cy^2$ and the class containing its “opposite” $ax^2 - bxy + cy^2$ are inverses.

This group is called the *class group*, and has cardinality $h(\Delta)$. The proof is long and technical, as might be expected; the results themselves represent an unprecedented level of abstraction for their time. Soon after discussing composition of classes, Gauss defines *duplication*: let K and L be proper equivalence classes of forms of discriminant D . If $K * K = L$, then we say that L is obtained by duplication of K . In Article 247, Gauss points out that the duplication of any class lies in the principal genus; in Articles 286-287 he shows the converse, stating that

“it is clear that any properly primitive class of binary forms belonging to the principal genus can be derived from the duplication of some properly primitive class of the same determinant”.

This fact is often referred to in the literature as the Principal Genus Theorem. While the statement is made rather casually (not even stated as a formal theorem), Gauss nonetheless describes it as “among the most beautiful in the theory of binary forms”. (See [12] for a discussion of the many generalizations of this result.)

We conclude with a description of Gauss’ proof of the Principal Genus Theorem. To demonstrate how duplication of any class is in the principal genus, Gauss defines composition of genera, and in doing so describes another group structure. In Article 246, he shows that if f, f' are primitive forms from one genus, and if g, g' are primitive forms from another genus, then the compositions $f * g$ and $f' * g'$ will be in the same genus. He then explains how one can determine the genus of $f * g$ using the characters for f, g respectively. First, he gives a multiplication table for the characters χ_0 ; then he describes multiplication of characters χ_i, χ'_i as Rp_i if $\chi_i = \chi'_i$ and as Np_i if $\chi_i \neq \chi'_i$.

The characters of $f * g$ are then the products of $\chi_i, \chi'_i, i = 0, 1, \dots, k$. If the discriminant Δ is odd, we can illustrate this by replacing the characters by their respective Legendre symbols. Let $\Delta = p_1 p_2 \cdots p_k$ be odd, and let f, g come from the genera G_1, G_2 respectively. Suppose that m is represented by f and that n is represented by g , so the total characters of the forms can be described as $\left(\frac{m}{p_1}\right), \left(\frac{m}{p_2}\right), \dots, \left(\frac{m}{p_k}\right)$ and

$\left(\frac{n}{p_1}\right), \left(\frac{n}{p_2}\right), \dots, \left(\frac{n}{p_k}\right)$ respectively. Then $G_1 * G_2$ is the genus with total character $\left(\frac{mn}{p_1}\right), \left(\frac{mn}{p_2}\right), \dots, \left(\frac{mn}{p_k}\right)$. Note that the principal genus always represents 1, which is a

quadratic residue modulo any prime; that is, $\left(\frac{1}{p_i}\right) = 1$ for all i . Thus the principal genus G is the genus in which all the characters have value 1. On the other hand, if G_i is any

other genus and m is an integer represented by G_i , the characters for $G_i * G_i$ will be

$\left(\frac{m^2}{p_1}\right), \left(\frac{m^2}{p_2}\right), \dots, \left(\frac{m^2}{p_k}\right) = 1, 1, \dots, 1$. Hence $G_i * G_i = G$. Moreover, it follows that the

genera form a group of order 2, whose identity is the principal genus.

BIBLIOGRAPHY

1. M. Beintema and A. Khosravani, "Binary Quadratic Forms: A Historical View", *Mathematics and Computer Education*, **40** (2006), p. 226-236.
2. D.A. Cox, *Primes of the Form $x^2 + ny^2$* , Wiley-Interscience, John Wiley and Sons, New York, 1989.
3. L.E. Dickson, *Theory of Numbers Vol. III: Quadratic and Higher Forms*, Chelsea, New York, 1952.
4. L. Euler, *Oeuvres Vol II*, Gauthier-Villars and Sons, Paris, 1894.
5. P. de Fermat, *Oeuvres Vol II*, Gauthier-Villars and Sons, Paris, 1894.
6. D. Flath, *Introduction to Number Theory*, Wiley, New York, 1989.
7. C.F. Gauss, *Disquisitiones Arithmeticae*, Springer-Verlag, New York, 1986.
8. J.L. Lagrange, "Recherches d'arithmétique", *Oeuvres III*, Gauthier-Villars, Paris, 1867.
9. A.M. Legendre, *Theorie des Nombres*, Paris, 1830; reprint, Blanchard, Paris, 1955.
10. F. Lemmermeyer: "The Development of the Principal Genus Theorem", ArXiv Mathematics e-prints, math/0207306, 2002.
11. W. Scharlau and H. Opolka, *From Fermat to Minkowski, Lectures on the Theory of Numbers and Its Historical Development*, Springer-Verlag, New York, 1985.
12. J.P. Serre, $\Delta = b^2 - 4ac$, *Math. Medley* **13** (1985), pp. 1-10.

THE IMPACTS OF UNDERGRADUATE MATHEMATICS COURSES ON COLLEGE STUDENTS' GEOMETRIC REASONING STAGES

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ABSTRACT

The purpose of this study is to investigate possible effects of different college level mathematics courses on college students' van Hiele levels of geometric understanding. Particularly, since logical reasoning is an important aspect of geometric understanding, it would be interesting to see whether there are differences in van Hiele levels of students who have taken non-geometry courses that emphasize or focus on logic and proofs (Category I) and those that don't (Category II). We compared geometric reasoning stages of students from the two categories. One hundred and forty nine college students taking various courses from the two categories have been involved in this study. The Van Hiele Geometry Test designed to find out students' van Hiele levels was used to collect data. After the collection and analysis of the quantitative data, the participants' van Hiele levels are reported and the reasoning stages of two groups are compared. The results show that students taking logic/proof based courses attain higher reasoning stages than students taking other college level mathematics courses, such as calculus. The results may have implications that are of particular interest to teacher education programs. Finally, the results also confirm a previous assertion about correlation between van Hiele levels and proof writing.

Key Words: van Hiele levels; logic; mathematics courses; college students; geometry; teacher education programs

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INTRODUCTION

Since the mid 1980s there has been a growing interest in the area of teaching and learning geometry (e.g., Crowley, 1987; Gutierrez, Jaime, & Fortuny, 1991; Clements & Battista, 1990; Mason, 1997; Lappan, Fey, Fitzgerald, Friel, & Phillips, 1996; Halat, 2006/2007). The National Council of Teacher of Mathematics (NCTM) (2000) recommends that new ideas, strategies, and research findings be utilized in teaching in order to help students overcome their difficulties in learning mathematics. Knowledge of theoretical principles gives teachers an opportunity to devise practices that have a greater possibility of succeeding (e.g., Swafford, Jones, & Thornton, 1997). Based on over twenty years of research, the van Hiele theory is a well-known and well-regarded theory in geometry, structured and developed by Pierre van Hiele and Dina van Hiele-Geldof between 1957 and 1986. It has its own reasoning stages and instructional phases in geometry.

The van Hieles described five levels of reasoning in geometry. These levels, hierarchical and continuous, are level-I (Visualization), level-II (Analysis), level-III (Ordering), level-IV (Deduction), and level-V (Rigor) (Van Hiele, 1986).

Description of the levels:

Level-I: Visualization or Recognition. At this level students recognize and identify geometric figures according to their appearance, but they do not understand the properties or rules of figures. For example, they can identify a rectangle, and they can recognize it very easily because of its shape, which looks like the shape of a window or the shape of a door.

Level-II: Analysis. At this level students analyze figures in terms of their components and relationships among components and perceive properties or rules of a class of shapes empirically, but properties or rules are perceived as isolated and unrelated. A student should recognize and name properties of geometric figures.

Level-III: Ordering. At this level students logically order and interrelate previously discovered properties and rules by giving informal arguments. Logical implications and class inclusions are understood and recognized.

Level-IV: Deduction. At this level students analyze relationships of systems between figures. They can prove theorems deductively, construct proofs, and they can understand the role of axioms and definitions. A student should be able to supply reasons for steps in a proof.

Level-V: Rigor. At this level students are able to analyze various deductive systems like establishing theorems in different axiomatic systems, and they can compare these systems. A student should be able to know, understand and give information about any kind of geometric figures (e.g., Crowley, 1987; Fuys, Geddes, and Tischler, 1988).

The existence of level-0 is the subject of some controversy (e.g., Usiskin, 1982; Burger & Shaughnessy, 1986). Van Hiele (1986) does not talk about or acknowledge the existence of such a level. However, Clements and Battista (1990) have described and defined level-0 (Pre-recognition) as “Children initially perceive geometric shapes, but attend to only a subset of a shape’s visual characteristic. They are unable to identify many common shapes” (p. 354). For example, learners may see the difference between triangles and quadrilaterals by focusing on the number of sides the polygons have but not be able to distinguish among any of the quadrilaterals (Mason, 1997).

EMPIRICAL RESEARCH ON THE VAN HIELE THEORY

Research has been completed on various components of this teaching and learning model. For instance, Wirszup (1976) reported the first study of the van Hiele theory, which attracted educators' attention at that time in the United States. In 1981, Hoffer worked on the description of the levels. Usiskin (1982) affirmed the validity of the existence of the first four levels in geometry at the high school level. In 1986, Burger and Shaughnessy focused on the characteristics of the van Hiele levels of development in geometry. They stated "students in the study who appeared to reason at different levels used different language and different problem solving processes on the tasks"(p.46). Furthermore, they said that students showed different levels of reasoning on different tasks. Fuys, Geddes, and Tischler (1988) examined the effects of instruction on a student's predominant Van Hiele level. Senk (1989), Mason (1997), and Gutierrez & Jaime (1998) evaluated and assessed the geometric abilities of students as a function of van Hiele levels. The study of Gutierrez, Jaime, & Fortuny (1991) with 9 eighth-grade pupils and 41 future primary school teachers was on an alternative way of analyzing the van Hiele levels of geometric thinking in the solid geometry. According to their study, most future primary teachers' van Hiele levels were level-I (recognition) and -II (analysis), but none of the participants showed level-IV (deduction) reasoning stage.

Mayberry (1983) conducted a study with 19 pre-service elementary school teachers. The tasks employed in her study were designed for the first four levels including seven geometric concepts that were squares, right triangles, isosceles triangles, circles, parallel lines, similarity, and congruence. According to the results of her study (1983), "the finding that 70% of the response patterns of the students who had taken high school geometry were below level-IV (deduction)" (p.68-69). In addition, the response of patterns showed that students who took part in the study were not at the suitable level to understand formal geometry, and that the instruction they had taken had not brought them to level IV (Deduction). The students' responses implied that the typical student in the study was not ready for a formal deductive geometry course (Mayberry, 1983).

Moreover, there have been some studies with pre-service elementary and secondary mathematics teachers regarding their reasoning stages in geometry. For instance, Knight (2006) conducted a research exercise with a total of 68 pre-service mathematics teachers, 46 elementary and 22 secondary. She found that the pre-service elementary and secondary mathematics teachers' reasoning stages were below level-III (informal deduction) and level-IV (deduction), respectively (Knight, 2006). Her findings are surprising because the van Hiele levels of pre-service elementary and secondary mathematics teachers are lower than the level expected of students completing grade 8 and grade 12, respectively. These results are consistent with the findings of Gutierrez, Jaime, & Fortuny (1991), Mayberry (1983), Duatepe (2000), and Olkun, Toluk, & Durmuş (2002). In all of these studies, none of the pre-service elementary and secondary mathematics teachers showed a level-V (Rigor) reasoning stage in geometry. Clearly, this is not a desirable outcome in teacher education.

According to van Hiele (1986), level-III is a transitional stage between informal and formal geometry. Geometry knowledge at this level is constructed by short chains of reasoning about properties of a figure and class inclusions. A student who functions at this level is able to follow a short proof based on properties gained from concrete experiences, but s/he is unable to construct a proof by her/himself. If students perform at the level-IV or -V geometry knowledge

then they will be able to do and write formal proofs. The study showed that although there is no individual van Hiele level that guarantees future success in proof writing, Level-III seems to be the critical entry level. Senk (1989) concluded, “the predictive validity of the van Hiele model was supported. However, the hypothesis that only students at level-IV or-V can write proofs was not supported” (p.309). According to Usiskin & Senk (1990) statements based on the study of Senk (1989), there was a positive correlation between students’ van Hiele levels and proof writing success.

Usiskin & Senk (1990) expressed their surprise at the results of the Senk (1989)’s study about the positive correlation between van Hiele levels and proof writing. They said that the van Hiele geometry test (25-item multiple-choice test) could be used to predict the student’s ability to write proofs. Van Hiele (1986) expressed two implications of the theory: a) students cannot show adequate performances at a level without having had experiences that enable students to reason intuitively at each preceding level. b) a student will not understand the instruction if the student’s reasoning level is lower than the language of instruction. Mayberry (1983), Burger & Shaughnessy (1986) and Fuys et al. (1988) support statements (a) and (b). Van Hiele levels are hierarchical, and the progress from one level to the next is continuous. Furthermore, students’ performance may vary from one concept to another in van Hiele theory. Concept formation in geometry may occur over long periods of time and requires specific interaction (Mayberry, 1983; Gutierrez et al., 1998). Moreover, Burger & Shaughnessy (1986) said that the van Hiele levels of reasoning could function as a basis for constructivist teaching experiments in geometry.

It is also shown that reform-based or NSF-funded standards-based curricula (e.g., Connected Mathematics Project, MATH Thematics, University of Chicago School Mathematics Project, Core-Plus Mathematics Project, and Everyday Mathematics) have more positive effects on students’ learning of mathematics than conventional ones (cf., Fuson, Carroll, & Drucek, 2000; Romberg & Shafer, 2003; Reys, Reys, Lapan, Holliday, & Wasman, 2003; Senk & Thompson, 2003). Moreover, according to the Halat (2007), reform-based geometry curricula had a very favorable impact on the acquisition of the van Hiele levels and motivation in learning geometry.

Burger & Shaughnessy (1986) and Halat (2006) found mostly level-I reasoning in grades K-8 while Fuys et al. (1988) found no one performing above level-II in interviewing sixth and ninth grade average and “above average” students, which supports the idea that most younger students and many adults in the United States reason at levels-I (Visualization), -II (Analysis), -III (Ordering) and -IV (Deduction) of Van Hiele theory (e.g., Usiskin, 1982; Hoffer, 1986; Mayberry, 1983; Knight, 2006). Mayberry (1983) and Fuys, Geddes, & Tischler (1988) stated that content knowledge in geometry among pre-service and in-service middle school teachers is not adequate. There are many factors, such as gender, peer support, age, type of mathematics course, instruction, and so forth that appear to be affecting pre-service mathematics teachers’ or college students’ performance and motivation in mathematics.

The purpose of the Study

The aim of this current study was to investigate possible effects of different college level mathematics courses on college students’ van Hiele levels. Particularly, since logical reasoning is an important aspect of geometric understanding, we were interested in testing whether there are differences in van Hiele levels of students who have taken non-geometry courses that emphasize or focus on logic and proofs (Category I courses) and those that don’t (Category II courses). More information about the two categories is given in the method section.

Furthermore, Usiskin (1982), Mayberry (1983), Burger & Shaughnessy (1986) and Fuys et al. (1988) confirmed the validity of first four levels of the geometric thought (visualization, analysis, abstract, and deduction). They all agreed that the last level, rigor (level-V), was not often seen in high school students. It was more appropriate for college students. This study also aimed to examine this argument. Finally, the results of the study can be interpreted as supporting the previous assertion by Usiskin & Senk (1990) about correlation between van Hiele levels and proof writing.

METHOD

Participants

In this study the researcher followed the “convenience” sampling procedure defined by McMillan (2000), where a group of participants is selected because of availability. Participants in the study were 149 college students divided into two groups, group-I and -II. The group-I consisted of 41 students from Category I courses, those courses that directly use or teach logic or proof writing. Category I courses in this study consist of

- a) An Introduction to Computer Programming course. This is an introductory course with no formal prerequisites. It teaches and uses the programming language C++.
- b) An Introductory Course on Logic, Set Theory and Proof Techniques. This is a sophomore level course and is required for mathematics majors and minors.
- c) A number of advanced mathematics courses that have the introductory proof writing course (Category I-b course above) as a pre-requisite, such as Real Analysis, Abstract Algebra, and other advanced elective mathematics courses (including a course on Euclidean and non-Euclidean geometry which had only 5 students).

The group-II included 108 students taking Category-II mathematics courses, those courses that do not directly use or teach logic or proof writing. In this study, Category II courses included the following courses

- a) An introductory course on statistics
- b) Each one of the three semesters of calculus

It is important to note that

- almost all participating students in both categories took geometry in high school
- None of the students took a geometry course at college level (with a small exception described below)
- None of the courses involved in the study directly teaches any geometry content, except for an advanced mathematics course on Euclidean and non-Euclidean geometry which only had 5 students. This course is only offered every two or three years at this college.

The study took place in a small liberal arts college in a Midwestern state of the US.

The general student profiles for each group were as follows:

Category I-a: There were 12 students in this group. About 40% of students were females, and about 60% were males. Most of the students were first year students, though there were a few students from each class. There were no students who declared mathematics as a major in this course. Several of them were undecided and the rest of them had various non-mathematics majors from a variety of divisions of the liberal arts disciplines.

Category I-b: This was a small group of 6 students with the male-female ratio being exactly equal. All students were either mathematics majors or minors (or strongly considering one).

Category I-c: This group consisted of 23 students. Most of the students were juniors or seniors, with few sophomores. Virtually all of them were mathematics majors (in many cases they were double majors with one of the sciences, Economics, or English). The male-female ratio was around 65% to 35% favoring males.

Category II-a: Though some mathematics majors do take this introductory course, they were not included in this group of size 47. So none of the students included in this group had mathematics as a major, neither did they take any advanced mathematics courses (or any courses from category I). The male-female ratio was around 66% to 34% favoring males. Majority of students were first or second year students, with a few students from the upper classes. There were some undecided students and declared majors spanned a wide spectrum of disciplines.

Category II-b: A total of 61 students included in this group. The male-female ratio was almost equal and a great majority of students were first or second year students. A small percent of students, most of them from the third semester multivariable Calculus, were declared mathematics majors.

Data Sources

The researchers gave participants a geometry test called *Van Hiele Geometry Test (VHGT)*. The VHGT was administered to the participants by the researchers during a single class period. The *Van Hiele Geometry Test (VHGT)* consists of 25 multiple-choice geometry questions. The VHGT is designed to measure students' van Hiele levels in geometry (Usiskin, 1982). The VHGT was given to the participants at or near the completion of the courses at the end of the semester Fall-2006. Due to time limitations and other constraints, we were not able to administer pre-tests to the participants. This is a limiting factor on the conclusions and implications of the study. While we get interesting suggestions from the results of the study, the reader should be cautious about making generalizations from the results in this study. Nevertheless, this study poses some questions and issues for further investigation. A few possible ways to strengthen the study are discussed at the end.

Test Scoring Guide

In this study, the 1-5 scheme was used for the levels. This scheme allows the researchers to use level-0 for students who do not function at what the van Hieles named the ground or basic level. It is also consistent with Pierre van Hiele's numbering of the levels. For this report, all references and all results from research studies using the 0-4 scale have been changed to the 1-5 scheme.

All participants' answer sheets from VHGT were read and scored by the investigators. All participants received a score referring to a van Hiele level from the VHGT guided by Usiskin's grading system.

"For Van Hiele Geometry Test, a student was given or assigned a weighted sum score in the following manner:

- 1 point for meeting criterion on items 1-5 (level-I)
- 2 points for meeting criterion on items 6-10 (level-II)
- 4 points for meeting criterion on items 11-15 (level-III)
- 8 points for meeting criterion on items 16-20 (level-IV)
- 16 points for meeting criterion on items 21-25 (level-V)" (1982, p. 22)

Analysis of Data

The data were responses from students’ answer sheets. In the process of the assessment of participants’ van Hiele levels, the criterion for success at any given level was four out of five correct responses. The researchers ran the independents-samples t-test to compare two groups’ van Hiele levels and to see the effects of the courses from both categories on the participants. Then they constructed frequency tables to get detailed information about distributions of participants’ van Hiele levels.

RESULTS

Table 1 presents the descriptive statistics and the independent samples t-test for college students’ van Hiele levels in both groups, Category-I and –II. According to the table 1, the mean score of group-I (4.02) is numerically higher than that of group-II (2.64). The independent-samples t-test showed that the difference between the groups is statistically significant, [$p < .001$, significant at the $\alpha/2 = .025$ using critical value of $t_{\alpha/2} = 1.96$], favoring the students who took Category-I mathematics courses.

Table 1

Descriptive Statistics and the Independent-Samples T-Test for Students’ van Hiele Levels

	N	M	SD	SE	df	t
Category-I	41	4.02	.90	.14	99.010	7.45*
Category-II	108	2.64	1.24	.12		
Total	149					

Note. * $p < .001$, significant at the $\alpha/2 = .025$ using critical value of $t_{\alpha/2} = 1.96$.

According to Burger & Shaughnessy (1986), the progress through the levels is continuous and not discrete. Despite the fact that students generally are assigned to a single van Hiele level, there may be students who cannot be assigned to a single van Hiele level. Gutierrez, Jaime, & Fortuny (1991) used a 100 - point numerical scale to determine the van Hiele levels of students who reason between two levels. This numerical scale is divided into five qualitative scales: “‘Values in interval’ (0%, 15%) means ‘No Acquisition’ of the level. ‘Values in the interval’ (15%, 40%) means ‘Low Acquisition’ of the level. ‘Values in the interval’ (40%, 60%) means ‘Intermediate Acquisition’ of the level. ‘Values in the interval’ (60%, 85%) means ‘High Acquisition’ of the level. Finally, ‘values in the interval’ (85%, 100%) means ‘Complete Acquisition’ of the level’” (p. 43).

The mean score 4.02 of the group-I can be explained with the scale described above. The score .02 can be placed into the interval named “No Acquisition” of the upper level. In other words, students who were in the group-I completed the level-IV (Deduction), but they have not attained the level-V (Rigor). At level-IV, students analyze relationships of systems between figures. They can prove theorems deductively, construct proofs, and they can understand the role of axioms and definitions. A student should be able to supply reasons for steps in a proof.

On the other hand, the interpretation of the mean' score 2.64 for the group-II would be that students' average van Hiele level falls between levels-II (Analysis) and-III (Informal Deduction) . Using the interval scale, the .64 indicates that there is high acquisition of level -III understanding, but not completed.

Table 2 indicates the participants' reasoning stages in detail. According to the frequency table 2 below, none of the students in group-I showed levels-0 (pre-recognition) and -I (visualization) reasoning stages. Mostly they demonstrated higher levels of thinking, level-IV (34.1%) and -V (36.6%) (see figure 1 below). However, students in group-II showed all geometric thinking stages in different percentiles. Mostly they showed level-III (47.2%) geometry knowledge on the test (see Figure 2).

Table 2
Frequency Table for Students' van Hiele Levels

Groups	N	Level-0	Level-I	Level-II	Level-III	Level-IV	Level-V
		%	%	%	%	%	%
Category-I	41	0	0	4.9	24.4	34.1	36.6
Category-II	108	7.4	13.9	11.2	47.2	15.7	4.6
Total	149						

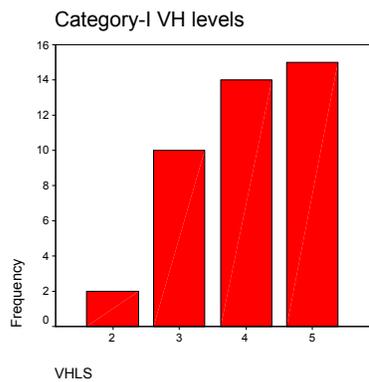


Figure 1

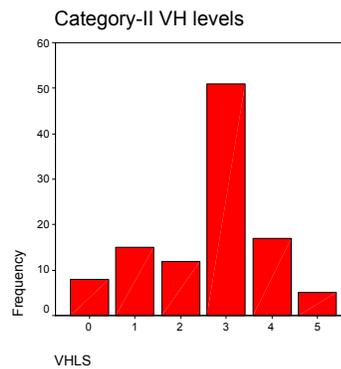


Figure 2

DISCUSSION & CONCLUSION

Students' Van Hiele Levels

Usiskin (1982), Mayberry (1983), Burger & Shaughnessy (1986) and Fuys et al. (1988) confirmed the validity of first four levels of the geometric thought (visualization, analysis, abstract, and deduction). Burger & Shaughnessy (1986) and author (2007) found mostly level-I reasoning in grades K-8 while Fuys et al. (1988) found no one performing above level-II in interviewing sixth and ninth grade average and "above average" students, which supports the idea that most younger students and many adults in the United States reason at levels-I (Visualization), -II (Analysis), -III (Ordering) and -IV (Deduction) of Van Hiele theory (i.e., Usiskin, 1982; Hoffer, 1986; Knight, 2006). They all agreed that the last level, rigor (level-V),

was not suitable for high school students. They stated that it was more appropriate for college students.

However, there were some studies with college students, or pre-service elementary and secondary mathematics teachers regarding their reasoning stages in geometry (e.g., Gutierrez, Jaime, & Fortuny, 1991; Mayberry, 1983; Duatepe, 2000; Olkun, Toluk, & Durmuş, 2002; Knight, 2006). These studies showed that none of the pre-service elementary and secondary mathematics teachers showed a level-V (Rigor) reasoning stage in geometry. This result is in contrast with the argument mentioned above. According to this current study, on the other hand, there were some college students (36.6%) in group-I and (4.6%) in group-II showing level-V (Rigor) reasoning stages.

Based on over twenty years of research, the van Hiele theory is a well-structured and well-known theory having its own reasoning stages and instructional phases in geometry. Many researchers have studied and confirmed different aspects of the theory since proposed by the van Hieles. This current study adds to the set of studies by examining the validity of the level-V (Rigor).

This study supports the research findings claiming that level-V (Rigor) is more appropriate for college students than for high school students. The results can also be interpreted as confirming the previous assertion about correlation between van Hiele levels and proof writing abilities of students.

The Impacts of Taking Higher Level Mathematics Courses on College Students' Van Hiele Reasoning Stages

The main point of this current study was to examine possible effects of different college level mathematics courses on college students' van Hiele levels. Particularly, since logical reasoning is an important aspect of geometric understanding, we were interested in testing whether there are differences in van Hiele levels of students who have taken non-geometry courses that emphasize or focus on logic and proofs (Category I courses) and those that don't (Category II courses).

The analysis of the data revealed that students taking logic/proof based courses attain higher reasoning stages than students taking other college level mathematics courses. The difference between two groups might be attributable to such factors as students' pre-existing knowledge, age, types of courses, and so forth. None of the participants have taken a geometry course since high school, except for five students. Moreover, we collected data for participants' SAT/ACT Math scores in the introductory courses (for the advanced mathematics courses we decided that information was not necessary). The results suggest that SAT-Math scores cannot fully explain the difference. For example, comparing the introductory programming course and second semester of Calculus, the average SAT-Math score in the programming course (704) is lower than the average SAT-Math score in Calculus II (~710), yet average van Hiele level in the programming course is higher. A similar comparison exists between average GPAs of participants in the two courses. We consider this to be an indication that logic/proof based courses enhance students reasoning stages.

Being in different levels in terms of ages or years in school might influence students reasoning stages. For example, most of the students taking category-I courses were juniors or seniors, except for the students in the introductory programming course, but students taking category-II courses were mostly first or second year students. However, when we look at the introductory course from Category I, that is the computer programming course, the ages/years of

the students in that course is comparable to those in Calculus I yet the van Hiele levels of students in the programming course is significantly higher than that of students in Calculus I, 4.09 and 2.62 respectively. On the other hand, according to Fuys, Geddes, & Tischler (1988), students' success in mathematics depends on instruction more than student's age or biological maturation. Putting these two findings together, the difference in terms of geometric reasoning stages between the two groups may be more attributable to the impact of the courses than the ages of the students.

Implications for Teacher Education Programs

This current research has several possible suggestions for both instructors and pre-service teacher education programs. According to Usiskin & Senk (1990), there is a positive correlation between van Hiele levels and proof-writing success in geometry. The results of the current study support their conclusion, assuming that students in Category I courses have better proof writing abilities than students in Category II courses. Though we have not conducted formal measurement of proof writing abilities in either category of students, it is reasonable to assume that students in advanced mathematics courses, all of which require the logic/proof writing course as a prerequisite, who make up the majority of the students in Category I have much better facility in writing mathematical proofs. Moreover, although the Category-I courses are non-geometry courses and the contents of the courses in Category-I are not related to the Euclidean-geometry (except for the small class of 5 students in the geometry course), constructing formal proofs or dealing with logical issues greatly affected students geometric reasoning levels. Therefore, this study suggests that knowledge of students' van Hiele levels might help instructors to better anticipate their students' proof writing abilities.

Furthermore, several studies have shown that many of the prospective teachers do not attain a level of geometry that they are expected to teach (i.e., Gutierrez, Jaime, & Fortuny, 1991; Knight, 2006). This is clearly unacceptable. This study indicates that logic/proof based courses might have a strong positive impact on geometric understanding of students, even without any additional geometry content. Therefore, teacher education programs may want to consider adding such a course to their program requirements.

LIMITATIONS & FUTURE RECOMMENDATIONS

The findings of this study have a limited scope and should not be immediately generalized because of several reasons. Firstly, this study has been carried out in a small, private and selective liberal arts college in the US. Therefore, it would be useful if a future study tests whether similar conclusions hold at a larger scale.

Secondly, we have been able to administer only post-tests to the students. A way to strengthen this study would be to apply both a pre-test and a post-test to see the effects of individual courses more clearly. Regarding prospective teachers in education, a future study might want to specifically target students in such a program. It would be interesting to see if there is a pre-service mathematics teaching program where some students take logic/proof based courses and some not; and if there is such a program whether there is a difference between van Hiele levels of students in each group.

Thirdly, we note that most of the students in Category I courses are mathematics majors and most of the students in Category II are not (either they have different majors or are

undeclared). It would be interesting to investigate if similar results hold amongst students who are not mathematics majors or minors. Researchers might want to compare students taking logic courses, e.g. in philosophy, who have not taken college level mathematics courses with those who have taken neither logic nor mathematics courses in terms of their van Hiele levels.

Finally, the results may be interpreted as confirming a previous assertion by Usiskin & Senk (1990) about positive correlation between van Hiele levels and proof writing abilities of students. We consider the assumption of the hypothesis “students in advanced mathematics courses have better proof writing abilities than students in introductory mathematics courses” to be a reasonable one to take for granted. However, researchers of a future study may want to more explicitly measure the proof writing abilities of students in two categories and the correlation stated above.

REFERENCES

Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17, 31-48.

Clements, D., & Battista, M. (1990). The effects of logo on children’s conceptualizations of angle and polygons. *Journal for Research in Mathematics Education*, 21(5), 356-371.

Crowley, M. (1987). The van Hiele model of development of geometric thought. In M. M. Lindquist, (Ed.), *Learning and teaching geometry, K-12* (pp.1-16). Reston, VA: NCTM.

Duatepe, A. (2000). *An investigation of the relationship between van Hiele geometric level of thinking and demographic variables for pre-service elementary school teachers*. Unpublished Masters’ Thesis, Middle East Technical University.

Fuson, K. C., Carroll, W. M., & Drucek, J. V. (2000). Achievement results for second and third graders using the standards-based curriculum everyday mathematics. *Journal for Research in Mathematics Education*, 31(3), 277-295.

Fuys, D., Geddes, D., & Tischler, R. (1988). The Van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education: Monograph Number 3*.

Gutierrez, A., Jaime, A., & Fortuny, J. (1991). An alternative paradigm to evaluate the acquisition of the van Hiele levels. *Journal for Research in Mathematics Education*, 22, 237-251.

Gutierrez, A., & Jaime, A. (1998). On the assessment of the van Hiele levels of reasoning. *Focus on Learning Problems in Mathematics*, 20(2, 3), 27-45.

Halat, E., (2006). Sex-related differences in the acquisition of the van Hiele levels and motivation in learning geometry. *Asia Pacific Education Review*, 7 (2), 173-183.

Halat, E., (2007). Reform-based curriculum & acquisition of the levels. *Eurasia Journal of Mathematics, Science and Technology Education*, 3 (1), 41-49.

Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher*, 74, 11-18.

Hoffer, A. (1986). Geometry and visual thinking. In T. R. Post (Ed.), *Teaching mathematics in grades K-8: Research based methods* (pp.233-261). Newton, MA: Allyn and Bacon.

Knight, K.C. (2006). *An investigation into the change in the van hiele level of understanding geometry of pre-service elementary and secondary mathematics teachers*. Unpublished Masters Thesis. University of Main.

Lappan, G, Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (1996). *Shapes and design. Two-dimensional geometry*. Palo Alto, CA: Dale Seymour Publications.

Mayberry, J. (1983). The Van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education*, 14, 58-69.

Mason, M. M. (1997). The van Hiele model of geometric understanding and mathematically talented students. *Journal for the Education of the Gifted*, 21(1), 39-53.

McMillan, J. H. (2000). *Educational Research. Fundamentals for the consumers* (3rd ed.). New York: Addison Wesley.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

Olkun, S., Toluk, Z. ve Durmuş, S. (2002). *Sınıf öğretmenliği ve matematik öğretmenliği öğrencilerinin geometrik düşünme düzeyleri*. Orta Doğu Teknik Üniversitesi'nce düzenlenen 5. Ulusal Fen Bilimleri ve Matematik eğitimi Kongresi'nde sunulmuş bildiri, 16-18 Eylül: ODTÜ, Ankara.

Reys, R., Reys, B., Lapan, R., Holliday, G., & Wasman, D. (2003). Assessing the impact of standards-based middle grades mathematics curriculum materials on the student achievement. *Journal for Research in Mathematics Education*, 34(1), 74-95.

Romberg, T. A., & Shafer, M. C. (2003). Mathematics in context (MiC)-Preliminary evidence about student outcome. In S. L. Senk & D. R. Thompson (Eds.), *Standards-based school mathematics curricula. What are they? What do students learn?* (pp. 224-250). Lawrence Erlbaum Associates: NJ.

Senk S. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education*, 20(3), 309-321.

Senk, S. L., & Thompson, D. R. (2003). Middle school mathematics curriculum reform. In S. L. Senk & D. R. Thompson (Eds.), *Standards-based school mathematics curricula. What are they? What do students learn?* (pp.181-192). Lawrence Erlbaum Associates: NJ.

Swafford, O. J., Jones, G. A., & Thornton, C. A. (1997). Increased knowledge in geometry and instructional practice. *Journal for Research in Mathematics Education*, 28(4), 467-483.

Usiskin, Z. (1982). *Van Hiele Levels and Achievement in Secondary School Geometry*. (Final report of the Cognitive Development and Achievement in Secondary School Geometry Project.) Chicago: University of Chicago. (ERIC Document Reproduction Service No. ED220288).

Usiskin, Z., & Senk, S. (1990). Evaluating a Test of van Hiele levels: A response to Crowley and Wilson. *Journal for Research in Mathematics Education*, 21(3), 242-245.

Van Hiele, P.M. (1986). *Structure and insight: A theory of mathematics education*. New York: Academic Press.

Wirszup, I. (1976). Breakthroughs in the psychology of learning and teaching geometry. In J. I. Martin and D. A. Bradbard (Eds.). *Space and geometry: Papers from a Research Workshops*. Columbus, Ohio: ERIC Center for Science, Mathematics and Environment Education.

A LONGITUDINAL STUDY OF STUDENTS' REPRESENTATIONS FOR DIVISION OF FRACTIONS

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Abstract: The representations that students use as part of their mathematical problem solving can provide us with a window into their grasp of the concepts they are exploring and developing. In this paper, the author indicates how these representations can evolve over time and enrich the understanding of division of fractions, often thought to be the most difficult of elementary school mathematical topics. The results of this research suggest that when appropriate problems are provided for students, in a meaningful context, they can demonstrate understanding of division of fractions that is durable over time, and that they are able to flexibly move back and forth between and among representations, choosing what they deem to be appropriate forms for a particular situation.

Keywords: elementary school; division of fractions; representations; problem solving; longitudinal research;

Introduction and Theoretical Framework

This research was designed to investigate two intertwined issues: the manner in which students build powerful ideas about fractions, division of fractions, in particular, and the importance of having students build, use, and connect different types of representations for these ideas. Specifically, this study investigates how the same group students developed and made sense of several different types of representations as part of their investigation into concepts involving division of fractions. Also under study was how these investigations helped them to avoid some of the common difficulties frequently experienced by others.

Difficulties with Fractions and Division of Fractions

The difficulties that many students have experienced while solving problems involving fractions have been well documented (for example: Davis, Alston, and Maher, 1991; Davis, Hunting, and Pearn, 1993; Steffe, Cobb and von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards and Cobb, 1983; Tzur, 1999;). Therefore, it is of great importance to find ways to help students overcome these difficulties. Ma (1999) states that division is the most complex of the mathematical operations and that fractions are the most complicated numbers to deal with in arithmetic. Therefore, she considers division of fractions to be the most difficult topic in elementary mathematics, for both teachers and their students. As a case in point, she notes that only forty-three percent of the United States teachers in her study were able to perform the computation successfully and only one out of twenty-three teachers was able to give a correct

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representation for a problem involving division of fractions. Specifically, she found that it was common to confuse dividing by a unit fraction with dividing by the whole number in the denominator; that is, dividing by one-half was often confused with dividing by two. She also found it common to confuse division by a fraction with multiplication by a fraction, for example, dividing by one-half and multiplying by one-half. In some cases, there was also confusion about dividing by one-half, multiplying by one-half and dividing by two.

In order to explore the types of errors that students make in dividing fractions, we can refer to the work of Tirosh (2000), who states that these errors can be classified into three categories. She refers to the work of other researchers (Ashlock, 1990; Barash & Klein, 1996; Fischbein, Deri, Nello, & Marino, 1985; Graeber, Tirosh, & Glover, 1989; Hart, 1981; Kouba, 1989; Tirosh, Fischbein, Graeber, & Wilson, 1993) who have studied elements of each of these categories. These categories are: 1) algorithmically based mistakes, 2) intuitively based mistakes and 3) mistakes based on formal knowledge.

Algorithmically based mistakes are errors made in the computational process (Ashlock, 1990; Barash & Klein, 1996). Tirosh indicates that this type of error is often made when students are taught the algorithmic procedure and confuse a step in the procedure. An example of such an error might be taking the reciprocal of the dividend instead of the divisor.

Intuitively based mistakes are errors based upon misconceptions associated with the operation of division. Most children only understand the partitive model of division and therefore cannot understand how it would be possible for one to divide a dividend by a larger divisor. In the partitive model of division, one is asked to divide a quantity into equal groups and then told to find how many in each group. For example, if I have twelve apples and I want to divide them equally among four friends, how many would each friend get? Kouba (1989) suggests three intuitive models for division based upon partitive division of whole numbers in her study. These are: 1) sharing by dealing, 2) sharing by repeated taking away and 3) sharing by repeated building up. These intuitive models are not easily extended to fractions. Since partitive division is commonly used to introduce the operation, it becomes a strong model for the operation. When models are initially constructed to serve in a specific context, remnants of those early models remain in generalized and transferred problems (Lesh, Lester & Hjalmanson, 2003). Consistent with this notion, Tirosh continues her discussion of intuitively based mistakes by saying that children's experience with the partitive model limits their ability to extend their understanding to division to fractions (Fischbein, Deri, Nello, & Marino, 1985; Graeber, Tirosh, & Glover, 1989; Tirosh, Fischbein, Graeber, & Wilson, 1993). This is especially true of problems where the divisor is larger than the dividend. Ott, Snook & Gibson (1991) note that textbooks and classroom examples further limit the experiences of students and their ability to extend their knowledge of partitive division to division of fractions. In order for students to be successful with division of fractions, they must also be familiar with the quotative model of division. This model originates with repeated subtraction. In this model, one is asked to formulate groups of a certain amount and to find out how many groups there would be. For example, if I have 12 apples and I want to give groups of four apples to my friends, how many friends will receive a group of apples?

The third type of error may be categorized as mistakes based on formal knowledge. These errors are based on misconceptions about the nature of fractions and misconceptions about the nature of operations. For example, a student might think that division is commutative and argue that $1 \div \frac{1}{2} = \frac{1}{2} \div 1$, which is equal to $\frac{1}{2}$ (Hart, 1981).

In describing children's early encounters with fractions, Burns (2000) states, "From their experiences, a number of ideas about fractions take shape informally in children's minds. However, children's understanding of fractions typically is incomplete and confused." (p. 223) Some examples that Burns cites as adult use of the language of fractions are, "I'll be back in three-quarters of an hour", "I need two sheets of quarter-inch plywood", and "The dishwasher is less than half-full." Children use this language as well when they say, "You can have half of my cookie", "Here, use half of my blocks" and "It's a quarter past one." Young children are often overheard talking about "the bigger half". Thus children come to school with some familiarity with fractions, but their knowledge is often incomplete.

Children also meet difficulty with fractions because they are unable to see a fraction as something to be counted as well as something that is a quantity. Conjectures have been made that a similar trend occurs at some early stage in the development of ideas about whole numbers as young children learn that a nickel and two pennies, three coins, is called seven cents. Difficulties with whole numbers seem to be overcome with greater ease than those involving fractions. In their study, Alston, Davis, Maher & Martino (1994) encountered a similar situation when students used Cuisenaire Rods® to build representational models for fractions. When a five-centimeter long rod was given the number name one, children were able to call a one-centimeter long white rod, one-fifth, and to call the length of two white rods, each one centimeter in length, two-fifths. However, when a single two-centimeter rod, replaced two one-centimeter rods, some of the children who were videotaped did not call the single rod two-fifths, because they did not see two objects.

Lamon (2001) attributes some of the difficulties students have with fractions to their limited ability to extend the meaning of a fraction to various interpretations. She states that a fraction, such as $\frac{3}{4}$ can be interpreted as 1) a part/whole comparison 2) an operator 3) a ratio or rate 4) a quotient or 5) a measure. She suggests that students be involved in a variety of activities that will enable them to experience the meaning of fraction in a wide range of ways.

Fortunately, many researchers have also documented instances in which students have successfully been able to build ideas relating to fractions (Bulgar, Schorr & Maher, 2002; Cobb, Boufi, McClain & Whitenack, 1997; Kamii & Dominick, 1997; Ma, 1999; Reynolds, 2005; Steencken, 2001; Steencken & Maher, 2002). In particular, in my previous work and work done with others (2002; 2003a; 2003b;) the conceptual development of ideas relating to division of fractions amongst fourth grade students participating in a teaching experiment, was reported. Within this teaching experiment, children experienced problem solving involving division of fractions prior to formal algorithmic instruction. Further, I reported that when the task and methodology used in the teaching experiment were replicated as part of the regular teaching practice in another classroom (my own), similar outcomes were achieved. It is this latter group of students, those who studied division of fractions as part of their regular classroom experience, with the author as their regular mathematics teacher, who are the subjects of this research.

Both the students in the above-mentioned teaching experiment and the subsequent classroom replication had essentially used the same three main strategies to solve a particular series of problems (Bulgar, 2002; 2003a; 2003b; Bulgar, Schorr & Maher, 2002). There were no strategies other than these three observed in either the classroom-based study or the teaching experiment. All three of the strategies were based upon existing counting schemes. (For a full description of these strategies, see Bulgar, 2002; 2003a; 2003b.)

These strategies consisted of the following:

- Reasoning involving natural numbers
- Reasoning involving measurement
- Reasoning involving fraction knowledge

The predominant solution method observed in the fourth grade teaching experiment, (Bulgar, 2002, 2003a, 2003b) consisted of reasoning involving natural numbers. Essentially, these students built representations that converted the meters to centimeters, thereby substituting the fraction division with natural number division, a topic generally prominent in fourth grade mathematics curricula in New Jersey (See NJ Mathematics Coalition and NJ State Department of Education, 2002). However this solution method was seen in the work of only one fifth grader in the replicated classroom study, and even when it did appear, there was a claim by the student that it was developed after the problem was solved using reasoning involving fraction knowledge (Bulgar, 2003a, 2003b). All of those students in the fifth grade who drew representations, created linear models to represent the division of a piece of ribbon into various-sized bows.

In an effort to underscore the difficulty that young students have with fractions, DeTurck (2005) has suggested that because the conceptualizations necessary for truly understanding fractions are too difficult for elementary school children, the topic should be eliminated entirely and postponed until much later in the curriculum. This is contrary to the Standards set forth by the National Council of Teachers of Mathematics (NCTM 2000), which suggest that basic fraction concepts be introduced as early as kindergarten.

Representations

The representations that children use when solving mathematical problems provide us with a gateway to understanding their thinking (NCTM, 2000). The representations that were used by the subjects of this study included concrete materials, such as Cuisenaire® rods, string, ribbons; drawings; and language to explain their thinking in their own words. Therefore, it is essential to examine closely some of the existing literature surrounding the very important notion of representations.

Cramer and Henry (2002) state that in studies surrounding The Rational Number Project (Bezuk & Cramer, 1989; Post, Ipke, Lesh & Behr, 1985), the most important pedagogical belief evoked is that most children need to use concrete models to represent fractions in order to build cognitive representations of fractions. In addition, these representations must be used over time. Other major beliefs that grew from this project are that children need to be engaged in discourse to strengthen their ideas and that conceptual knowledge must precede the formal use of algorithms. This underscores the importance of closely looking at the representations that the students in this study built and using the existing opportunity (described under methodology) to examine how their representations evolved over time.

In an effort to better understand the cognitive role of representations, one can look at the work of Speiser and Walter (2000), who, in agreement with Davis (1984), base their assertion on the previous work of Minsky (1975) and others, when they claim that mathematical knowledge is cognitively represented symbolically, often in the form of representations that are referred to as frames. When students think about a mathematical situation, they must first build a representation, which is usually done in the form of a mental representation. The building of

these representations can be assisted by the use of pencil and paper or manipulative materials. The construction of these representations is often so rapid and so instinctive that students are not aware that they have come into existence. Then they do a memory search or construction of relevant knowledge in order to proceed. The mapping that they construct between the data representation that was the input and the knowledge representation that was there gets checked and revised and ultimately is used to solve a problem (Davis & Maher, 1990; Davis, Maher & Martino, 1992). The data representations are saved in frames (Davis, 1984; Minsky, 1975), which carry information and are arranged hierarchically according to their stability (Minsky, 1975). There are several subframes, which together form a counting construction frame. The representations for these counting frames can easily be interchanged between integers and fractions, making a counting frame for fractions a natural extension of the one for integers (Speiser & Walter, 2000).

Speiser and Walter (2000) further state that it is the binary partitions subframe which is used to assist in the development of representations involving iterated processes, including counting. The binary subframe is a representation for a set that is partitioned into two subsets. For example, when we add or subtract we are connecting to this type of representation. A special type of binary subframe is one where there is only one element in one of the subsets. It is this special type of binary subframe that helps create representations for iterated processes. Partitioning of a unit into fractions requires that a student be able to take a unit comprised of subunits and operate on it while simultaneously dividing into equal partitions (Tzur, 1999).

Maher, Davis and Alston (1993) studied one student, Brian, over a period of time. They used videotapes of him doing problem-solving activities involving fractions and noted that as he worked with a partner, he was fluent in the use of different types of representations, using diagrams and concrete objects to help him solve problems. He represented his ideas in great detail. For example, when solving a problem involving the sharing of pizza, Brian used pattern blocks to build models and assigned specific students' names to the pattern block representations. After studying Brian in fifth and sixth grade, it was concluded that he insisted upon making sense of the models that he constructed rather than relying upon the ideas of others. The researchers also note the significance of an appropriate classroom environment, one that enhances students' opportunities to be engaged in thoughtful mathematics.

In their study, Watson, Campbell and Collins (1993) examined how four fraction problems were solved by children from kindergarten to grade ten to analyze the work of children's use of images, reality and experience. They found a developmental progression in the iconic reasoning, the ability to reason involving images and drawings, was developed in building ideas about fractions. They say that there is a connection between the development of iconic reasoning and of concrete symbolic reasoning. As a result of their findings, they urge schools to incorporate more problems that would give children an opportunity to develop their iconic reasoning so that the development of concrete symbolic reasoning can be supported. They also say that more study of this issue is needed in the form of teaching experiments.

In her examination of the various subconstructs of fractions described above, Lamon (2001) noted that they lend themselves to different types of representations and that significant understanding of these subconstructs is related to the use of continuous, discrete and unitary, solitary or composite representations. Area models or regions, sets of discrete objects, and the linear models such as number lines are the models most commonly used to represent fractions in

the elementary and junior high school (Behr & Post, 1992). Consistently, Van de Walle (2004) suggests that when studying fractions students should be provided with opportunities to be engaged in activities that employ a variety of representations.

The ability to use a variety of representations for the same concept, in this case, fractions, requires flexible thinking. Researchers such as Warner, Alcock, Coppolo and Davis (2003) and Warner and Schorr (in progress) emphasize that a critical aspect of mathematical flexibility is the ability of students to use multiple representations for the same idea and to link, extend and modify those representations to a broader range of situations, involving a more extensive array of models. Since the goal of instruction in this study was not merely to have students retrieve facts or procedures, or to display understanding only for very specific situations or for limited time periods, the notion of mathematical flexibility is of significant relevance. This type of mathematical flexibility is particularly important if students are to use knowledge across a wide spectrum of ideas. Fosnot and Dolk (2001) note, "The generalizing across problems, across models, and across operations is at the heart of models that are tools for thinking." (p.81). They report on a class in New York City wherein a third grade teacher provided students with three different contexts that lent themselves to different models while all three resulted in the same answer. In each case the children produced different representations that were closely linked to the context. Fosnot and Dolk go on to state that it is easy for students to notice that the answers are the same but that the important issue is for them to see the connections among and between the representations to develop a generalized framework for the operations.

Lamon (2001) also addresses the issue of students developing their own representations as opposed to adapting their teacher's representations, since the latter is not an indication of understanding. She goes on to say that:

When a student truly understands something in the sense of connecting or reconciling it with other information and experiences, the student may very well represent the material in some unique way that shows his or her comfort with the concepts and processes. (P. 156)

From this, one can infer that being able to move flexibly among and between representations for division of fractions designates a deeper understanding of the concept.

It is important to address that representations can be both internal and external. Goldin and Shteingold (2001) note the importance of the distinction between internal and external representations as well as the significance of the connection between internal and external representations as a fundamental element of teaching and learning. They add that the internal models or representations that one builds are not observable, but can only be inferred from students' interaction with materials, discourse and /or the external representations they create. These researchers go on to say:

Through interaction with structured external representations in the learning environment, students' internal representational systems develop. The students can then generate new external representations. Conceptual understanding consists in the power and flexibility of the internal representations, including the richness of the relationships among different kinds of representation. (P. 8)

Before continuing, it is important to distinguish between the conceptual models that are embodied in the representational media that students use, and the cognitive models that reside inside the minds of learners (Lesh & Doerr, 2003). In this work, both will be addressed, with an

emphasis on the nature of the cognitive models that are born out in the representational media that the students use, especially as evidenced in their mathematical flexibility. The nature of the models that the students have built in terms of their mathematical flexibility, not just during or shortly after the instruction took place, but rather over a more extended period of time is documented here. Also addressed is flexible thought in the context that follows, because it is relevant to this study.

Research Questions

The primary question under study in this research is, given appropriate conditions, how do students extend, modify, revise and refine their representations involving division of fractions over time? Additionally, how do they demonstrate flexibility so that they can fluidly move back and forth between and among representations, being selective about what models or representations will help them to create meaningful problem solutions without experiencing some of the many difficulties normally encountered by others?

A significant goal of this study is to better understand how a group of students extended, modified, revised, refined and ultimately generalized their ideas relating to division of fractions during the school year following their initial experience with problem solving activities related to this topic. This is done with a focus on mathematical flexibility, the nature of the representations that were used and the evolution of these representations during the following school year. In particular, the focal point is on how students initially used linear models, how these models evolved into discrete area models and how these students moved easily to continuous linear models when they found them to be more appropriate.

The research that frames this study includes investigations of fractions and more specifically division of fractions. This topic traditionally has involved much complexity and difficulty for students. Additionally, in order to trace the evolution of the representations of a particular population, it was necessary to examine studies regarding representations in general and specifically related to fractions.

Methods and Procedures

Background, Setting and Subjects

In the fall of 1993, a teaching experiment, including the study of fractions, was conducted in a small suburban New Jersey public school, under the direction of Carolyn Maher of Rutgers University, and other researchers from the University¹. Fourth grade was selected for this teaching experiment because it is the year prior to the one in which students are traditionally taught fraction algorithms in New Jersey. The premise of this study was that the fraction knowledge that these fourth grade students built could therefore be attributed to the work done within the project rather than to classroom instruction. It was expected that careful monitoring of the children's development of ideas would give the researchers insight into how students built mathematical knowledge about fractions. The instructional design of this teaching experiment was the model for the regular classroom instruction in the classes that are the subject of the study reported upon here. (Some salient elements of the classroom instructional practices are described below, but for further descriptions of the instructional practices used in the teaching experiment see Bulgar 2002; Reynolds, 2005; Steencken 2001.)

In the school years 2000 – 2001 and 2001 – 2002, a unique opportunity presented itself. While investigating the work of the students in the above-mentioned research study, the author had the opportunity to teach the same mathematical ideas to a second group of students who were in fifth grade, and later on, in sixth grade (also taught by the author). Although the content explored during these two years cover a wide range of topics, this paper addresses only ideas related to division of fractions. The particularities of this situation allowed the author to document the growth of these ideas over the course of two school years in the context of everyday teaching in the company of the students' regular mathematics teacher (as opposed to a project led by visiting University researchers). This will be more clearly described below.

The students being studied attended a small private parochial school in New Jersey that attracts children from several surrounding communities. This academically heterogeneous class consisted of 13 girls. These students had experienced a very traditional² classroom-learning environment prior to the fifth grade. They were used to being told whether or not their answers were correct and being shown procedures for doing mathematics. The predominant goal of mathematics had been to get the right numerical answer. In contrast, upon entering fifth grade and being taught mathematics by the author, the students were encouraged to take responsibility for convincing others that their solutions were correct and they were expected to write about their thinking on a regular basis. They began doing mathematics with block scheduling, meeting for one 40-minute period per week and two 80-minute periods per week. Discourse was of great importance. Responsive questioning took place to encourage mathematical thinking by attempting to elicit verbalization of mathematical thought (See Goldin & Shteingold, 2001 above). Predominantly, students worked in pairs or triads and collaboration was promoted. The classroom community was one in which students' ideas were always highly respected. Alternate strategies were encouraged, shared and discussed. The students were invited to discuss their thinking and to submit ideas in writing or via email. The goal was to achieve deep understanding of the mathematics embedded in the problems that the students experienced. Students were not taught algorithms. When they recognized patterns and could justify that these patterns were valid for the examples that they observed, they created generalizations, which they would apply to future problems. (See Bezuk & Cramer, 1989; Cramer and Henry, 2002; Post, Ipke, Lesh & Behr, 1985, above.) A fundamental characteristic of the instructional environment was the facilitation of mathematical problem solving. This was based on the premise that students needed to be engaged in mathematical activities that promote understanding (Cobb, Wood, Yackel & McNeal, 1993; Davis & Maher, 1997; Klein & Tirosh, 2000; Maher, 1998; NCTM, 2000; Schorr, 2000; Schorr & Lesh, 2003). There was a strong effort by the teacher and assistants, when present, not to lower the cognitive demand of the problem solving activities so that the goal remained that students be "doing math" as described by Stein, Smith, Henningsen, and Silver, (2000). Therefore, conditions established during the fifth grade were set up to create a classroom community in which student inquiry and discovery were of paramount importance. Once this community was established, students actively participated and remained engaged in the work they did, often posing extensions and hypothetical situations to the problems they were assigned, indicating that their thinking and their personal goals went beyond just getting a numerical solution.

In the second year of this study, when the students were in the sixth grade, one of the original students had left the school, but another new student had joined, thus leaving the classroom population the same in number. The routines and classroom community that had been instituted the previous year remained in place, so that the semester began with the expectation that

mathematics class would involve inquiry and discovery. Students were enthusiastic and eager to continue with the type of work they had been engaged in during the previous year.

Data Collection

The primary data that were examined for this study consist of artifacts of actual student work, which were collected during two school years, when the subjects were in the fifth and sixth grades. These data pertained to the study of division of fractions. After the work was collected, written notes from the teacher were attached to some of the work, usually in the form of questions. When work was returned, students had the opportunity to answer these questions before the final submission of their papers. The written work was examined qualitatively for its relationship to representations used to solve a variety of problems involving division of fractions.

In addition, during the time that these students were introduced to division of fractions in the fifth grade, there were two mathematics education graduate students present, serving as assistants. They interacted with the students, questioning them about what they were doing and listening to their explanations. They were familiarized with the type of learning environment that was the norm of this class and therefore knew not to interfere with children's thinking. Their field notes are also included in the data collection. One of these graduate students was with the class all semester, as part of a university practicum experience. The other visited because of interest in the particular topic of division of fractions. Because of the author's association with the university, faculty and graduate student visitors had stopped by on various occasions. Also, other teachers from the school had come in to observe the teaching of mathematics. Therefore, these students were used to having other adults in the classroom as they worked. When the class was in the sixth grade, the classroom was located in a somewhat isolated supplemental trailer, so visitors from the school faculty were very rare. During the sixth grade activity reported here there was no one present other than the students and the author.

Tasks and Tools

During the fifth grade, students were assigned the task called Holiday Bows.³ In this task they were provided with a meaningful context for understanding division of a natural number by a fraction. This topic is part of the fifth grade mathematics curriculum and appears in most fifth grade mathematics textbooks. The task involved finding out how many bows of several fractional lengths could be made from various sizes of ribbon. For example, one of the questions was how many bows, each one-third meter in length, could be made from a piece of ribbon that is six meters in length. Students had access to actual ribbons, pre-cut to the specified sizes, meter sticks, string and scissors to help them form concrete representations of their thinking. (See Fig. 1) This was the students' first classroom introduction to division of fractions.

HOLIDAY BOWS

- (1) Red ribbon comes packaged in 6 meter lengths;
- (2) Gold ribbon comes packaged in 3 meter lengths;
- (3) Blue ribbon comes packaged in 2 meter lengths; and
- (4) White ribbon comes packaged in 1 meter lengths.

Bows require pieces of ribbon that are different lengths. Your job is to find out how many bows of particular lengths can be made from the packaged lengths for each color ribbon.

I. White Ribbon	Ribbon Length of Bow	Number of Bows
1 meter	$\frac{1}{2}$ meter	
1 meter	$\frac{1}{3}$ meter	
1 meter	$\frac{1}{4}$ meter	
1 meter	$\frac{1}{5}$ meter	
II. Blue Ribbon	Ribbon Length of Bow	Number of Bows
2 meters	$\frac{1}{2}$ meter	
2 meters	$\frac{1}{3}$ meter	
2 meters	$\frac{1}{4}$ meter	
2 meters	$\frac{1}{5}$ meter	
2 meters	$\frac{2}{3}$ meter	
III. Gold Ribbon	Ribbon Length of Bow	Number of Bows
3 meters	$\frac{1}{2}$ meter	
3 meters	$\frac{1}{3}$ meter	
3 meters	$\frac{1}{4}$ meter	
3 meters	$\frac{1}{5}$ meter	
3 meters	$\frac{2}{3}$ meter	
3 meters	$\frac{3}{4}$ meter	
IV. Red Ribbon	Ribbon Length of Bow	Number of Bows
6 meters	$\frac{1}{2}$ meter	
6 meters	$\frac{1}{3}$ meter	
6 meters	$\frac{1}{4}$ meter	
6 meters	$\frac{1}{5}$ meter	
6 meters	$\frac{2}{3}$ meter	
6 meters	$\frac{3}{4}$ meter	

Figure 1. The Holiday Bows Task

During the beginning of the sixth grade, the students worked on the task called Tuna Sandwiches. This task was created by the author with the intention of being similar in structure to Holiday Bows, which had been experienced during the previous year. Students had access to various materials such as Unifix Cubes®, pattern blocks, Cuisenaire Rods®, paper, pencils, graph paper, meter sticks, string, scissors at all times during mathematics class and were encouraged to choose what they deemed appropriate and helpful as needed. The task follows directly below.

Mr. Tastee's restaurant serves four different kinds of sandwiches. A junior sandwich contains $\frac{1}{4}$ lb of tuna; a regular sandwich contains $\frac{1}{3}$ lb of tuna; a large sandwich contains $\frac{1}{2}$ lb of tuna and a hero sandwich contains $\frac{2}{3}$ lb of tuna. Tuna comes in cans that are 1lb, 2 lb, 3lb and 5 lb. How many of each type of sandwich can you make from each size can? Find a clear way to record your information. You will need to write a letter to the restaurant owner, Mr. Tastee, and give him your findings.

One of the goals in creating the Tuna Sandwiches problem was for it to lend itself to be represented by an area model rather than a linear model, as was the case with Holiday Bows. That is, the intention was for the fractions to be based on a portion of a region, rather than a portion of a length as is the case in a linear model. Fosnot and Dolk (2001) state that just because we create a problem with certain models in mind, we cannot be assured that these models will be used by students. By creating a problem that was intended to be fundamentally similar in structure to the Holiday Bows yet embodied in a different type of representation, an area model, the notion of flexibility could be explored as well as an examination of the durability of the knowledge the students had demonstrated during the previous year.

In both the fifth and the sixth grades, these tasks and the whole class sharing and discussion associated with them were followed by problems that required students to find the values of expressions involving division of fractions using only symbolic notation. In the fifth grade, these problems consisted of only a natural number divided by a fraction. The problems were assigned upon completion and discussion of the Holiday Bows task. Students were asked to find the value of any two of the following three expressions:

- $5 \div \frac{1}{3}$
- $12 \div \frac{3}{4}$
- $7 \div \frac{2}{3}$

In the sixth grade, two problems involving only symbolic notation were assigned to the students approximately six weeks after they began working on the Tuna Sandwiches task. The problems, assigned one at a time, were to find the value of the expressions directly below.

- $2 \div \frac{3}{4}$
- $\frac{5}{8} \div 2 \frac{1}{2}$

The second of these expressions was the students' first exposure to finding the quotient of a common fraction divided by a mixed numeral. As stated in the theoretical framework (See Tirosh, 2000 above), this type of division involving fractions is especially difficult for students to understand. There were three significant goals of the assignment of the problems involving only symbolic notation during the sixth grade. The first was to see whether or not the knowledge demonstrated in the past regarding division of fractions was durable. The second was to see if the

knowledge was flexible enough to be able to be extended and applied to a problem involving division of a common fraction by a mixed numeral and thirdly, to see if the students would make use of previously employed representations when the specific context was removed.

Results and Discussion

Prior to their work on division of fractions, the fifth grade students had done extensive problem solving with fractions using manipulative materials. They were encouraged to construct models to support their solutions, draw their models and to justify those solutions in terms of the representations. For example, on November 16, 2000, students were given the following problem.

Which is greater, $\frac{2}{3}$ or $\frac{3}{4}$? By how much? Build a model to solve the problem and explain how your model can be used to find a solution.

At this early time in the semester, students were most familiar with Cuisenaire Rods® as a manipulative material for building models. All of the pairs were able to solve the problem using this linear model. In Figure 2, we see Samantha and Eve’s solution, which articulates a clear understanding of the problem.⁴

Q: Which is greater $\frac{2}{3}$ or $\frac{3}{4}$?

A: $\frac{3}{4}$ is greater by $\frac{1}{12}$.

M [models]:

P			P			P					
Dg				Dg							
Lg		Lg		Lg		Lg					
W	W	W	W	W	W	W	W	W	W	W	W

Lg		Lg		Lg		
P			P			W

E [Explanation]: $2 \text{ Dg} = 1$. $1 \text{ Lg} = \frac{1}{4}$. $1 \text{ P} = \frac{1}{3}$. We took 3 Lg ($\frac{3}{4}$) and 2P ($\frac{2}{3}$) and put them one on top of the other and saw that 3 Lg ($\frac{3}{4}$) are bigger. $12 \text{ W} = 2 \text{ Dg} (1)$. If you put 1W ($\frac{1}{12}$) next to the 2P ($\frac{2}{3}$) then they are the same length as the 3 Lg ($\frac{3}{4}$). So that is how we figured out our answer.

Fig. 2. Samantha and Eve’s solution.

Holiday Bows and Applications – Fifth Grade

In late May of the same school year, the fifth graders worked on the Holiday Bows problem for two double (80 minute) periods on consecutive days. They had received no prior formal instruction regarding division of fractions and were not told that this problem was relevant to that topic. They worked collaboratively in groupings of their choice, but each submitted an individually completed chart (See fig. 1) and an individually written explanation of the solutions.

Seven of the 13 students constructed representations, which they included with their solution explanations. All of the students who drew representations used linear models. The specific solutions of several students are worth mentioning. Some, such as Sarah, drew discrete linear models for each meter of ribbon. She used the solution method, *reasoning involving fractions* (See Bulgar 2002, 2003a, 2003b) to find out how many bows, each $\frac{1}{2}$ meter in length could be made from six meters of ribbon (see above). In this solution method, one recognizes that $\frac{1}{2}$ means one out of two equal parts that comprise the whole and therefore for each whole there will be two parts. The number of units (in this case meters) is therefore multiplied by two to find the number of halves in the entire amount. Using this solution method, Sarah states that in each of the six meters there would be two bows so in six meters there would be 6×2 bows or 12 (See Figure 3). Though the statement of the problem alludes to the ribbon being a continuous piece, six meters in length, her diagram indicates discrete representations for each of the six meters.

5/24/01

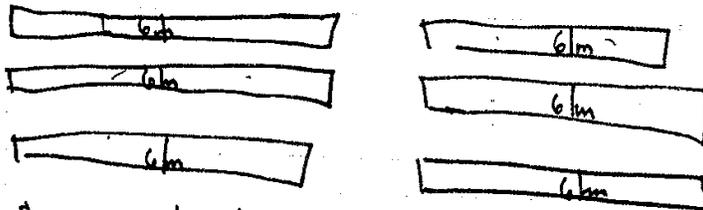
(Continued)

I counted how many sections there were & there was 15 and that made 15 bows.

Red:

Q: How many bows can you make out of different sizes to make 6 meters?

Ex: I took 6 meter:



I counted the sections & got 12 & each ~~bow~~ bow is supposed to be $\frac{1}{2}$ long so you can make 12 bows. You can also do $6 \times 2 = 12$.

Fig. 3. Sarah's explanation for how many bows, each $\frac{1}{2}$ meter in length can be made from 6 meters of ribbon.

In contrast, Nicole drew a continuous linear representation for each of the ribbon lengths. She also used *reasoning involving fractions* indicating that in each meter there would be two bows, each $\frac{1}{2}$ meter in length. However, unlike Sarah, who viewed the problem multiplicatively, Nicole imposed additive structures, adding two bows for each additional meter of ribbon. One might say that Nicole's solution makes use of *reasoning involving measurement* (See Bulgar, 2002; 2003a; 2003b) as well. In this solution method, students create a measurement tool, such as a piece of string, as long as the length of bow and then place it along the length of ribbon repeatedly, counting the number of times it fits on the ribbon. Though Nicole did not create such a tool, the additive structure of her solution implies that she is cognitively placing $\frac{1}{2}$ meter pieces along each meter and counting them. She is thereby making use of an internal or mental model for the measurement tool, counting the number of times the tool could be placed along the given number of meters of ribbon. As stated by Goldin and Shteingold (2001), internal models are borne of and developed through the use of external models. These students previously had experience with the use of Cuisenaire® rods and apparently created an internal model based upon the external measurement models they had used in the past.

Pg. 1

Math
Grade 5

5/24/01

Q: How many bows can you make out of a persons arm of ?

E/a: What I did for the 1 meter ribbons
 for 1 meter: if you have 1 meter + the length of a bow is $\frac{1}{2}$ a meter then obviously the 2... did the same thing for $\frac{1}{3}=3, \frac{1}{4}=4, \frac{1}{5}=5$.

M:

1 meter	
$\frac{1}{2}$ meter	$\frac{1}{2}$ m

E/a: What I did for 2 meters is: Since for 2 is $\frac{1}{2}$ of 2 I add the #s for 1.
 For example: for the $\frac{1}{2}$ meter an 1 meter ~~met~~ got 2. So I added $2+2=4$. So ~~the~~ was my answer

M:

2 m.			
$\frac{1}{2}$ m	$\frac{1}{2}$ m	$\frac{1}{2}$ m	$\frac{1}{2}$ m

E/a: What I did for 3 m. is: I added what I got for 3 m. 1 m. + 2 m.
 For example: $\frac{1}{2}$ of 1 meter was 2
 $\frac{1}{2}$ of 2 meters was 4 so I added the together $2+4=6$.

M:

3 M.					
$\frac{1}{2}$ M					

Figure 4. Nicole's solutions.

Both Gabriella and Stephanie did not draw representations, but each used *reasoning involving fractions* and multiplication. They both wrote lengthy and detailed explanations for their solutions. Stephanie wrote the following to explain her solution for how many bows, each $\frac{2}{3}$ meter in length could be made from two meters of ribbon.

...with 2 m. each meter had 3 thirds and 2m. would be $3 \times 2 = 6$ [thirds], so 6 would be the answer [for the number of bows $\frac{1}{3}$ meter in length that could be made from 2m of ribbon]. But 6 is to much even though 2m. $\times \frac{2}{3}$ [she circled the 3 in the denominator] = 6. I thought about that leftover $\frac{2}{3}$ [she circled the 2 in the numerator]. Then I realized that if you can divide $2 \div 6 = 3!$ That would be the answer because if you minus $\frac{1}{2}$ from the 6 it equals 3. If you have 6 and you $-\frac{1}{2}$ from every $\frac{1}{3}$ of 6, it = 3.

Stephanie has confused the language for subtraction, which she refers to as “minus”, and division. She has also reversed the dividend and the divisor when she states that $2 \div 6 = 3$, which is a common misconception observed in students of this age. However, she has evidently understood the inverse relationship created by enlarging the size of the bow and getting fewer bows. Understanding that as the divisor is doubled, the quotient divided in two is a very complex notion. Yet, it appears to be very clear to Stephanie from her explanation. Additionally, though she has confused the dividend and the divisor in her symbolic notation, her explanation indicates that she understands the role of each of the numbers conceptually.

Samantha clearly and concisely explains how she used *reasoning involving measurement* to solve the problems (See Figure 5). However, she does not make any mention of how she created the measurement tool. Most students did this by cutting a one meter piece of string and then folding it equally into the number of parts needed for the unit fractions. That is, to create a measurement tool that was $\frac{1}{3}$ meter in length, they would fold the meter length of string into three equal parts and cut it on the folds. When creating measurement tools for non-unit fractions, several began with the unit fraction tool. For example, to create a measurement tool that is $\frac{2}{3}$ meter in length, one would cut two $\frac{1}{3}$ pieces, place them end to end and then cut another piece of string the length of the combined pieces.

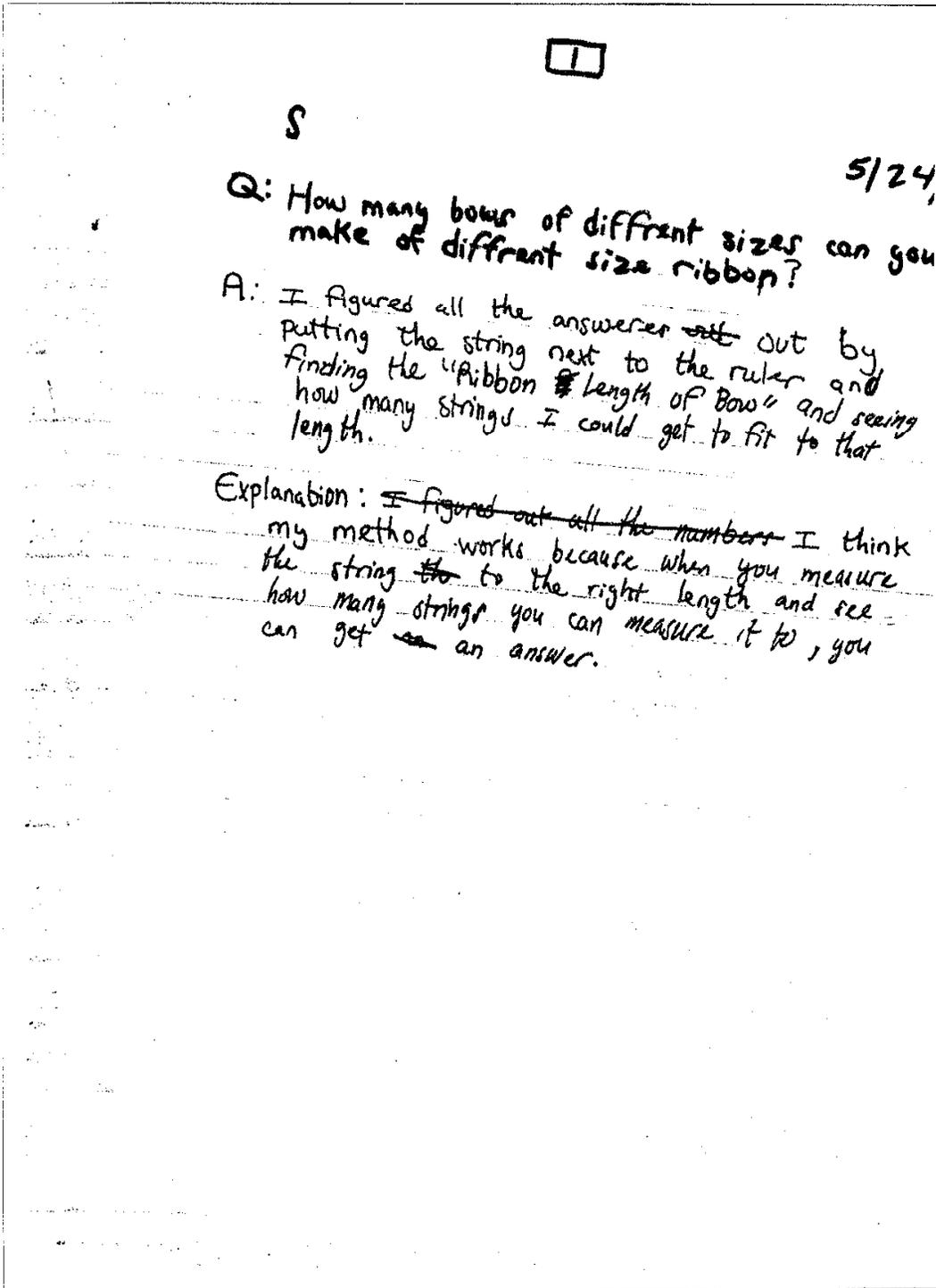


Figure 5. Samantha's solution

Several of the students indicated that they found the problems involving a non-unit divisor to be more difficult. In her description of how many bows, each $\frac{2}{3}$ meter in length could be made from two meters of ribbon, Brooke appears to have an internal representation of each meter discretely as well as each $\frac{1}{3}$ of a meter in each bow. She does not draw a representation, but she writes the following.

you take the ribbin which is 2m. long & cut it into half each is a m. and each half cut into 3 pieces and each 2 piece is a bow so you get 3 [bows].

We get a window into the internal representation that Brooke has created by examining her external representation (See Goldin & Shteingold, 2001 above) and the language she uses in her explanation. After she deduces that there are six $\frac{1}{3}$ -meter bows in the two meters of ribbon, she cognitively connects two such pieces to form a bow that is made of $\frac{2}{3}$ meter of ribbon.

In this population, Olivia was the only one to refer to *reasoning involving natural numbers* (See Bulgar 2002, 2003a, 2003b) and she did not come up with this type of solution initially. At the conclusion of her written work she says the following.

I figured out a shorter way to explain this & it makes more sense. It works as follows: 1 meter = 100 centimeters. You could change the amount of meters you have into centimeters. Thus, let's say you have to make bows each $\frac{1}{2}$ of a meter. Figure out how many centimeters = $\frac{1}{2}$ of a m. 50 centimeters = $\frac{1}{2}$ of a m because half of 100 is 50. Then see how many times 50 goes into 100. However many times 50 goes into 100 is how many bows you can make with each bow $\frac{1}{2}$ of a m. & with 1 m. You can also do this with $\frac{1}{3}$ of a m. or $\frac{1}{4}$... as long as you change $\frac{1}{3}$ or $\frac{1}{4}$ of a meter into a # amount of centimeters. You can also do this with 2 or 3m... of string as long as you change 2 or 3m... into centimeters. I think this works because you have to figure out how many $\frac{1}{3}$ rds or $\frac{1}{4}$ th s of a m. go into 1 m. That is saying the same thing as a certain # of centimeters go into 100 or 200 or 300. Or you could do $1 \div \frac{1}{4}$ and you would get 4. That is the same thing as $100 \div 25 = 4$. They both = the same thing which proves they both work.

As some of the students moved from the unit fractions to the non-unit fractions, they had to adjust their strategies. Linda solved the problems that required division by a unit fraction using *reasoning involving fractions*, multiplying the number of meters by the number of bows of each size in a one-meter piece of ribbon. She assumed this method was no longer valid when faced with a non-unit fraction divisor and therefore changed her mental representation and employed the strategy of *reasoning involving measurement*. The following appeared in the field notes of one researcher who was present.

When she got to the question of 6m ribbon and $\frac{2}{3}$ m bow, she started measuring. I asked her why she didn't just use her multiplying method, she replied, "cause there's a 2 there not a 1 [in the numerator], so you can't do it, you can only do it when there's a 1, so I have to measure it if there's another number there." It's ironic how she understands that the 2 in the numerator makes her method invalid, but she doesn't understand why. (C. Hayworth, unpublished notes, May 24, 2001)

After the students had all completed the work on Holiday Bows and submitted their individual papers, the problems were discussed in a whole group setting with various students sharing their solutions. A week later, they were asked to find the value of any two of the following expressions, which involve only symbolic notation.

Bulgar

(1) $5 \div \frac{1}{3}$

(2) $12 \div \frac{3}{4}$

(3) $7 \div \frac{2}{3}$

Students were asked to solve the problems and explain the solutions. Most of them chose the unit fraction divisor for one of the problems. Every submitted solution was correct, though there were a variety of explanations. Some students used approaches that focused on procedural explanations with some variety among them. For example, Samantha offers the following for her incorrectly copied example $5 \div \frac{3}{4}$.

you can get the answer by timzing (x) the number that you're dividing by. Like you would timez the 5 by the 4 and you would get 20. And the you'de divide 20 by 3 which is $6 \frac{2}{3}$.

Brooke writes the following.

before you do $12 \div \frac{3}{4}$ you have to find out how many [fractional] parts there are so you do 12×4 which equals 48 then you do $48 \div 3$ which equals 16.

While it appears that these students are relying upon algorithmic solutions, they seem to be using "Procedures with Connections" (See Stein, Smith, Henningsen & Silver, 2000) in that they refer to dividing and breaking into fractional parts. That is, they initially find the number of unit fractions in the length of ribbon and then "join" these discrete unit fractional parts to create the desired non-unit fractional piece.

Sarah was the only one in the class who drew area representations for her solutions. The students had worked with pattern blocks for other fraction activities in the past, but they were not available when the students worked on Holiday Bows. Her explanations are noteworthy. She states the following for $5 \div \frac{1}{3}$.

You are doing $5 \times 3 = 15$. You do 1 devided by 15 because it is only one third not three thirds.

Even though Sarah has confused the dividend with the divisor, which is not uncommon when children of this age use language to express their mathematical experiences, she indicates an understanding of the role of the number of thirds in the divisor. Therefore, it is expected that she would easily transition to dividing by a non-unit divisor. Regarding $12 \div \frac{3}{4}$, she writes the following.

You do how many times can four go into every one of the 12 (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) which is doing $12 \times 4 = 48$. Now you have to do $48 \div 3$ & get 16.

Her discrete area representation consists of twelve circles divided into fourths with a sequential multiple of four written above each one.

A few of the students made some reference to ribbon or bows during their explanations. For example, Gabriella drew a continuous linear model and stated the following when explaining her solution for $7 \div \frac{2}{3}$.

2 goes into 3 3 time so I do 3×7 because I want to see how much times 3 can go into 7 meters. $3 \times 7 = 21$. Now I want to devide 21 by 2 because if you have 7m. ribbon you devide it into 3 parts and I can make 2. (Her representation appears here in her explanation with the notation "2 parts out of 3 parts". She then draws a standard division problem that

indicates that $21 \div 2 = 10 \frac{1}{2}$.) A half is because 1 part out of 2 is $\frac{1}{2}$ and $2 \div 21$ is $10 \frac{1}{2}$.

Gabriella has also reversed the language of the dividend and the divisor, but she unmistakably understands the very difficult inverse relationship in division that when you divide by a larger dividend, you get a smaller quotient.

Stephanie, Olivia and Eve also make reference to understanding this inverse relationship. Eve states the following, which additionally indicates that she is thinking multiplicatively about the problem.

Now I will explain $12 \div \frac{3}{4} = 16$. There are 48 one thirds in 12. $\frac{3}{4}$ is 3 times as much as $\frac{1}{3}$. The answer will be three times less than 48 which is $(48 \div 3) 16$.

Stephanie indicates her understanding of the inverse relationship when she states the following.

The second one was $12 \div \frac{3}{4}$ so like the other problem you do 12×4 because you ask how many times does $\frac{3}{4}$ go into 12. It is 48 so now you divide by 48 because you aren't asking how many times does $\frac{1}{4}$ go into 12 but $\frac{3}{4}$. So $\frac{3}{4}$ is more ribbon. Now you divide $48 \div 3$ which is 16.

Stephanie has mentioned ribbon, which seems to indicate that although she did not draw any representation, she is referring back to an internal model that she has created, one that is a continuous linear model. Here the external representation in the form of language is leading us to believe that she has constructed an internal model of the ribbon to help her solve the problem.

Olivia also explained her solution in terms of the inverse relationship. She states the following in her explanation of $7 \div \frac{2}{3} = 10 \frac{1}{2}$.

First you must find out how many thirds are in one. You must find this out because you can't find out how many $\frac{2}{3}$ are in 7 if you can't find out how many $\frac{1}{3}$ go into 7. $\frac{1}{3}$ goes into one, three times because $\frac{3}{3}$ are 3 pieces put together to equal one. $\frac{1}{3}$ is one of these 3 pieces. Once you know that you have to multiply 7×3 . You multiply because $\frac{1}{3}$ goes into one three times, & you need to find out how many of these threes go into 7. $7 \times 3 = 21$. 21 would be how many $\frac{1}{3}$ go into 7. You must find out how many $\frac{2}{3}$ go into 7. $\frac{2}{3}$ is more than $\frac{1}{3}$, so therefore you would get less ribbons... you divide 21 in half or by 2. 21 divided by 2 is $10 \frac{1}{2}$ & that is your answer..

Olivia has gone into great detail to justify her solution and has also referred to the ribbons, alluding to her use of a cognitive model based on her work with Holiday Bows.

By the time the fifth graders had completed the tasks described above, they had demonstrated an understanding of division of fractions. In summary, there was indication of each of the following:

- Solving division of fractions problems within a concrete context.
- Solving division of fractions problems using symbolic notation.
- Understanding of the inverse relationship resulting in a decreased quotient when the dividend is increased.
- Understanding that the quotient is a count of how many times the divisor can be measured along the dividend.
- Understanding that the operation of division for fractions has the same meaning as the

operation of division in natural numbers so that one can fluidly move between these two forms to get the same solution.

- Even when a procedure for division of fractions is described it is rooted in the understanding of the conceptual basis for why this works. Not all students conceptualized the procedure in the same way.
- Even when students did not draw representations, their explanations provide clues to the internal models they used to solve problems.

Tuna Sandwiches and Applications – Grade 6

The classroom culture that had been established in the fifth grade was part of the sixth grade mathematics class from the onset. Therefore, when the students were asked to work on the task, Tuna Sandwiches, they knew they had to explore the problem on their own and with their peers without waiting for the teacher to “give” them an algorithm or procedure. Again, they were not told that this was a division problem. They were merely told to solve the problem and to use the letter that was required by the problem to justify their findings.

Not one of the thirteen sixth graders used a linear model to solve the Tuna Sandwich Problem. Ten of the thirteen students actually drew area models to represent their solutions and three of the thirteen explained their thinking without referring to any representation. It is interesting to note that all but one of the area models included discrete drawings for each pound of tuna. One would think that the problems involving the hero sandwiches, those which each required $\frac{2}{3}$ of a pound of tuna, would be more difficult to solve when using discrete representations. There was no mention, either verbally or in writing, of greater difficulty with the non-unit fractions, as had been the case in the fifth grade. In fact, several students stated that each one-pound of tuna would yield one and one-half hero sandwiches. It appeared that the shift in unit was made seamlessly by the sixth graders. One-third pound of tuna was recognized to be one half the quantity needed to make a hero sandwich, which required two-thirds pound of tuna. This kind of understanding was not demonstrated in the fifth grade.

Though they were not required to do so, most of the sixth graders spontaneously formed some kind of graphic organizer to structure their results (See Figure 6). Seven of the thirteen students formed a matrix indicating the amount of tuna required for each sandwich as one dimension and the different-sized cans of tuna as the other dimension. Four of the students indicated their solutions in an organized listing. One of these students had both an organized listing and a matrix.

	1 pound	2 pound	3 pound	5 pound
Junior Sandwiches	4	8	12	20
Regular Sandwiches	3	6	9	15
Large Sandwiches	2	4	6	10
Heroe Sandwiches	$1\frac{1}{2}$	3	$4\frac{1}{2}$	$7\frac{1}{2}$
<p>This is the chart of explanations. For the Heroe Sandwiches, first you make $1\frac{1}{2}$ of a Heroe Sandwiches from a 1 pound can of tuna. Add the half to the $2\frac{1}{2}$ Heroe sandwiches you get 3, then $4\frac{1}{2}$ + really you would get 6 if you used a 4 pound can but since you can't you get 7 or a $\frac{1}{2}$</p>				

Junior Sandwiches = $\frac{1}{4}$

Regular Sandwiches = $\frac{1}{3}$

Large Sandwiches = $\frac{1}{2}$

Heroe Sandwiches = $\frac{2}{3}$ fits into 1 pound like so =

Your welcome $\frac{2}{3}$ if you take a pound of tuna you could make 3 pieces of $\frac{2}{3}$ each. you need $\frac{2}{3}$ for a 3 Heroe sandwiches. So there for 1 or a half.

Michelle

②

Figure 6. Michelle's Graphic Organizer

Since the students specified their solutions using *reasoning involving fraction knowledge* by looking first at how many sandwiches of each type could be made from a one pound can of tuna, it is interesting to note that very few used proportional reasoning approaches (ie., they did not

use multiplicative structures to arrive at solutions involving multiple cans of tuna). Rather, most used additive structures. Stephanie begins by alluding to proportional reasoning when she writes the following as she explains her solutions for finding out how many regular sandwiches, those requiring $\frac{1}{3}$ pound of tuna, could be made from each of the various sized cans.

You can only make 3 sand. With one lb of tuna because 3 thirds make 1. ($3/3=1$) With one more lb of tuna (2lb) you can make twice as many sand. So you have 6 sand. With 3 lb of tuna you can make 3 more sand. (9 altogether) because you have one more lb of tuna which make 3 sand. Because 3 thirds ($3/3$) =1. Now with 5 lb. you add not 3 sand. But 6 because it is not 4 lb, but 5 lb of tuna.

Stephanie appears to be going back and forth between multiplicative and additive approaches, adding on multiples of three sandwiches. When Stephanie explains her solution to the hero sandwich problem, the one involving division by a non-unit fraction, she states the following.

So with a 1 lb can you can make 1 sand. and a $\frac{1}{2}$ of another because it is $\frac{2}{3}$ of a lb of tuna [required for each hero sandwich] so you have $\frac{2}{3}$ left which is $\frac{1}{3}$ left which is $\frac{1}{2}$ of $\frac{2}{3}$. A 2 lb can of tuna you can make 3 sand. easily and the excess is $\frac{1}{3}$ from both so that makes 3... Now for a 5 lb. can you can make 6 $\frac{1}{2}$ sand. because you can make 5 easily and 2 $\frac{1}{2}$ more with the extra of each lb.

Though Stephanie's solution of 6 $\frac{1}{2}$ sandwiches is not consistent with her explanation, she has evidently demonstrated an understanding that $\frac{1}{3}$ of a pound of tuna represents $\frac{1}{2}$ of a hero sandwich, an idea that students had more difficulty understanding the previous year when they worked with the linear model suggested by the Holiday Bows problem. This change in the unit is a very difficult one in general. It would appear that Stephanie is first counting the complete sandwiches that can be made from each pound, the ones she refers to as being made "easily", and then is gathering up the remaining $\frac{1}{3}$ pounds from each can to combine them in order to make additional sandwiches. While doing so, she made the error of recording 6 $\frac{1}{2}$ sandwiches as her final answer, though she says "...you can make 5 easily and 2 $\frac{1}{2}$ more with the extra of each lb." This kind of thinking was also observed in the representations of other students, such as Gabriella, Lynn, Amy, Sarah and Bea, who drew connecting lines to the "leftover" one-third pound of tuna in each representation of a one-pound can. (See Figure 7.)

P.S. When I was figuring this out for you I noticed something interesting. I noticed That by the junior sandwich ($\frac{1}{4}$ lb.) you added 4 by every can of tuna. This is because every time the can get bigger by 1 lb (from which you can make 4 sandwiches) so you just add another 4 and the 5 lb., it is 2 more lbs. So you add 8 instead of 4.

Though Eve used *reasoning involving fractional knowledge*, she applied additive reasoning to get the solutions, adding the number of sandwiches that can be made from each pound of tuna. It is interesting to note that Eve and other students who did this did not recognize the repeated addition of the same addend as multiplication.

Sarah used multiplicative reasoning in finding the solutions. She wrote the following.

Out of 3 pound you can make 12 junior. There is 4 in each [pound] and $4 \times 3 = 12$.

Sarah included a diagram of 3 circles divided into four sections or fourths. She numbered the sections from one to twelve. She used this representational structure for all of her solutions.

Gabriella also used multiplicative reasoning. She drew five circles, divided them in half vertically and stated the following.

How much large sandwiches can you make from 5 pounds. Let's try those imaginary pounds [her drawings]. Well 2 in each of the 5 pounds $5 \times 2 = 10!$

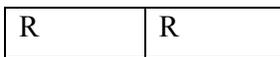
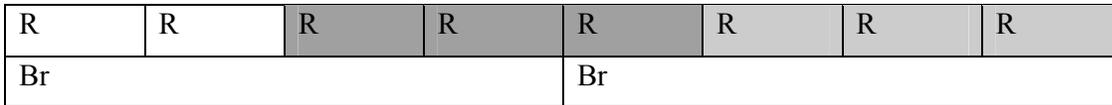
In the summative class discussion of the Tuna Sandwiches Problem, students talked about the problem and how it was like the problem they had done the previous year, called Holiday Bows. Those who did not recognize it at first agreed when their peers noted the structural similarity in the problems. They recognized that the problem required division of fractions and easily explained their solutions using symbolic notation. For example, when summarizing that three hero sandwiches could be made from two pounds of tuna, they were able to create the number sentence, $2 \div \frac{2}{3} = 3$. Some of the number sentences that the students provided were recorded on an overhead projector transparency. These number sentences are seen as solutions representing conceptual understanding derived from the use of student-generated representations and internal models, rather than as algorithmic answers. The students agreed that they had solved these problems using division of fractions.

Approximately six weeks after the students began working on the Tuna Sandwiches problems, they were assigned division of fractions problems using only symbolic notation, one at a time. During those six weeks, the class went on to explore other unrelated topics in the sixth grade curriculum such as practice in rounding decimals and surface area. There were a significant number of religious holidays (6 days plus a full week) for which the school was closed during this time creating both a lack of consistency and an aura of festivity. What is interesting to note when examining the students' work done with the decontextualized problems is that when drawing representations, students invariably went back to linear representations. Many referred specifically to Cuisenaire Rods® when they discussed their linear models. Thus components of the conceptual models they had built early in the fifth grade had endured, which is consistent with the conclusion drawn by Lesh, Lester and Hjalmarson (2003) that elements of initial models are retained.

The first of the two problems involving symbolic notation was to find the value of $2 \div \frac{3}{4}$. The students were told to build a model, to solve the problem and to explain how the model could be used to find the solution. Some (Michelle, Amy and Rose for example) wrote the problem as "How many $\frac{3}{4}$'s are in 2?" This would indicate an understanding of the meaning of division.

All of the students used linear representations and these representations were all continuous. The students referred to Cuisenaire Rods® in their explanations and descriptions. Even though Amy used a continuous linear model, she referred to sandwiches in her explanation. This would seem to indicate that she is comfortably moving back and forth between linear and area models cognitively to represent the fractions. She writes the following.

I made a train⁵ of 2 Brown's wich is 1. I wanted to make sandwiches with 3 scoops of margarine so I took 6 scoops then I had 2 extra wich was 2. and I had $2\frac{2}{3}$.



$$\frac{2}{3}$$

Figure 8. Amy's representation of her solution for $2 \div \frac{3}{4}$.

In order to create this representation, Amy needed to be able to select the appropriate length Cuisenaire Rods®, the ones that enable her to find a suitable solution. In this case, Amy chose the brown rod, which is eight centimeters in length to call “one”. She showed that each red rod, which is two centimeters in length, is therefore one fourth. She clearly indicated that three red rods (which form the number three – fourths) is now defined as one, indicating that each time this length is achieved, it represents one time that $\frac{3}{4}$ goes into 2. Amy's representation shows that she has seamlessly been able to move between the changes of unit necessary to interpret her representation. She darkened the outline of each set of 3 red rods (shaded in the diagram) to indicate a count of $\frac{3}{4}$ that goes into 2.

Subsequent to providing solutions for the problem above, students worked on the problem of finding the value of the expression $\frac{5}{8} \div 2\frac{1}{2}$. Though this problem is considerably more difficult, involving a quotative division model, every student in the class found the correct solution. Again, students used continuous linear models and referred to Cuisenaire Rods® in their explanations and their representations.

Most of the representations and explanations involved the reasoning that if $4 \times \frac{5}{8} = 2\frac{1}{2}$, then $\frac{5}{8} \div 2\frac{1}{2} = \frac{1}{4}$. Rearranging the equation in this way is a very complex notion. All of the representations that accompanied the explanations indicated the use of Cuisenaire® rods, using the brown rod to represent 1.

Olivia and Eve were the only students to submit a joint solution. They wrote the following, which is based upon the idea that $\frac{1}{4}$ of $2\frac{1}{2}$ is $\frac{5}{8}$.

Brown = 1. 8 whites go into brown so, each white = $\frac{1}{8}$. 5 whites are equal to one yellow, so

Bulgar

each yellow is $\frac{5}{8}$. 2 purples equal 1 brown therefore each purple equals $\frac{1}{2}$. According to what we just wrote, 2 browns and one purple would be $2\frac{1}{2}$. Yellow goes into $2\frac{1}{2}$ four times. Therefore, the answer is $\frac{1}{4}$ because $\frac{1}{4}$ of $2\frac{1}{2}$ is $\frac{5}{8}$.

Their representation also indicates the understanding that $\frac{5}{8}$ is $\frac{1}{4}$ of $2\frac{1}{2}$. They bracket each yellow rod and indicate that it is $\frac{1}{4}$ of $2\frac{1}{2}$.

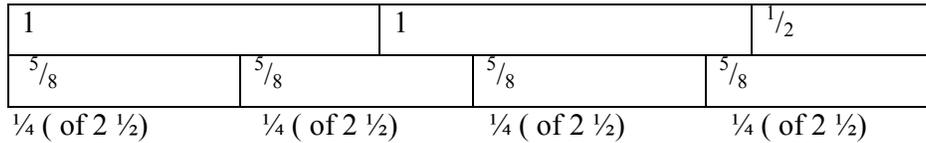
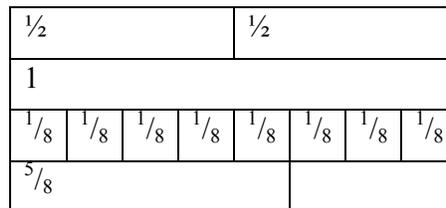


Figure 9. Olivia and Eve's representations for $\frac{5}{8} \div 2\frac{1}{2}$.

This solution, like the others, indicates conceptual understanding of division of fractions.

A summary of the representations created by the students over time is shown in the chart below.

Tasks:	Ordering Fractions	Holiday Bows	$5 \div \frac{1}{3}$ $12 \div \frac{3}{4}$ $7 \div \frac{2}{3}$	Tuna Sandwiches	$2 \div \frac{3}{4}$ $\frac{5}{8} \div 2 \frac{1}{2}$
Grade:	5	5	5	6	6
Lynn	Linear	None	None	Discrete Area****	Continuous Linear
Gabriella	Linear	None	Continuous Linear	Discrete Area****	Absent
Samantha	Linear	None	None	Continuous Area	Continuous Linear
Eve	Linear	Continuous Linear	None***	Discrete Area****	Continuous Linear
Sarah	Linear	Discrete Linear	Discrete Area	Discrete Area****	Continuous Linear
Amy	Linear	Continuous Linear	None	Discrete Area	Continuous Linear
Stephanie	Linear	None	None***	None	Continuous Linear
Nicole	Linear	Continuous Linear	None	Discrete Area****	Continuous Linear
Linda	Linear	**	Absent	-	-
Bea	Linear	Discrete Linear	None	Discrete Area****	Continuous Linear
Brooke	Linear	None	None	Discrete Area	Continuous Linear
Olivia	Linear	Discrete Linear*	None***	None	Continuous Linear
Rose	Linear	Continuous Linear	None	Discrete Area	Continuous Linear
Michelle	-	-	-	Graphic Organizer	Continuous Linear

*After having used fractional knowledge to solve the problem, Olivia used reasoning involving natural numbers. She was the only student in the 5th grade to come up with this form of solution.

**Linda was absent on the first day that the problem was assigned and therefore submitted only her completed chart. (See Figure 1.) Because she is a child with special needs, completion of the written explanations was not required.

***Though these students did not draw representations, they referred to ribbons and/or bows in their explanations. This would lead to the conclusion that the cognitive representations they had created were linear.

****Also included a graphic organizer.

Figure 10. Summary of the representations used by students.

Conclusions and Implications

There are several main points that can be gleaned from this investigation into the nature of the representations that students built and used as they developed an understanding of division of fractions in the absence of being taught the formal algorithms or procedures. The first, and perhaps the most significant, is that students can, under certain conditions, create and link representations that can help them to make sense of problems involving division of fractions, and in the process produce solutions that are mathematically correct. Furthermore, this can be done within the context of regular classroom practice. This study documents that the ideas that are built are robust and can be spontaneously retrieved even after long periods of time have elapsed. The students in this study were able to competently and easily recognize, retrieve, and use ideas that they had formulated several months earlier, and these ideas could be used to solve a variety of symbolic and decontextualized problems—all in the absence of formal instruction on the use of algorithms. Lesh, Lester and Hjalmarson (2003) indicate that when model eliciting problems are assigned, models are developed for specific purposes, just as the students in this study created their initial models as representations to solve particular problems. Moreover, they suggest that as the models become generalizable and transferable, they retain some characteristics of the original situated context. We see this in the students' references to ribbons and bows and Cuisenaire® rods as they construct solutions for the decontextualized problems.

A second significant point is that like the students cited in Bulgar, 2002; 2003a; and 2003b, the students in this study made use of three main methods to solve the problems, even though they had not been taught a procedure or algorithm to solve problems involving division of fractions. These methods are *reasoning involving natural numbers*, *reasoning involving measurement* and *reasoning involving fraction knowledge*. In *reasoning involving natural numbers*, students converted the units to other units, changing the division from a fraction problem to a natural number problem. For example students might convert fractional parts of meters to centimeters. In *reasoning involving measurement*, students created a measurement tool the size of the divisor and counted how many times they could place the tool along the object that is the dividend. In *reasoning involving fraction knowledge*, the students made use of their knowledge of the number of unit fractions in each “one”. Only one of the students in this group of subjects used *reasoning involving natural numbers*. This student claimed to have originally solved the problem differently, using *reasoning involving fraction knowledge*. Some students began by using

reasoning involving fraction knowledge, but then applied *reasoning involving measurement*. This was seen when students worked with non-unit fractions or when they applied additive structures to their solutions. The predominant method that students used when solving these problems, in this study, was *reasoning involving fractions*. These solutions evolved from the representations that the students created. All three of these fraction solution methods provide students with knowledge frames based upon counting schemes, which as stated by Speiser and Walter (2000) are easily interchanged between integers and fractions, making a counting frame for fractions a natural extension of the one for integers. Because of the contextualized problems, the students were not limited in their ability to extend their knowledge of division to division of fractions (Tirosh, 2000) and were able to construct meaningful solution strategies.

Finally, as the cited research suggests (Fosnot & Dolk, 2001; Warner, Alcock, Coppolo & Davis, 2003; Warner & Schorr, in progress), the ability to move among and between different representations for the same concept, indicated for these students a deeper understanding of ideas relating to fractions. The students in this study seamlessly moved among continuous and discrete linear representations and area representations when solving problems involving division of fractions, indicating a very meaningful grasp of division of fractions, often thought to be the most difficult topic in elementary school mathematics (Ma, 1999). They were able to generalize their ideas and apply them to problems using symbolic notation, thereby using their conceptual knowledge rather than algorithmic procedures to solve problems. The knowledge that they built about division of a natural number by a fraction was so robust, durable and flexible that they were able to extend their understanding to solve problems with fractional dividends and fractional quotients. Their solutions were rooted in their interpretation, extensions and revisions of the representations that they had created.

These findings have specific significance for the teaching of mathematics. They underscore the need for teachers to build a deep understanding of students' representations in order to choose and design appropriate tasks that become concrete contexts for the development of abstract ideas about division of fractions. Contextualization has been identified as one of the complex notions surrounding teacher knowledge (Doerr & Lesh, 2003). Teachers also need to understand how to interact with students as they encourage them to develop, use and build meaning for the ideas associated with division of fractions (Davis & Maher, 1997). Such interactions would include the creation of a classroom environment in which justification, sense making and meaningful discourse are encouraged. The research reported in this paper substantiates the importance of encouraging discourse as an important means of strengthening students' ideas (Bezuk & Cramer, 1989; Post, Ipke, Lesh & Behr, 1985). This study confirms the findings of other researchers (Bezuk & Cramer, 1989; Post, Ipke, Lesh & Behr, 1985), wherein discourse was also shown to be critical to the development of concrete models, which were then used as cognitive representations of fractions. Additionally the findings presented herein substantiate the need to create the appropriate learning environments to provide the building blocks that will assist students in constructing meaningful representations (Davis & Maher, 1997).

In conclusion, this research has important implications for both teachers and researchers. As indicated above, the students in this study were able to avoid, for all practical purposes, the main difficulties typically associated with the study of fractions and division of fractions in particular. By considering the classroom context, the problem situations posed, and the trajectory of ideas formulated by these students, teachers and researchers can gain insight into ways in which to help students make meaningful sense of this material. In closing, the author wishes to

underscore the fact that the research contained herein was done in these students' regular mathematics class, facilitated by their regular mathematics teacher, and not by a team of researchers under highly idealized circumstances. The type of activities described here, and the culture of this classroom were typical of how these students regularly received mathematics instruction. This should lend a note of encouragement to teachers who are in search of practices that may help their students build a deeper understanding of this very complex topic.

Endnotes

¹ The research cited here was supported in part by grant MDR 9053597 from the National Science Foundation and by grant 93-992022-8001 from The NJ Department of Higher Education. The opinions expressed here are those of the author and are not necessarily the opinions of the National Science Foundation, The NJ Department of Higher Education, Rutgers University or Rider University.

² Traditional, in this case refers to a more didactic environment of the type described in Cuban, 1993.

³ This task was originally developed by Alice Alston of Rutgers University. It has been studied extensively reported upon by Bellisio (1999), Bulgar (2002, 2003a, 2003b) and Bulgar, Schorr & Maher (2002).

⁴ The abbreviations used by the students in this model had been adopted by the class and represented the colors of the Cuisenaire Rods®. P represents purple, which is 4 cm in length; Dg represents dark green, which is 6cm in length; Lg represents light green, which is 3cm in length and W represents white, which is 1cm in length.

⁵ A train is the class-accepted term to denote two or more Cuisenaire Rods® that have been placed side by side along their shorter end. The resulting combined length is now treated as if it were one longer Cuisenaire Rod®.

References

- Alston, A.S., Davis, R.B., Maher, C.A., Martino, A.M. (1994). Children's use of alternative structures. *Proceedings of Psychology of Mathematics Education XVIII*. Lisbon, Portugal.
- Ashlock, R.D. (1990). *Error patterns in computation*. New York: Macmillan. (as cited in Tirosh, 2000).
- Barash, A., & Klein, R. (1996). Seventh grades students' algorithmic, intuitive and formal knowledge of multiplication and division of non negative rational numbers. In L. Puig * A. Gutierrez (Eds.) *Proceedings of the 20th conference of the International Group for the Psychology of Mathematics Education*, 25, 166-208. (as cited in Tirosh, 2000).
- Behr, M. & Post, T. (1992). Teaching rational number and decimal concepts. In T. Post (Ed.), *Teaching mathematics in grades K-8: Research-based methods* (2nd ed.) (pp. 201-248). Boston: Allyn and Bacon.
- Bellisio, C. (1999). A study of elementary students' ability to work with algebraic notation and variables. Unpublished doctoral dissertation, Rutgers University, NJ.

- Bezuk, N. & Cramer, K. (1989). Teaching about fractions: What, where and how? In P.R. Trafton (Ed.). *New Directions for Elementary School Mathematics: 1989 Yearbook*. National Council of Teachers of Mathematics: Reston, VA.
- Bulgar, S. (2002). Through a teacher's lens: Children's constructions of division of fractions. (Doctoral dissertation, Rutgers, The State University of New Jersey at New Brunswick, 2002) *Dissertation Abstracts International* 63/05. p.1754, AAT 3052715
- Bulgar, S. (2003a). Using research to inform practice: Children make sense of division of fractions. In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Twenty –seventh Conference of the International Group for the Psychology of Mathematics Education held jointly with the Twenty-fifth Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education: Vol. 2. Navigating Between Theory and Practice (157 – 164)*. Honolulu, HI: CRDG, College of Education, University of Hawai'i.
- Bulgar, S. (2003b). Children's sense-making of division of fractions. *The Journal of Mathematical Behavior: Special Issue on Fractions, Ratio and Proportional Reasoning, Part B*. 22(3), 319 – 334.
- Bulgar, S., Schorr, R.Y. & Maher, C.A. (2002). Teacher's questions and their role in helping students build an understanding of division of fractions. In A.D. Cockburn & E. Nardi (Eds.), *Twenty – sixth Annual Conference of the International Group for the Psychology of Mathematics: Vol. 2. Learning From Learners (161 – 167)*. Norwich, UK: School of Education and Professional Development University of East Anglia.
- Burns, M. (2000). *About teaching mathematics*. Sausalito, CA: Math Solutions Publications.
- Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education*, 28(3), 258-277.
- Cobb, P., Wood T., Yackel, E & McNeal, E.(1993) Mathematics as procedural instructions and mathematics as meaningful activity: The reality of teaching for understanding. In R.B. Davis & C.A. Maher (eds.) *Schools, mathematics, and the world of reality*. (pp.119-133). SMA: Allyn and Bacon.
- Cramer, K. & Henry, A. (2002).Using Manipulative models to build number sense for addition of fractions. In B. Litwiller & G. Bright (Eds.). *Making Sense of Fractions, Ratios and Proportions: 2002 Yearbook*. National Council of Teachers of Mathematics: Reston, VA.
- Cuban, L. (1993). *How teachers taught: Constancy and change in American classrooms, 1890-1980*. (2nd ed.). New York: Teachers College Press.
- Davis, G.E., Hunting, R.P., & Pearn, C. (1993). What might a fraction mean to a child and how would a teacher know? *The Journal of Mathematical Behavior*, 12(1), 63-76.
- Davis, R.B. (1984). *Learning Mathematics: The cognitive science approach to mathematics education*. Norwood, NJ: Ablex (as cited in Speiser & Walter, 2000).
- Davis, R.B., Alston, A, & Maher, C. A. (1991). Brian's number line representation of fractions. *Proceedings of Psychology of Mathematics Education XV*, Assisi, Italy.
- Davis, R.B., & Maher, C.A., (1990). The nature of mathematics: What do we do when we do mathematics. In R.B. Davis, C.A. Maher & N. Noddings (Eds.) *Constructivist views on the*

- teaching and learning of mathematics*. (pp.65-78). Reston, Va.: National Council of Teachers of Mathematics
- Davis, R.B., Maher, C.A. (1997). How students think: the role of representations. In L. D. English (ed.) *Mathematical reasoning: analogies, metaphors, and images*. (pp. 93-115). NJ: Lawrence Erlbaum.
- Davis, R.B., Maher, C.A. & Martino, A.M. (1992). Using videotapes to study the construction of mathematical knowledge by individual children working in groups. *Journal of Science Education and Technology*, 1(3), 177-189.
- DeTurck, D. (2005). Down with fractions. *Penn Arts & Sciences Magazine*. University of Pennsylvania School of Arts and Sciences. (Pp.12-14). Spring, 2005.
- Doerr, H. M. & Lesh, R. (2003). A modeling perspective on teacher development. In R. Lesh & H. Doerr (Eds.) *Beyond constructivism: a models and modeling perspective on teaching, learning, and problem solving in mathematics education*. Lawrence Erlbaum, Hillsdale, NJ.
- Fischbein, E., Deri, M., Nello, M. & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 16, 3-17. (as cited in Tirosh, 2000).
- Fosnot, C.T. & Dolk, M. (2001). *Young mathematicians at work: Constructing multiplication and division*. Portsmouth, NH: Heinemann.
- Goldin, G. & Shteingold, N. (2001). Systems of representation and the development of mathematical concepts. In A.A. Cuoco & F. R. Curcio (Eds.). *The Roles of Representation in School Mathematics: 2001 Yearbook*. National Council of Teachers of Mathematics: Reston, VA.
- Graeber, A.O., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 20, 95-102. (as cited in Tirosh, 2000).
- Hart, K. M. (Ed.) *Children's understanding of mathematics: 11-16*. London: John Murray. (as cited in Tirosh, 2000)
- Kamii, C., & Dominick, A. (1997). To teach or not to teach algorithms. *The Journal of Mathematical Behavior*, 16(1), 51-61.
- Klein, R. & Tirosh, D. (2000). Does a research based teacher development program affect teachers' lesson plans? In T. Nakahara & M. Koyama (Eds.) *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*. Hiroshima, Japan.
- Kouba, V.L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education*, 20, 147-158.
- Lamon, S.J. (2001). Presenting and representing: From fractions to rational numbers. In A.A. Cuoco & F. R. Curcio (Eds.). *The Roles of Representation in School Mathematics: 2001 Yearbook*. National Council of Teachers of Mathematics: Reston, VA.

- Lesh, R., & Doerr, H. (2003). In what ways does a models and modeling perspective move beyond constructivism? In R. Lesh & H. Doerr (Eds.) *Beyond constructivism: a models and modeling perspective on teaching, learning, and problem solving in mathematics education*. Lawrence Erlbaum, Hillsdale, NJ.
- Lesh, R., Lester, F. K., Jr. & Hjalmarson, M. (2003). A models and modeling perspective on metacognitive functioning in everyday situations where problem solvers develop mathematical constructs. In R. Lesh & H. Doerr (Eds.) *Beyond constructivism: a models and modeling perspective on teaching, learning, and problem solving in mathematics education*. Lawrence Erlbaum, Hillsdale, NJ.
- Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Maher, C.A., Davis, R.B., Alston, A.S. (1993). Brian's representation and development of mathematical knowledge; The first two years. In R.B. Davis, C.A. Maher, (Eds.). *Schools, mathematics, and the world of reality*, Boston: Allyn and Bacon.
- Minsky, M. (1975). A framework for representing knowledge. In P.H. Winston (Ed.), *The psychology of computer vision* (pp. 211-277), New York: McGraw-Hill (as cited in Speiser & Walter, 2000).
- National Council of Teachers of Mathematics. Curriculum and Evaluation Standards for School Mathematics. Reston, Va.: The Council, 2000.
- New Jersey Mathematics Coalition and New Jersey State Department of Education. *New Jersey Mathematics Curriculum Framework, 2002*.
- Ott, J., Snook, S.L. & Gibson, D. L. (1991). Understanding partitive division of fractions. *Arithmetic Teacher*, 39(2), 7-11.
- Post T., Wachsmuth I., Lesh R., & Behr M. (1985). Order and Equivalence of Rational Number: A Cognitive Analysis. *Journal for Research in Mathematics Education*, 16(1), 18-36.
- Reynolds, S. (2005). A study of fourth-grade students' explorations into comparing fractions. (Doctoral dissertation, Rutgers, The State University of New Jersey at New Brunswick, 2005) *Dissertation Abstracts International* 66/04. p. 1305. AAT 3171003.
- Schorr, R.Y. (2000). Impact at the student level. *Journal of Mathematical Behavior* 19, 209-231.
- Schorr, R.Y., Lesh, R. (2003) A models and modeling perspective on classroom-based Teacher Development. In R. Lesh & H. Doerr (Eds.) *Beyond constructivism: a models and modeling perspective on teaching, learning, and problem solving in mathematics education*. Lawrence Erlbaum, Hillsdale, NJ.
- Speiser, R. & Walter, C. (2000). *Five women build a number system*. Stamford, CT: Ablex Publishing Corporation.
- Steencken, E.P. (2001). Studying Fourth Graders' Representations of Fraction Ideas. (Doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick, 2001) *Dissertation Abstracts International* 62/03. p.953, AAT 3009381.

- Steencken, E. P. & Maher C. A. (2002). Young children's growing understanding of fraction ideas. In B. H. Littwiller & G. Bright (Eds.), *2002 NCTM Yearbook: Making Sense of Fractions, Ratios, and Proportions*, pp. 49-60. Reston, VA: National Council of Teachers of Mathematics.
- Steffe, L. P., Cobb, P., & von Glasersfeld, E (1988) *Young children's construction of arithmetical meanings and strategies*. New York: Springer Verlag (as cited in Davis, Hunting & Pearn, 1993).
- Steffe, L. P., von Glasersfeld, E. Richards, & Cobb (1983). *Children's counting types: Philosophy, theory and applications*. New York: Praeger Scientific (as cited in Davis, Hunting & Pearn, 1993).
- Stein, K.S., Smith, M.S., Henningsen, M.A. & Silver, E.A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press
- Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, 31(1), 5-25.
- Tirosh, D., Fischbein, E. Graeber, A. & Wilson, J. (1993). *Conceptual adjustments in progressing from whole to rational numbers*. Final report of the United States – Israel Binational Science Grant, Tel Aviv, Israel: Tel-Aviv University. (as cited in Tirosh, 2000).
- Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education*, 30(4), 390-416.
- Van de Walle, J.A. (2004). *Elementary and middle school mathematics: Teaching developmentally (5th ed.)*. Boston, MA: Pearson Education Incorporated.
- Warner, L.B., Alcock, L. J., Coppolo Jr., J. & Davis, G. E. (2003). "How does Flexible Mathematical Thinking Contribute to the Growth of Understanding?" In N.A. Pateman, B.J. Dougherty & J. Zillox (eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA*, July 13 – 18, 2003, Honolulu, Hawaii (Vol. 4, pp.371-378).
- Warner, L.B. & Schorr, R.Y. (in progress). *Flexible Mathematical Thought: Watching an Idea Grow*.
- Watson, J.M., Campbell, K.J., Collis, K.F. (1993). Multimodal functioning in understanding fractions. *The Journal of Mathematical Behavior*, 12(1), 45-62.

ELEMENTARY SCHOOL PRE-SERVICE TEACHERS' UNDERSTANDINGS OF ALGEBRAIC GENERALIZATIONS

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Abstract: It is critical for all students to learn algebra, including the ability to generalize, to function in our increasingly complex world. This pretest/intervention/posttest study of preservice elementary teachers (N = 63) in their math methods course assessed their knowledge of writing and applying algebraic generalizations using instructor-made rubrics along with analysis of work samples and reported insights. Initially, although most subjects could solve a specific case, they had considerable difficulty determining an algebraic rule. After a problem-solving-based teaching intervention, students improved in their ability to generalize, however, they encountered more difficulty with determining the algebraic generalization for items arranged in squares with additional single items as exemplified by x^2+1 , than with multiple sets of items, as exemplified by $4x$.

Keywords: algebra; generalizations; intervention; pre-service elementary teachers

Overview

It is critical for all school-aged students to learn algebra, including the ability to generalize, to function in our increasingly complex world (National Council of Teachers of Mathematics [NCTM], 2000; RAND, 2003). Preservice elementary teachers play a critical role in initiating and developing algebraic reasoning in grades K-6, however the research base of teachers' knowledge regarding algebraic instruction is rather limited (Doerr, 2004; Kieran, 2006). Many call for increased attention to algebraic reasoning in the elementary grades to ease the transition from arithmetic and build understanding of the abstract concept of variables (Kieran, 1992; Kaput, 2000). At the same time, teachers' weak conceptual understanding of essential subject-matter knowledge is well known (Ma, 1999). The transition from a procedural approach in arithmetic to a structural understanding of algebra does not come easily (Kieran, 1992). Without the prerequisite content knowledge on the part of preservice elementary teachers, meeting these objectives for students is unlikely. To meet the goals of teaching algebraic reasoning in the elementary school curriculum, we need to understand more about how preservice teachers are prepared for this undertaking.

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Algebraic reasoning at the elementary level takes many forms, including extending pictorial and number patterns, doing and undoing, understanding equivalence, solving for an unknown, and writing a generalization for a pattern (Carpenter, Franke, & Levi, 2003; Kaput, 2000; NCTM, 2000). It is this latter aspect of algebra that we will address. Because students' understanding of writing generalizations is enhanced using pictorial geometric patterns (Bishop, 1997), we investigate how writing algebraic generalizations from pictorial patterns affects *preservice teachers'* understanding. Therefore, this study examines the following questions: given a pictorial pattern, how do preservice elementary teachers interpret the pattern and write a corresponding generalization? And after completing the activities, how do preservice teachers perceive their ability to teach algebraic generalizations?

Theoretical Framework

The literature is replete with studies documenting both students' and elementary preservice teachers' difficulty with beginning algebraic reasoning and writing generalizations. MacGregor and Stacey (1997) investigated students' algebraic learning and found that students did not easily learn how to express simple relationships in algebraic notation. Students also misused algebraic symbols and syntax in relatively basic problems (allowing, for example, the letter *h* to represent height). Mac Gregor and Stacey found that misleading teaching materials reinforced the erroneous concept that a letter represents an object. Students extend patterns numerically more easily than they can generalize about them (Mac Gregor & Stacey, 1997; Zaskis & Liljedahl, 2002). Approaching algebraic expressions and equations from a contextual vantage point, Bishop (1997) asked seventh and eighth grade students to model perimeter and area problems with pattern blocks and tiles, and then generalize the relationships symbolically. Bishop found that the use of mathematical patterns promoted algebraic reasoning, but not all students were able to generalize. Gray, Loud, and Sokolowski (2005) found that students in college algebra classes and calculus classes had difficulty using variables as generalized numbers.

In contrast, students from classrooms that were a part of intensive staff development projects for in-service elementary teachers were found to be capable of algebraic reasoning. Third graders were able to generalize and formalize their mathematical thinking about even and odd numbers (Kaput & Blanton, 2000). In that study, students initially used computation to solve problems about even and odd numbers; later, they used the terms even and odd as placeholders (or variables). Although they were not at a formal symbolic level, the students in this study also perceived even numbers as multiples of two. On a state assessment, third graders in this project outscored fourth graders from a classroom not involved in the effort to improve the teachers' algebraic instruction (Kaput & Blanton, 2001).

Bishop and Stump (2000) examined preservice elementary and middle school teachers' conceptions of algebra. In a semester course, the preservice teachers engaged in college-level algebraic experiences involving generalization, problem solving, modeling, and functions. They found that many preservice teachers did not understand what distinguishes arithmetic from algebra, and of those that did make the distinction, a majority held a procedural perspective even at the end of the semester course.

Methodology

Sixty-three white undergraduate elementary preservice teachers (53F, 10M) enrolled in a mathematics methods class participated in the study. 79% of the participants completed and 11% were currently enrolled in a college level math course. All students took a pretest on the first day of class and an identical posttest nine weeks later, after the intervention had been completed. This instrument consisted of two problem sets in which drawings depicted the pattern described in the problem. The pre- and posttest consisted of two problems. The first problem set focused on writing a rule for the number of legs in sets of four-legged tables, $4n$; the second problem set presented a progressively larger design that could be described as x^2+1 , consisting of boxes arranged in a square with one additional box. For each problem, subjects were asked to: 1) solve for a specific case; 2) describe the generalization in words; 3) write an algebraic generalization; and 4) describe their strategy. A scoring rubric was developed by qualitatively categorizing student responses on the assessment at four levels: proficient, basic, developing, and poor.

The intervention involved two forty-minute activities conducted on different days where students worked in small groups to generate algebraic generalizations for sets of symbols (Sharp & Hoiberg, 2005). The instructional sequence was taught through problem solving. The launch of the lesson occurred as the instructor demonstrated her thinking in analyzing the first pattern set. Then, during the exploration, groups tried to solve the remaining problems cooperatively and the instructor provided hints and suggestions but not solutions. During the summary, a student from each group came to the front of the classroom, presented the group's solution to a problem, and discussed strategies. After input of ideas from other groups, the key points for each pattern were summarized.

Results

Pretest results showed that preservice teachers could continue a pattern and solve numerically for the next case. They had difficulty expressing ideas in words, writing a generalization, recognizing a pattern of square numbers, and explaining a strategy. The posttest revealed that the preservice teachers made significant growth in their understandings of algebraic generalizations as a result of the intervention activities. In addition to what they could do on the pretest, they could now express ideas in words, write a generalization, recognize a pattern of square numbers, and explain a strategy.

Our results corroborate prior research regarding the ability to generalize. Results differed for each type of question and the performance was stronger for the generalization $z=4n$ than for $z=n^2+1$. Preservice teachers were more successful at generalizing the pattern for the first problem set as shown in Table 1. Initially, 98% students could extend this pattern numerically and 89% could write a generalization. After the posttest, the percent of students able to write the generalization increased to 95%. Preservice students' ability to explain how they arrived at the answer, write a generalization, and explain strategies all improved.

Table 1.
Percent of students (N=63) scoring at each level for $Z=4n$

	Type of Q	Proficient (3)	Basic (2)	Developing (1)	Poor (0)
		Pretest-Posttest	Pretest-Posttest	Pretest-Posttest	Pretest-Posttest
1	Extend Pattern Numerically	98-97	0-0	0-0	2-3
2	Explain in Words	65-86	25-13	8-0	2-1
3	Generalization	89-95	5-2	3-0	3-3
4	Strategy	65-86	25-13	8-0	2-1

The second question, to extend the pattern of a number of boxes arranged in a square pattern plus one additional square proved to be more difficult for preservice elementary teachers, however, increases in ability to solve the problems occurred during the study. Initially, only 79% could extend the pattern numerically, and 41% could write a generalization. After the posttest, 97% of the students could extend the pattern numerically, and 98% could successfully write a generalization.

Table 2.
Percent of students (N=63) scoring at each level for $Z=n^2+1$

	Type of Q	Proficient (3)	Basic (2)	Developing (1)	Poor (0)
		Pretest-Posttest	Pretest-Posttest	Pretest-Posttest	Pretest-Posttest
5	Extend Pattern Numerically	79-97	8-0	0-1.5	13-1.5
6	Explain in Words	43-95	30-5	10-0	18-0
7	Generalization	41-98	6-0	11-2	41-0
8	Strategy	26-90	22-8	17-2	33-0

Preservice students were finally asked what they learned from the unit on algebra with a written survey. Responses were coded into three categories. The most frequent category of response addressed increased knowledge of techniques and strategies for writing a generalization. Students commented, "I was able to learn different strategies to show the problem," "There are many ways to solve them," and "Making up problems helped." About half of the students expressed a better understanding of the importance of teaching algebra as a result of the activities. The third category centered on improved ability to solve for a generalization. Many students commented that presenting and sharing strategies with the class helped them better understand how to arrive at a generalization. Almost half the students volunteered that they perceived an improvement in ability.

Discussion and Conclusion

Consistent with prior research, even though 79% of the students had completed a college level mathematics course, the pretest indicated that writing generalizations was difficult for

many preservice students. The posttest results indicated that preservice elementary teachers' ability to write a generalization of the type $y=4n$ and $y=n^2+1$ increased throughout the study. This is a difficult area of the curriculum for preservice elementary teachers, however, when problems were placed in a context and taught in a problem solving environment, understanding improved. As would be expected, more students were successful at the $y=4n$ type of problem. This is the type of question that is most typically found on grades 3-6 state assessments.

Students' work and comments during the practice showed they enjoyed the work but found it challenging. Inquiry, problem solving, and critical thinking occurred as students devised algebraic equations for sets of symbols. We recommend that instruction in algebraic generalization include group inquiry following a launch, explore, and summarize sequence. We also believe that projects that are complex and require analysis of the work of others be part of project-work in mathematics for preservice teachers. Many states now have adopted NCTM recommendations for teaching more algebraic reasoning in the elementary grades. An area for continued study is to see if incoming groups of preservice teachers improve on their initial understanding of writing generalizations.

References

- Bishop, J. W. (1997). *Middle school students' understanding of mathematical patterns and their symbolic representations*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL. (ED410107).
- Bishop, J. W., & Stump, S. L. (2000). Preparing to teach in the new millennium: Algebra through the eyes of preservice elementary and middle school teachers. In M. Fernandez (Ed.), *Proceedings of the 22nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: ERIC.
- Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.
- Dobrynina, G., & Tsankova, J. (2005). Algebraic reasoning of young students and preservice elementary teachers. In G.M. Lloyd, M. Wilson, J. L. M. Wilkins, & S. L. Behm (Eds.) *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Retrieved May 21, 2006 from http://convention2.allacademic.com/index.php?cmd=pmena_guest.
- Doerr, H. M. (2004). Teachers' knowledge and the teaching of algebra. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra: The 12th ICMI study* (pp. 267-290). Norwood, MA: Kluwer Academic Publishers.
- Gray, S. S., Loud, B. J., & Sokolowski, C. P. (2005). Undergraduates' errors in using and interpreting variables: A comparative study. In G.M. Lloyd, M. Wilson, J. L. M. Wilkins, & S. L. Behm (Eds.) *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Retrieved May 21, 2006 from http://convention2.allacademic.com/index.php?cmd=pmena_guest.
- Kaput, J. J. (2000). *Teaching and learning a new algebra with understanding*. U.S.; Massachusetts: National Center for Improving Student Learning and Achievement in Mathematics and Science Dartmouth MA.

- Kaput, J. & Blanton, M. (2000). Generalizing and progressively formalizing in a third grade mathematics classroom: Conversations about even and odd numbers. In M. Fernandez (Ed.), *The Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 115-119). Columbus, OH: ERIC.
- Kaput, J. & Blanton, M. (2001). Student achievement in algebraic thinking: A comparison of 3rd graders performance on a 4th grade assessment. In R. Speiser, C. Maher, & C. Walter (Eds.), *The Proceedings of the 23rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 99-107). Columbus, OH: ERIC.
- Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan Publishing Company.
- Kieran, C. (2006). Research on the learning and teaching of algebra: A broadening of sources of meaning. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 11-49). Rotterdam: Sense Publishers.
- Ma, L. (1999). Knowing and teaching elementary mathematics teachers' understanding of fundamental mathematics in China and the United States. Mahwah, NJ: Lawrence Erlbaum Associates.
- MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics*, 33, 1-19.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education/RAND Mathematics Study Panel*, Deborah Loewenberg Ball, Chair. Santa Monica, CA: RAND Corporation. (ED-476809)
- Sharp, J., & Hoiberg, K. (2005). *Learning and teaching K-8 mathematics*. New York: Pearson Education.
- Zazkis, A., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 49(3), 379-402.

COMPARISON OF HIGH AND LOW ACHIEVERS: A Discussion of Juter's (2007) article¹

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Abstract: *The use of questionnaires and interviews to compare the responses to a mathematical task of high achievers with low achievers has limitations. The partial information that they have provides a way of comparing high and low achievers. Some references are given here to relevant task formats and theories. An example is given of how examinees' performance with unusual task formats (specifically, answer-until-correct) may be analysed to throw light on the mathematical description of partial information.*

Keywords: answer until correct tasks; research methodology; questionnaire analysis; task formats;

1. Introduction

Juter (2007) compared high achieving students with low achieving students in respect of performance on problems concerned with limits of functions. Juter made use of questionnaires and interviews, and results were presented in the form of examples of responses given by high achievers and by low achievers. Presenting results in that way, and concluding that "high achievers have richer concept images" and their "abstraction abilities were more highly developed" (Juter, 2007, p. 64) does have some interest. But these descriptions do not say much more than that students who knew more about limits did, indeed, know more about limits --- the attempt at analysis is almost circular. Section 2 below will suggest some ways of comparing high achievers with low achievers that avoid this circularity. Section 3 gives an example of how empirical results (specifically, in an answer-until-correct task) may be compared with theories.

2. Discussion of Juter (2007)

The concern noted above may also be expressed as follows. On any single performance measure, high achievers are likely to score better than low achievers. This is unlikely to be of great interest on its own. An interesting research question is likely to involve two measures, and to concern

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how the between-group difference on one measure relates to the between-group difference on another measure. Exactly what these questions are and how they can be answered will naturally depend on what the different observed measures are; an example will be given in Section 3.

Instead of taking a numerical difference, one might put high achievers and low achievers on an equal footing in one respect, and then compare something else: instead of comparing responses by high achievers with responses by low achievers, compare wrong responses by high achievers with wrong responses by low achievers, and (separately) compare correct responses by high achievers with correct responses by low achievers.

- Do high achievers and low achievers differ in the wrong responses they give? In cases where one wrong response is considered less wrong than another, do high achievers tend to give the less wrong response? (Juter refers to embodied, proceptual, and formal modes of mathematical thinking. This might be the basis for classifying one wrong response as less wrong than another.) If a second attempt is permitted following a wrong response, do high achievers tend to do better than low achievers?
- If an explanation of a correct response is asked for, is its quality better for high achievers than for low achievers? If confidence in a correct response is asked for, is it higher for high achievers than for low achievers?

Once like is being compared with like, then it is reasonable to ask about richness of concept or abstraction, provided they can be operationalized and measured.

Most research into different wrong responses, performance at second attempt, and supplementary questioning has been based on multiple-choice items. Selection of different wrong responses by different ability groups in multiple-choice tests is discussed by Green et al. (1989), Price (1964), Wainer (1983), Wainer et al. (1984), and Hutchinson (1991, Sections 5.17, 8.6, 9.3, 9.4). When responses are generated (constructed) by examinees, there are often so many possible wrong responses that it is difficult to aggregate and classify them. However, Cairns et al. (2002) found evidence that some wrong responses are disproportionately generated by examinees of high ability and others disproportionately generated by examinees of lower ability.

As well as the nature of wrong responses and performance at second attempt, other topics that have been studied include performance when “don’t know” is one of the available options, performance when “none of the above” is one of the available options, performance when none of the available options are correct, performance when more than one option is correct, the changing of responses by examinees, and the confidence that the examinee expresses. Unusual task formats are sometimes considered to have both practical utility and psychological interest, in that they reveal more about the examinee than “choose the one correct, or best, answer” does. But the practical utility is debatable, as administration time tends to be longer --- if extra time is feasible, it might be better to set more items of conventional format. (Concerning confidence, it may be noted that although there is some plausibility in the idea that high achievers will tend to know they are correct, and low achievers will not, there are great complications: people may differ in how they use the scale of confidence, how well they know themselves, and how honest they are in reporting. For example, among the four students discussed by Juter, 2005, it was the best student who was the only one who was unsure whether she had control over the notion of a limit.)

Two further comments are worth making. (a) Some researchers have asked examinees to describe their thought processes. Unfortunately, it is particularly difficult to say anything about low achievers. Williams and Jones (1972) reported an interview survey of 15 schoolchildren who had taken a mathematics test, but found that not much information could be gleaned from the weaker students. See also Section 5.18 of Hutchinson (1991). (b) As previously noted, examinees' performance with unusual task formats holds some value for psychological theory. I urge those who have used such task formats to look carefully at the resulting dataset for any implications it may hold. Examples of comparing datasets with a theory that seeks to operationalize the notion of partial information are in Chapters 6 and 7 of Hutchinson (1991), and another is given in Section 3 below. (However, the data typically need to be aggregated --- e.g., over all examinees within a certain band of abilities --- and it is not certain that what is seen at the aggregate level is also the case for an individual examinee. For other limitations of the approach, see Chapter 8 of Hutchinson, 1991.)

Thus it seems that methods concentrating on individual examinees (discussing responses to particular questions, as Juter did, or asking about thought processes), and methods that employ large samples and aggregated data, each have strengths and weaknesses.

3. Example of quantitative study of partial information

Suppose there is data on examinees' performance with an unusual task format. Sometimes a simple feature of the data is directly of interest. For example, is second-choice performance only at the chance level? Or, do the proportions with which different incorrect options are selected differ when examinees are grouped according to ability? On other occasions, a quantitative prediction is the centre of attention, as in the following example.

In answer-until-correct (AUC) tests, the examinee is given immediate feedback as to whether the response is correct; if it is wrong, then the examinee chooses another option, and again is given immediate feedback; the examinee continues until the correct option is chosen, then moves on to the next item. The dataset to be discussed is from Abplanalp (1995). That paper had much about the practicalities of AUC testing, and some interesting data, but lacked any theory to give context to the data. Consider the relationship between the number of errors when the test is scored conventionally and when using the AUC method. Figure 1 shows Abplanalp's data, which was from a test of 22 items having 5 options each, taken by 74 examinees. The horizontal axis shows the average number of wrong options chosen per item when using the AUC format, and the vertical axis shows the proportion of items answered correctly at first attempt.

Let y be the probability of answering correctly at first attempt, and x be the average number of wrong options chosen per item. Further, let m be the number of options per item. The limits on the relationship between x and y are as follows.

- If, whenever a second attempt is needed, the examinee is always correct at second attempt, $x = 1 - y$.
- If the examinee always chooses the correct option *last* whenever it is not chosen first, $x = (m-1)(1-y)$.

The two extremes are shown as dashed lines in Figure 1, as in Abplanalp's Figure 3.

The simplest theory for the relationship between x and y is based on assuming that the examinee's subsequent attempts are equivalent to random guesses whenever the first choice is wrong. Then $x = m(1-y)/2$. When m is 5 (as in Abplanalp's test), this leads to $y = (5 - 2x)/5$, and this is the straight line in Figure 1. It can be seen that most of the data points lie below this. That is, the examinees, if they are wrong at first attempt, take fewer attempts to find the correct response than they would if they had no knowledge. (At any given y , we can look across and see that the data points have a smaller x than would be expected if the examinees had no knowledge.) We might say the examinees have some degree of *partial information* about the item.

Alternative predictions arise from the following approach. (For more details, see Hutchinson, 1982, 1991, 1997.) Suppose that the examinee considers each option within each item, and that each option within each item gives rise to some feeling as to the degree to which it fails to match the question posed. At first attempt, the examinee will choose the option generating the lowest feeling of mismatch. If the first choice turns out to be wrong, so that a second attempt is necessary, the option generating the second-lowest feeling of mismatch is chosen. And so on. Now suppose that the mismatch for the correct options is taken from some probability distribution, and that the mismatches for the wrong options are taken independently from some other probability distribution. The distribution for the wrong options will have a higher mean than that for the correct options. Indeed, the difference between the means is a measure of the examinee's ability. Let the probability of the mismatch exceeding z be $F(z)$ for correct options and $G(z)$ for wrong options. Further, let f be the probability density of mismatch for correct options, $f = -dF/dz$.

- The probability of being correct at first attempt is the probability that the mismatch from the correct option is some value z , multiplied by the probability of all of the mismatches from the wrong options (there are $m - 1$ of them) being greater than z , integrated over all z .
- If the mismatch from the correct option is z , the proportion of mismatches from wrong options that are less than z is $1 - G(z)$, and the expected number of them is $(1 - G(z))(m - 1)$. Averaging over different values of z is achieved via another integration.

Let λ be some measure of how different G is from F , that is, a measure of the examinee's ability --- it might, for instance, be the difference between the means of the distributions. Once an assumption has been made about what F and G are, the integrations referred to above lead to an equation for y in terms of λ and an equation for x in terms of λ . Then the equation for y in terms of x is obtained by elimination of λ .

To get a definite prediction, it is necessary to make some specific assumption about what F and G are. Three examples that are easy algebraically are as follows.

- Exponential distributions. Here, mismatch is taken to have an exponential distribution with mean 1 in the case of a correct option, and an exponential distribution with mean λ (this being greater than 1) in the case of wrong options. In the case of $m = 5$, this leads to $y = (4 - x)/(4 + 3x)$.

- All-or-none knowledge. Mismatch is taken to have uniform distributions, with the upper end of the range being the same for the wrong options as for the correct option. For $m = 5$, $y = (5 - 2x)/5$, as given earlier.
- Recognisable distractors. Mismatch is taken to have uniform distributions, with the lower end of the range being the same for the wrong options as for the correct option. For $m = 5$, $y = [1 - (1 - x/2)^5]/(5x/2)$. This is quite the opposite to the previous model, in the sense that now a wrong option is sometimes recognised as being wrong, but the correct option is never positively identified as such. This assumption has found occasional application in the psychological literature (Murdock, 1963; see also Section 4.6 of Hutchinson, 1991).

It may be asked how different are the three models, and whether Abplanalp's data favour one in preference to the others. The relationships between y and x are plotted in Figure 1. It appears that examinees have some degree of information when they give a wrong answer initially, but that it is rather less useful than is implied by the "recognisable distractors" and "exponential" theories.

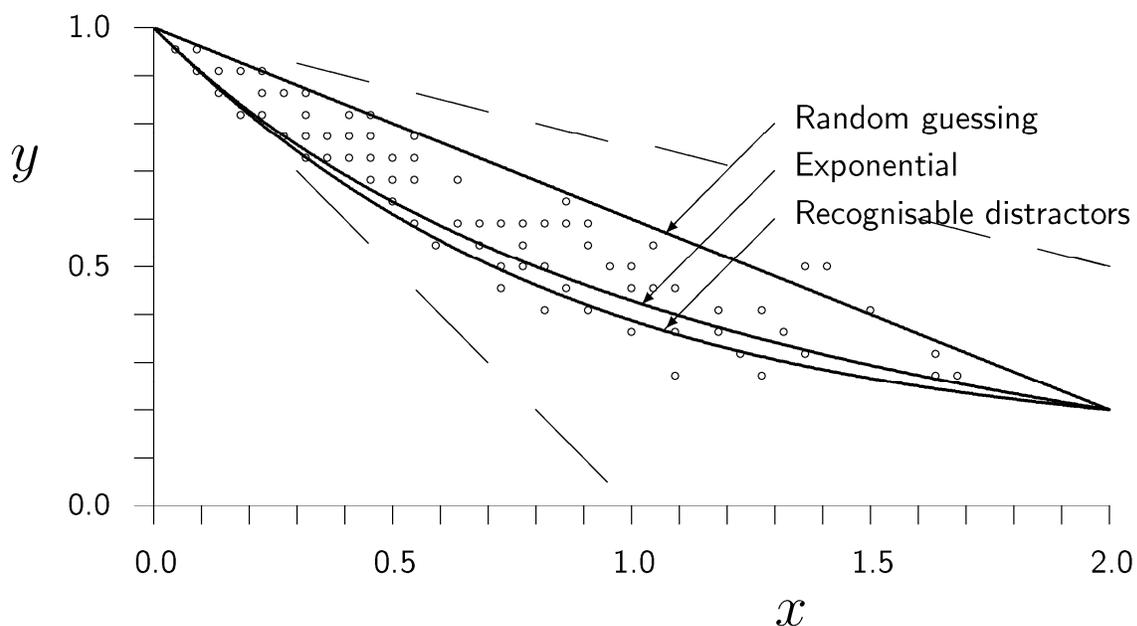


Figure 1. Data from Abplanalp (1995) compared with the predictions of three theories; y is the probability of answering correctly at first attempt, and x is the average number of wrong options chosen per item

Acknowledgements

The Centre for Automotive Safety Research, University of Adelaide, receives core funding from the Motor Accident Commission (South Australia) and the Department for Transport, Energy and Infrastructure (South Australia). The views expressed in this paper are those of the author, not necessarily of the University of Adelaide or the sponsoring organizations.

References

- Abplanalp, P. (1995). Application of latent image technology to the multiple-choice test format. *Optometric Education*, 20, 118-123.
- Cairns, D., Chekaluk, E., & Hutchinson, T. P. (2002). The wrong responses to a very difficult item: A comparison of high-scoring and low-scoring examinees. *International Journal of Mathematical Education in Science and Technology*, 33, 839-842.
- Green, B. F., Crone, C. R., & Folk, V.G. (1989). A method for studying differential distractor functioning. *Journal of Educational Measurement*, 26, 147-160.
- Hutchinson, T. P. (1982). Some theories of performance in multiple choice tests, and their implications for variants of the task. *British Journal of Mathematical and Statistical Psychology*, 35, 71-89.
- Hutchinson, T. P. (1991). *Ability, Partial Information, Guessing: Statistical Modelling Applied to Multiple-Choice Tests*. Adelaide: Rumsby Scientific Publishing.
- Hutchinson, T. P. (1997). Mismatch models that permit partial information to be shown. In W. J. van der Linden & R. K. Hambleton (Eds.), *Handbook of Modern Item Response Theory* (pp. 481-494). New York: Springer.
- Juter, K. (2005). Limits of functions --- How do students handle them? *Pythagoras*, 61, 11-20.
- Juter, K. (2007). Students' conceptions of limits: High achievers versus low achievers. *The Montana Mathematics Enthusiast*, 4, 53-65. Retrieved from <http://www.math.umt.edu/TMME/vol4no1/TMMEv4n1a3.pdf>
- Murdock, B. B. (1963). An analysis of the recognition process. In C. N. Cofer & B. S. Musgrave (Eds.), *Verbal Behavior and Learning: Problems and Process* (pp. 10-22). New York: McGraw-Hill.
- Price, D. B. (1964). A group approach to the analysis of individual differences in the randomness of guessing behavior on multiple-choice tests and the development of scoring methods to take such differences into account. M.Ed. thesis, Rutgers, The State University, New Brunswick, New Jersey. Circulated as Research Bulletin 64-59, Educational Testing Service, Princeton, New Jersey.
- Wainer, H. (1983). Pyramid power: Searching for an error in test scoring with 830,000 helpers. *American Statistician*, 37, 87-91.
- Wainer, H., Wadkins, J. R. J., & Rogers, A. (1984). Was there one distractor too many? *Journal of Educational Statistics*, 9, 5-24.
- Williams, S. I., & Jones, C. O. (1972). A comparison of interview and normative analysis of mathematics questions. Test Development Report 71-4, Educational Testing Service, Princeton, New Jersey.

FOSTERING CONNECTIONS BETWEEN THE VERBAL, ALGEBRAIC, AND GEOMETRIC REPRESENTATIONS OF BASIC PLANAR CURVES FOR STUDENT'S SUCCESS IN THE STUDY OF MATHEMATICS

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Abstract: We discuss the significance of making connections between the verbal, algebraic, and geometric representations of basic mathematical objects for students' understanding of mathematical instructions. Our survey of 499 students enrolled in a pre-calculus university course reveals that such connections are not always present, even if the objects themselves are familiar to the students. We stress that the ability of making these connections needs to be specifically addressed in teaching mathematics at various levels. A proper attention to the matter contributes to the formation of students' mathematical background, which makes a difference for their success in study of calculus, in particular.

Keywords: line, circle, semicircle, parabola, hyperbola, ellipse, planar curve, graphical image, prototype, algebraic formula, algebraic transformation, mathematical definition, concept formation.

Introduction.

The words we use have different degrees of precision and clarity; they have different capacities to identify various concepts and express certain images and feelings we may experience. Consequently, some rare words may evoke fuzzy and uncertain images, if any at all. Even if a word sounds familiar it may produce nothing but a blank image in one's mind. It may also produce a poor or inadequate association featuring some restrictive interpretation or a very specific situation. The ability to retrieve a complete, adequate, and flexible image associated with a given word is essential for our communication. The development of this ability depends on the frequency of using the word in a conversation, as well as the context, personal experience and practices related to the word. In order to enhance this development it is important to reflect upon and

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adjust the image through observations of how others use the word or respond to it. The attachment of a word to an idea or object does not appear at once. There are various cognitive processes leading towards the formation of a word's meaning:

- Categorization in a very rough way.
- Recognition and assigning some meaning within a context only.
- Evocation of a related image without a context.
- Frequent use in speech or writing.
- Recollection of the word, given a definition or description of it (like in a crossword).
- Recognition of synonyms and antonyms.

Consequently, there are various levels of familiarity with a particular word:

- Never heard.
- Heard but do not know exact meaning.
- Can guess the most appropriate meaning from a variety of given descriptions.
- Can give an example or counterexample.
- Can list properties.
- Can explain the meaning with various representations and contexts.

Our everyday casual words and words used in a mathematical context do not differ much in the sense specified above: they all carry a certain meaning, which develops through their use in conversations and is accompanied by formation of certain images. A non-understanding of a sentence starts from a non-understanding or inadequate understanding of a word. However, in a mathematical conversation the situation becomes more complex due to the fact that many words have a precise formal definition, which can be expressed in mathematical symbols and formulas. The formulas may also have a geometrical or pictorial representation to accompany them and to add to the formation of a complete image. Thus, in a mathematical conversation one often needs a three-way linkage between words, formulas, and graphs. Lack or weakness of one of those associations leads to poor understanding and failure to grasp the meaning of a mathematical sentence.

In this study we worked with 499 first year university students enrolled in a precalculus course. We collected data concerning students' ability to match names, formulas, and graphs of basic planar curves, as the ability developed in high school courses. We express a concern about an unreasonable assumption, frequently occurring in teaching practices, about the presence of those three-way links in students' cognitive schemas. In order to be effective, an instruction shall not rely on the assumption about the presence of those links. Instead, it shall reinforce and strengthen the links by means of repetitive juxtaposition of the same ideas in the three different representations.

The paper is organized as follows. In section 1 we have a brief discussion about concept formation and acquisition in terms of how the concept is introduced within a field of professional knowledge and internalized by a particular learner, who is new to the field. In section 2 we describe our experimental setting, and research questions. Section 3

summarizes the expected mathematical background and abilities of the students of our sample. Section 4 contains the results of our survey. We conclude the paper with a discussion in section 5 about the importance of forming proper connections between names, images and formulas of basic plane curves, particularly for the students' future success in study at the university level.

1. Words and images in mathematics.

While it is questionable whether or not many fundamental concepts are fully expressible in words and images, an ability to do so, or at least a desire to do so with a certain precision is essential for clear communication of our understanding of them. Words and images play a dual role in this process: we use the words to define the concepts formally, but we often rely heavily on the images when it comes to internalizing the meaning.

Tall & Vinner (1981) define *concept image* as a “total cognitive structure that is associated with the concept”. In their description it is important that the image must include processes and properties besides all mental pictures associated with the concept. They contrast the notion of image with *formal concept definition* as “a form of words used to specify a concept”, and argue that in thinking the concept image will almost always be evoked while formal definition “will remain inactive or even forgotten”.

Furthermore, it was observed that mental pictures associated with a concept contain special examples that are highly significant for the grasp of the concept. Such examples, often called prototypes, are used by the learner as “cognitive reference points”. The prototypical thinking was identified in the study of natural semantic categories (Rosch & Mervis, 1975), as well as in a geometrical context (Hershkowitz & Vinner, 1983). In visual prototypical thinking “the shape of the prototype serves as a criterion for judgment” (ibid). Besides that, the thinking could be based on self-attributes of a prototype, i.e. on the features and properties this particular prototype possesses. The drawback of a prototypical judgment is that while some features of a prototype are not characteristic for the category or concept the prototype represents, they can nevertheless be considered as being essential. In this case, the student may “reject an instance as an exemplar of a concept because the instance lacks the self-attribute of the prototype” (Schwarz & Hershkowitz, 1999).

Another serious problem with prototypical thinking is that the degree of rigor is insufficient to carry on a mathematical derivation. As noted by Poincaré (1996) in his discussion on a role of the definition in mathematics, “many learners will not have understood, unless they find around them the object of such and such mathematical nature. Under each word they want to put a sensible image; the definition must call up the image, and at each stage of a demonstration they must see it transformed and evolved. On this condition only will they understand and retain what they have understood. These

often deceive themselves: they do not listen to the reasoning, they look at the figures; they imagine that they have understood when they have only seen”.

A concept image appeals to a student's intuition, “but intuition cannot give us exactness, not even certainty, and this has been recognized more and more”. The exactness cannot be introduced in arguments unless it is “first introduced in definitions”. These observations lead us to a conclusion that formal definitions are essential for mathematical culture but often become an obstacle for mathematical teaching and learning. Initially, students need to be given an image of a concept, a prototype, an example, a framework for developing their intuition. But after this stage “they should be made to see that they do not understand what they think they understand, and brought to realize the roughness of their primitive concepts, and to be anxious themselves that it should be purified and refined” (Poincaré, 1996).

In the introduction of Polya's celebrated book *How To Solve It*, we find a similar idea: mathematics has two faces, it is presented by rigorous definitions and proofs, but it is discovered or invented by guessing and intuition. This fact is reflected in the existence of radically different approaches to its teaching and learning and in extensive discussion among educators taking opposite sides of the debate.

An analysis of the interplay between rigor and intuition brings us to the following important goal of mathematical teaching. That is, helping the learners to establish and be in control of a strong connection between the words and formulas used in mathematical reasoning, and the images produced in the learners' minds. The students ought to develop an awareness of their mental actions and the degree of adequacy of their mathematical prototypes, reinforcing their reasoning.

2. The sample, the procedure and the research questions.

2.1 The sample.

The sample consisted of 499 students enrolled in a precalculus undergraduate course at a large Atlantic Canadian University. This course is offered by the Department of Mathematics and Statistics for those students who, according to their Mathematical Placement Test scores, need to improve their mathematical skills in order to study calculus and other courses offered by the department. These students have previously studied mathematical concepts tested in our questionnaire in senior high school. The questionnaire was administered before these concepts were reviewed and used in the precalculus course.

According to the provincial curriculum, the most advanced mathematical course, which is not required for graduation but is desirable for students planning mathematics related university study, is Mathematics 3207. Students in the advanced stream normally graduate from high school with Mathematics 3205, and students in academic stream – with Mathematics 3204. The same core curriculum and textbook is used for both

Mathematics 3204, and Mathematics 3205, but the latter course covers the material in more depth.

Upon labelling what was the highest-level senior high school mathematics course and year of graduation, the students were divided into representative categories. There were 73 students graduated with Mathematics 3207, 52 students with Mathematics 3205, 222 students with Mathematics 3204, and a mixed sample of 152 students who did not specify the highest-level mathematics course taken.

2.2 The survey and the procedure.

The questionnaire shown in Appendix A was administered in English to the subjects of the sample. The students were not asked to provide their names, but they were asked to state the highest mathematics course taken in high school and the year of completion.

There was no review or any special activity aimed at refreshing students' memory about the subject of the survey. The students did not know prior to the survey what types of questions are going to be asked and were not specifically prepared for them. The students were asked to perform to the best of their ability, but they were not motivated by any reward for showing good results. We speculate that many of them were working at the level of knowledge recall and did not try to analyze in any way the information given. In this sense, the results of the survey reflect the true state of the concepts' knowledge as they were formed and retained by the students.

The questionnaire was administered for 25 minutes, during regular class time. The first question was designed to reveal the students' concept images. Within the first question, six words were provided: line, circle, semicircle, ellipse, parabola, and hyperbola. The students were asked to draw what first comes to their mind upon reading the given words. The Cartesian coordinate axes with no division scale were given. The second question asked the students to state how many functions can be drawn through three given points. The Cartesian coordinate system was provided and did not contain a division scale. The three points were positioned in the first quadrant. This question is the same as in a study of Schwarz & Hershkowitz (1999). We do not provide results for this question, as its purpose was to act as a separator between the first and the third question. The third and last question was designed to test the students' understanding of correspondence between algebraic and graphical transformations. Within the third question, the provided graphs incorporated scaled axes. The students were asked to match the formulas and names with the provided images. The questionnaire specifically addressed the fact that there might be several correct formulas for one graph, e.g. $x = -2$ and $x + 2 = 0$; $xy = 1$ and $y = \frac{1}{x}$; $y = |x|$ and $y = \sqrt{x^2}$ for line, hyperbola, and absolute value, respectively. Students could have matched one of two or both formulas for the same graph.

2.3 The research questions.

Based on the results obtained from the survey, we aim to address the following questions:

1. What are the students' prototypes associated with the words: line, circle, semicircle, parabola, hyperbola, and ellipse?
2. What is the most frequently encountered example in each case?
3. How well are the students able to recognize and name the graphs of the curves listed in question one?
4. How well are the students able to match the graphs of the curves with the associated algebraic equations, and to recognize the corresponding algebraic and geometric transformations, such as shifts and stretching, applied to the standard form of a curve?

3. Mathematical context to be tested in the survey.

3.1 General principles and approaches introduced in high school.

The objects we work with have a strong visual aspect: they all are plane curves, which can be defined as a locus of points in the plane with certain geometric properties. While the curves can be introduced through those characteristic properties, or otherwise as conic sections, they are also graphs of algebraic equations in the Cartesian coordinate plane. According to the high school curriculum, the students we surveyed were supposed to be familiar with only the latter aspect of the curves. Needless to say, this reduces the richness of the concepts along with the broadness of possible applications, but we leave this matter for another discussion.

The important fact that should be known to students is that behind each of the tested mathematical object such as line, circle, parabola, etc., there is a whole family of curves. Usually one can talk about the principal member of the family equipped with a number of parameters. Varying the parameters, one can describe all other members of the family, including some degenerate or untypical cases, and even bifurcations of the family. This fact can be viewed as an application of a more general principle: starting from an arbitrary curve one can transform it by stretching and shifting to another curve of the same algebraic kind. Alternatively one talks about rescaling and shifting the system of coordinates while leaving the curve unchanged. In any case the core of the general principle is the correspondence between the algebraic and geometric transformations: the horizontal/vertical translations of the curve produce the shift of the arguments in the algebraic equation of the curve $F(x, y) = 0 \rightarrow F(x - a, y - b) = 0$, while the horizontal/vertical stretching of the curve corresponds to the rescaling $F(x, y) = 0 \rightarrow F(ax, by) = 0$. Note that both operations are linear with respect to the arguments x and y .

Our first question aims to find out whether or not the name of a curve evokes any graphical images in the minds of the students. Considering that there is an infinite number of possible responses, we are also interested whether some of them are more popular than others, and how broad or narrow is the set of all produced examples in the case of each curve.

On a separate page we tested the students' ability to name an algebraic curve given in the Cartesian plane and to choose an appropriate formula from a pool of algebraic equations. Besides knowing the prototypical shape of curves, another principle appears to be very helpful for matching a Cartesian graph with a formula, i.e. the curve consists of those and only those points whose coordinates satisfy the algebraic equation of the curve. Consequently, it helps to look at some special points, such as the x - and y -intercepts and the origin, as well as to investigate the boundaries of a curve, and to identify special features of the domain and range.

Thus, besides the basic knowledge and comprehension, this task requires analysis and synthesis to some degree. The latter comes into play particularly when a student is asked to recognize an algebraic formula for a non-traditional (for senior high school) but intuitively familiar curve, such as a semicircle. Acquiring the skills of analysis and synthesis is possible if "elements are not presented as meaningless statements to be learned at the level of Knowledge, but where emphasis is on "why" of each point" (Whilhoyte, 1965 as cited in Furst, 1981). "Thus, the student may not know what a principle means until understanding occurs at least at the next level (Comprehension). But even under knowledge of specific there is necessarily embedded a variety of intellectual abilities and skills" (Pring 1971, & Sockett, 1971 as cited in Furst, 1981).

For the purpose of illustration we present few examples of reasoning useful for matching the equation $y = \sqrt{1-x^2}$ with corresponding graph (see Appendix).

Method 1. Analyzing domain and range of function $y = \sqrt{1-x^2}$ students notice that $y \geq 0$ and $y \leq 1$ and that $x^2 \leq 1$. Thus, the entire curve is constrained by the rectangle $-1 \leq x \leq 1$, $0 \leq y \leq 1$. This makes the choice of graph obvious.

Method 2. If the students start from the graph, they notice that the following integer points $(1,0)$, $(-1,0)$, and $(0,1)$ belong to the graph. Thus, they can choose $x = 1$ and $y = 0$ and substitute these values into the provided equations, until one gives an identity. If more than one graph are selected this way, then other integer points will help to single out the answer.

Method 3. Students square both sides of the equation $y = \sqrt{1-x^2}$ and obtain the familiar equation of the unit circle. Then, observing that $y \geq 0$, they choose the graph of the upper semicircle.

3.2 Particular notions introduced in high school.

This section gives a brief overview of when and how the curves of our interest are introduced in the textbooks currently used in the province. In this respect, we refer to *Mathematical Modeling, Book 1* (Barry, Small, Avard-Spinney, & Wheadon, 2000) used for study Mathematics 1204, which is a level-one senior high school course, normally taken by students in grade 10, and *Mathematical Modeling, Book 3* (Barry, Besteck-Hope, Hope, Pilmer, Small, Avard-Spinney, & Wheadon, 2002), used for both Mathematics 3204 and Mathematics 3205, which are graduation level courses. For the most advanced mathematical course Mathematics 3207, *Mathematical Modeling, Book 4* (Barry, Besteck-Shaw, Brown, & Avard-Spinney, 2002) is used.

1. The line

The line is formally introduced in Mathematics 1204 in the slope y -intercept form $y = mx + b$, where m represents the slope and b is the y -intercept. In Book 1, the concept of line is mainly used in applications of linear behaviours, e.g. economy-cost issues.

2. The circle

The name and the shape of the circle are introduced as early as elementary school. However, neither the equation nor the coordinate axes are present until Mathematics 3204/3205. In Book 3, the circle is defined as “the set of points in a plane that are at the same distance (radius) from a fixed point called the centre” (Barry et al., 2002). A unit circle is introduced via equation $x^2 + y^2 = 1$ as a circle with radius 1 and centered at the origin. Any circle is viewed as an image of the unit circle under one of the following mapping rules $(x, y) \rightarrow (rx, ry)$ and $(x, y) \rightarrow (x + h, y + k)$, or their combination. As a result, the general equation of a circle in standard form is $(x - h)^2 + (y - k)^2 = r^2$. It can

be rewritten in the transformational form as $\left[\frac{x - h}{r}\right]^2 + \left[\frac{y - k}{r}\right]^2 = 1$.

3. The absolute value function

In the high school course Mathematics 1204, the notion of absolute value $|x|$ is introduced as the distance between a number x and the origin. The algebraic description of this function is $y = |x|$. In Book 1, students are encouraged to “construct a table of values for this function using x -values between -4 and 4 ” (Barry et al., 2000), to graph the function and to describe the shape of the function in their own words. A variety of examples are listed and the theoretical results of their investigations are summarized succinctly as vertical and horizontal translations. For example, in Book 1 “the graph of $y - q = |x|$ is the image graph of $y = |x|$ after a vertical translation of q units; and the graph of $y = |x - p|$ is the image graph of $y = |x|$ after a horizontal translation of p units” (Barry et al., 2000). Reviewing the absolute value in Book 4, a more elaborate image is

presented, i.e. “the graph is composed of two segments, each described by a different linear equation” (Barry et al., 2002). The notion of a piecewise-linear function and algebraic formula, $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ are introduced. Therefore, a complete connection between name, algebraic definition and graphical image is established in Mathematics 3207.

4. The parabola

In Book 1, the investigation technique is used in introducing the concept of a graph of a quadratic function. Students are being asked to “construct a table of values” for $y = x^2$ “using x -values between -4 and 4 ”, and then they are asked to graph the function (Barry et al., 2000). Oftentimes, the emphasis is placed on the study and recognition of elementary functions, e.g. “if you can recognize the graphs of the basic functions like $f(x) = x$ or $f(x) = x^2$, you can often use these basic shapes to sketch the graphs of more complex functions” (Barry et al., 2000). With reference to the material studied before, the term parabola is introduced in Book 3, as “the graph of any quadratic function” (Barry et al., 2002). Details pertaining to the vertex, axis of symmetry and the transformational form are discussed. The transformational form of a quadratic function is expressed as $a(y - k) = (x - h)^2$, where parameters a, k, h are real numbers and $a \neq 0$. The transformational form is used as early as Mathematics 1204, together with the standard form $y = a(x - h)^2 + k$, where $a \neq 0$. In both Book 1 and Book 3, the first example introduced is $y = x^2$ and is often used for further comparison with transformed shapes.

5. The ellipse

We notice that the shape of the ellipse appears as early as Mathematics 1204 (Barry et al., 2000), but no proper identification is attached to the shape. During Mathematics 3204/3205, the name oval is used for the first time in conjunction with the shape of an ellipse (Barry et al., 2002). Further along Book 3, the ellipse is explored as being a stretching transformation of the unit circle with possible translation. The

transformational form of the equation of the ellipse is given as $\left[\frac{x - h}{a}\right]^2 + \left[\frac{y - k}{b}\right]^2 = 1$.

6. The hyperbola

As early as Mathematics 1204, students have the opportunity to see hyperbolas, although the actual name of the curve is not revealed in Book 1. The shape of a hyperbola occasionally appears, e.g. in the “equipping your function toolkit” section (Barry et al., 2000). In Mathematics 3207, the simple rational functions are formally introduced. The

first example of such function appears in Book 4 and has the form $f(x) = \frac{c}{x}$ (Barry et al., 2002). In the same book, the hyperbola is defined as follows. “These functions (i.e. $y = \frac{c}{x}$) are examples of rational functions and their graphs can form a conic section called a *hyperbola*” (Barry et al., 2002). The notions of horizontal and vertical asymptotes are also introduced and discussed at this level.

4. Results.

4.1 Evoking images.

The first question of our survey stated: “Draw what comes to mind when you read the following words: line, circle, semicircle, ellipse, parabola, and hyperbola”. The data collected address our first two research questions: what are the students’ prototypes and what is their frequency? The results obtained for the first question are presented in the following charts.

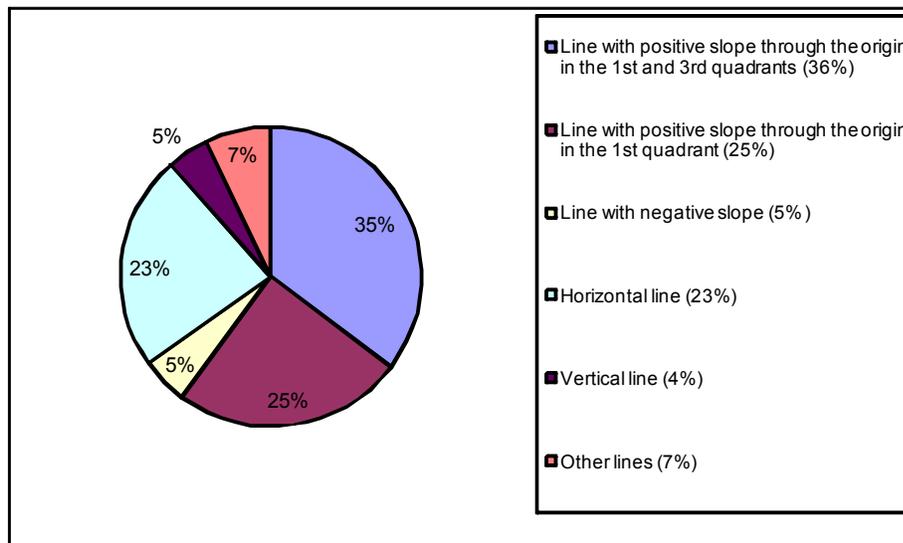


Figure 1. The variety and frequency of images of *line* evoked by the entire sample of the precalculus students.

With respect to drawing lines, 61% of the students draw lines with positive slope; while only 5% draw lines with negative slope. We infer that the apparent prototype is the line with positive slope, passing through the origin. The lines with positive slopes drawn followed the pattern of $y = x$, or small variations of it, e.g. $y = cx$ and $c > 0$. We believe that the observed apparent prototype is influenced by both the frequency of similar

examples and the nature of the very first example students encountered while the concept was introduced.

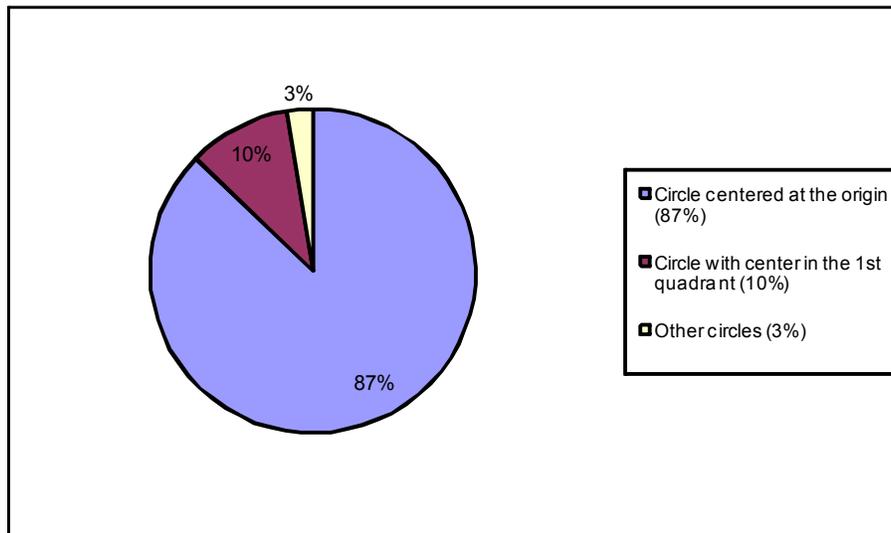


Figure 2. The variety and frequency of images of *circle* evoked by the entire sample of the precalculus students.

With respect to drawing the circle, 87% of the entire sample did draw a circle centered at the origin. We infer that the obvious prototype is the circle centered at origin.

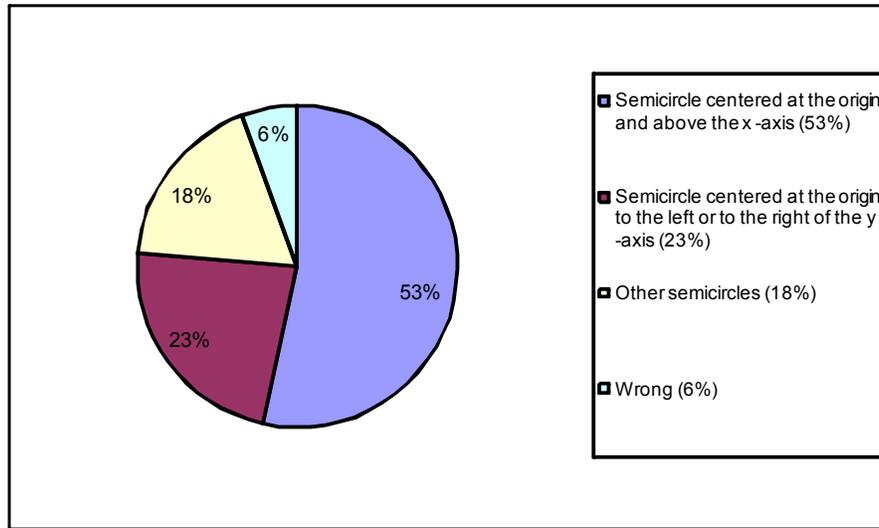


Figure 3. The variety and frequency of images of *semicircle* evoked by the entire sample of the precalculus students.

In terms of the semicircle concept, 76% of the entire sample decided to split in half the prototype circle either above the x -axis or to the left or right of the y -axis. Therefore, we infer that the semicircle prototype is directly connected to and derived from the circle prototype. Only 18% of the entire sample decided to draw other types of semicircles.

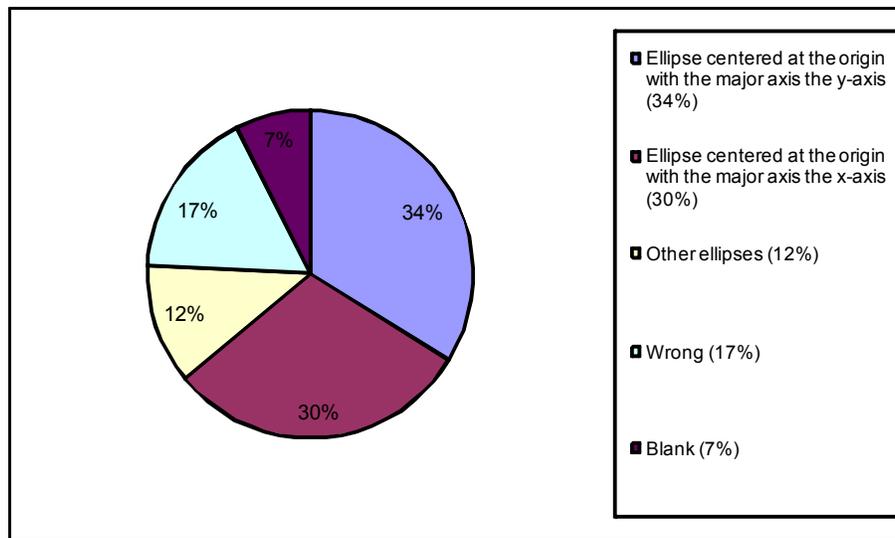


Figure 4. The variety and frequency of images of *ellipse* evoked by the entire sample of the precalculus students.

With regards to the ellipse, there is no clear winner in terms of the prototype used; since 34% draw an ellipse stretched along the y -axis, while 30% draw an ellipse stretched along the x -axis.

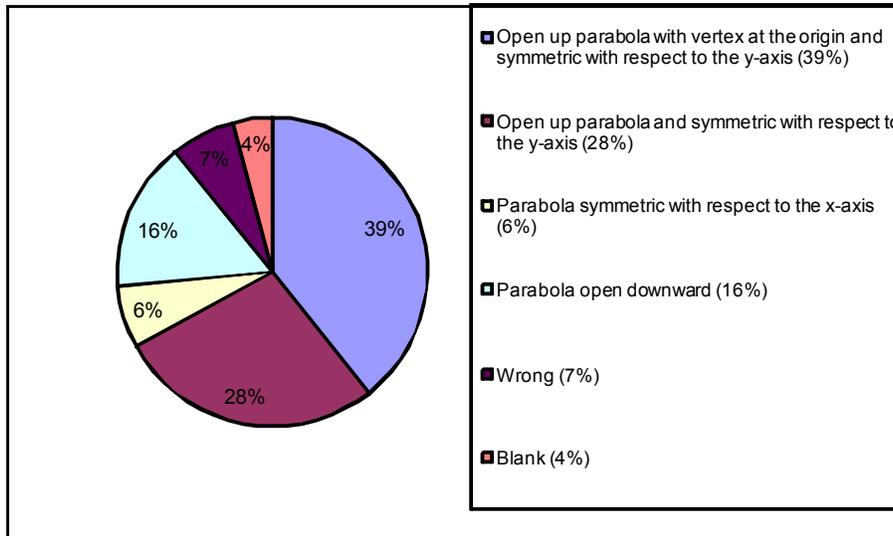


Figure 5. The variety and frequency of images of *parabola* evoked by the entire sample of the precalculus students.

With respect to drawing parabolas, 67% of the entire sample's preference was related to drawing an open upward parabola, while 16% of the students draw open downward parabolas. We infer that the evident prototype is an open upward parabola, passing through the origin. The drawn open upward parabolas followed the pattern of $y = ax^2$, $a > 0$; while the open downward parabolas followed the pattern of $y = ax^2$, $a < 0$. In other words both types of parabolas had vertex at the origin. We believe that the observed prototype coincides with the first example of the graph of a quadratic parabola presented in Mathematics 1204.

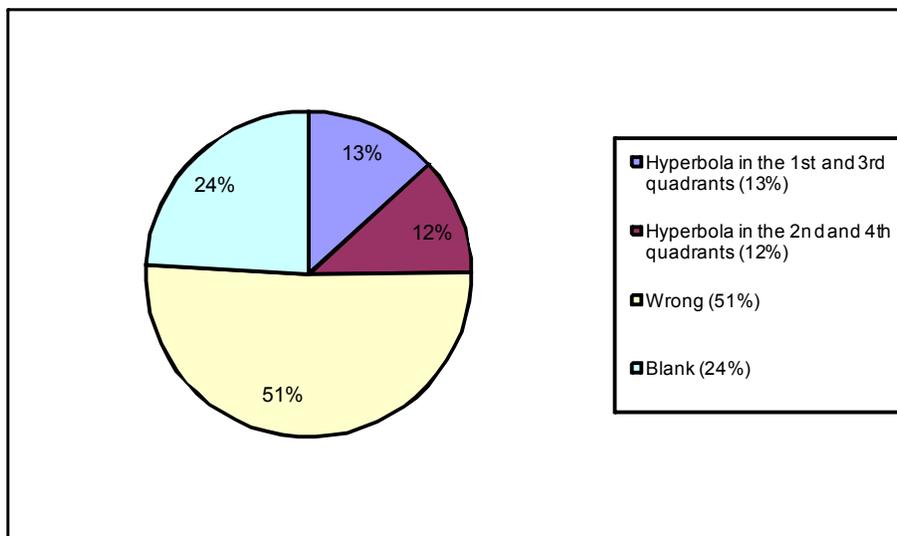


Figure 6. The variety and frequency of images of *hyperbola* evoked by the entire sample of the precalculus students.

The diagram on Figure 6 clearly reflects the absence of the hyperbola from the high school curriculum. As we pointed out earlier, only students completed Mathematics 3207 receive proper knowledge in relation to this curve. Such students constitute about 15% of our sample, so the fact that 25% of the sample nevertheless is familiar with the curve, is an evidence of random occurrence of this object in earlier mathematical courses.

It is noticeable that the majority of the graphs produced by the students are either centered around the origin (circle, semicircle, ellipse and hyperbola) or pass through the origin (line and parabola). It is hard to say whether this is an evidence of the rigidity of students' prototypes having an irrelevant feature such as reference to the origin of the Cartesian plane. Probably, this is just a natural result of frequent exposition of the students to the origin-centered graphs, so that images having this attribute are indeed "what comes to mind first" but this does not exclude the familiarity of the students with other less typical examples. Having said that, we still see a potential danger of the frequent use of the origin centered examples, as this may cause the formation of a distorted view and restricted prototypes, and is particularly undesirable for students planning to study future mathematical courses that require more flexibility and adaptability of the images. The students' ability to juggle with the visual and graphical aspects of basic curves will be essential in grasping more elaborate mathematical objects.

But what makes understanding of the curves flexible? The whole idea that a parabola remains a parabola even if it is translated and rotated in a plane is not difficult. Far less obvious is the connection of a curve transformation with corresponding algebraic manipulations, and we claim that this very connection is often not well established as it will follow from the results of the second page of our questionnaire.

4.2 Matching graphs and formulas with names.

In this section we report the results obtained from the responses occurred on the second page of our questionnaire where students were provided with twelve graphs and were asked to match them with equations and names from a given list (see Appendix). The following table contains information on each curve for entire sample as well as for each category of students who identified their highest mathematical course as Mathematics 3207, Mathematics 3205, or Mathematics 3204. In the last column, for a purpose of comparison, we also give data collected for a group of randomly selected students who had completed six or more undergraduate mathematical courses including calculus stream at least two years prior to the survey date. We call them the senior math group. This group of 27 students also was not specifically prepared or informed about the types of questions prior to the survey, so their performance is, in the same way as with the precalculus students, a true measurement of the students current state of knowledge.

	Entire Sample		3207 Sample		3205 Sample		3204 Sample		Mixed Sample		Senior Sample	
Number of Students	499		73		52		222		152		27	
	Name	Formula	Name	Formula	Name	Formula	Name	Formula	Name	Formula	Name	Formula
Vertical line	80%	27%	86%	44%	88%	27%	77%	17%	77%	32%	100%	85%
Horizontal line	86%	46%	94%	66%	84%	42%	88%	40%	80%	45%	100%	92%
Line with positive slope	82%	17%	94%	33%	81%	17%	83%	10%	75%	20%	100%	96%
Line with negative slope	84%	7%	94%	12%	86%	13%	85%	3%	77%	7%	100%	89%
Parabola opened upward	80%	27%	86%	44%	88%	27%	77%	17%	77%	32%	100%	85%
Parabola opened downward	66%	17%	76%	22%	77%	15%	69%	12%	54%	20%	96%	74%
Hyperbola Quadrants 1 & 3	42%	6%	50%	7%	48%	8%	45%	4%	31%	9%	93%	71%
Hyperbola Quadrants 2 & 4	66%	17%	76%	22%	77%	15%	69%	12%	54%	20%	96%	74%
Circle	85%	31%	92%	37%	88%	50%	88%	26%	78%	29%	100%	93%
Semicircle	72%	2%	81%	5%	82%	2%	74%	2%	61%	2%	81%	67%
Ellipse	61%	24%	70%	31%	77%	38%	65%	21%	47%	21%	96%	89%
Absolute Value	63%	51%	78%	65%	69%	57%	64%	47%	52%	47%	89%	89%

Some of the observations from the table are:

- Students recognize the names of curves much better than their formulas.

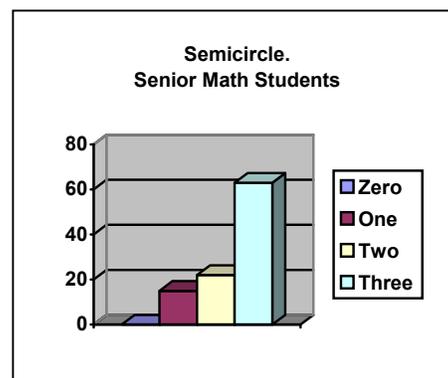
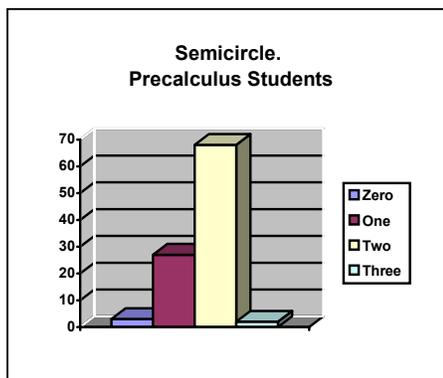
- Students more correctly recognize formulas for lines with positive slope than for lines with negative slope, and parabolas opened upward than parabolas opened downward.
- There is a noticeable increase in percentage 10-17-33% for students enrolled in Mathematics 3204-3205-3207 in recognizing the formulas for the lines with positive slope. However, we would expect a much better match for a line with equation $y = x$. The best match was done for the formula corresponding to a horizontal line, i.e. 40-42-66% for Mathematics 3204-3205-3207.
- The order of preference in recognizing the line formulas is: horizontal, vertical, line with positive slope and line with negative slope.
- Some matching assignments were less straightforward than others because they require a few algebraic manipulations in order to be compared to standard forms. Consequently, the performance in such cases was less successful. Particularly, recognition of the line with negative slope and the semicircle presented difficulty for many students.
- The parabola with positive leading coefficient is a preferred example over the parabola with negative leading coefficient for both formulas and names. This is in accordance with the way parabola was introduced in high school. We conclude that the prototype is the parabola with positive leading coefficient.
- Although hyperbola does not belong to the Mathematics 3204 or Mathematics 3205 curriculum, we found out that a significant percentage of students 45%, respectively 48% know the name of the hyperbola in quadrants 1 and 3, and that 69%, respectively 77% know the name of the hyperbola in quadrants 2 and 4.
- Mastering the formula for ellipse shows less successful performance than mastering the formula for the circle.
- The absolute value function proved to have relatively good results in terms of terminology, matching formula and graphs.

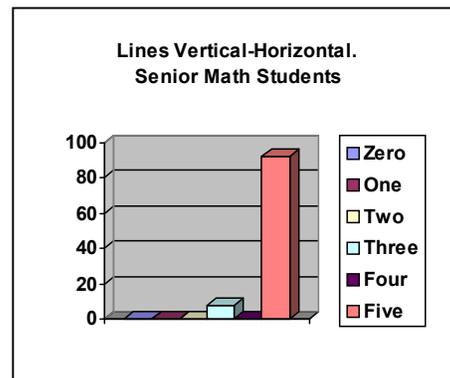
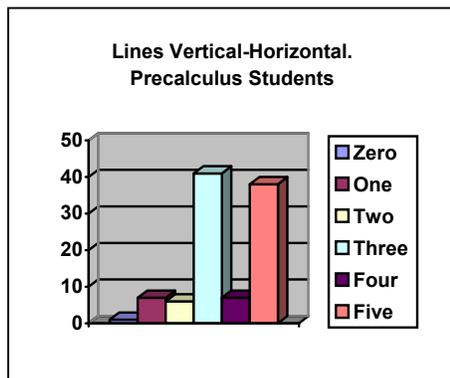
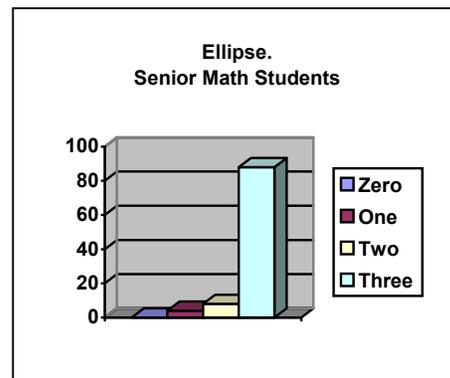
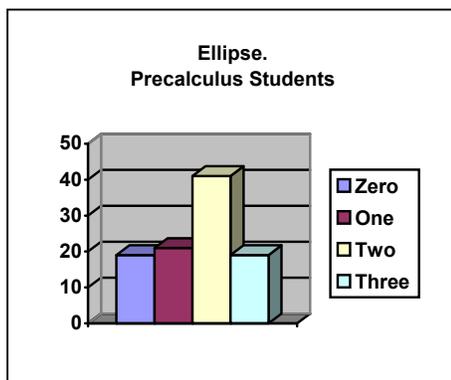
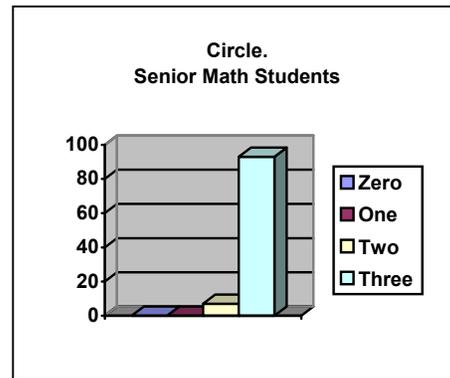
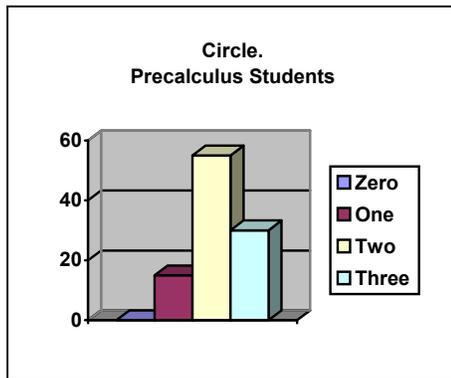
In order to characterize the level of students' knowledge about each particular mathematical object we use a graphical bar-diagram representation of the results collected. For this purpose we used the following marking schema: if both equation and name were written correctly under a graph on page 2, the student was given 2 points; if only name or only formula were identified correctly, the student was given 1 point; zero points were given for either incorrect or no answer; an additional point was given for a correct image of the same object drawn on the first page. This way for each of circle, ellipse and hyperbola a student could collect at most three points (two on the second page and one on the first), and for parabola – at most five points (four on the second page and one on the first). We separated lines in two subcategories: vertical or horizontal, and lines

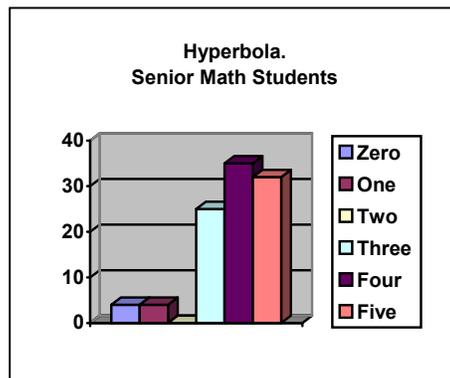
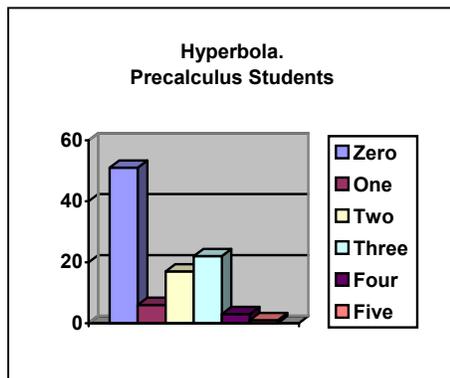
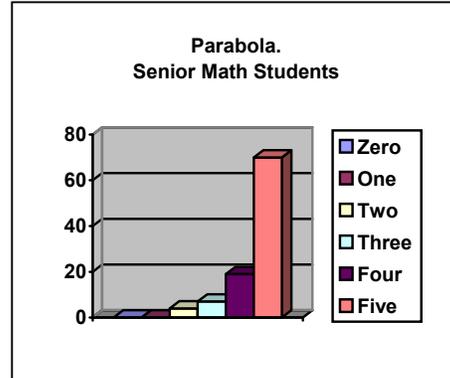
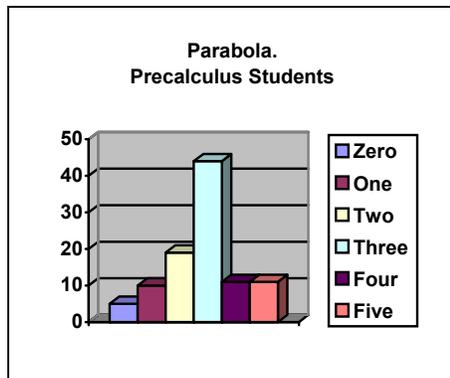
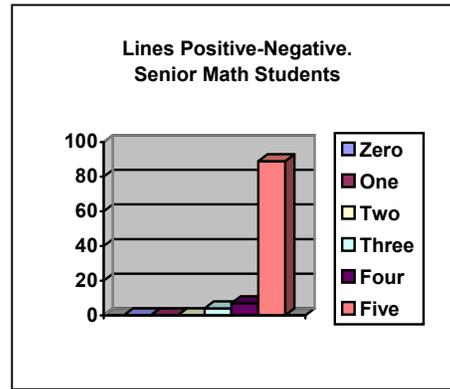
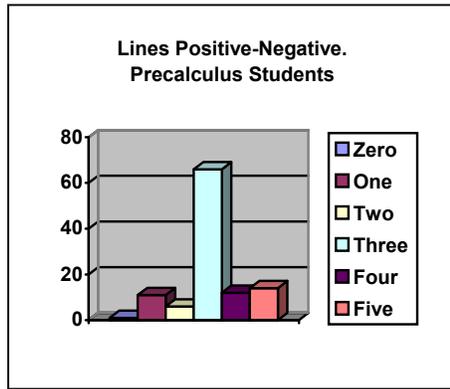
with positive or negative slope. In this way at most five points were collected for each subcategory of lines (four on the second page and one point on the first page; any image of line drawn on the first page contributed one point into each subcategory of lines).

For each object we created a bar-diagram which shows the percentage of the total number of students who collected zero points, one point, two points, or three points (extended to four points and five points in case of parabola and the two subcategories of lines). Obviously, there are two extreme profiles with 100% of a sample at zero points, and 100% of a sample at the maximum possible points, which correspond to complete non-familiarity and perfect performance, respectively. In reality, the profile of the bar-diagram is somewhat in-between the extreme shapes, but closeness to one extreme or another characterizes the degree of success in performance with respect to a particular object (curve). The profile also shows the degree of homogeneity of a particular group of students in terms of their familiarity with a particular object of study. For example, it turned out that the sample in our study was more homogeneous in performance with circle, and lines with positive or negative slope, compared to their performance with the ellipse, the horizontal or the vertical lines.

For a comparison purpose, we give bar-diagrams created for the senior math group described above. We observe that, while for this latter group of students with stronger mathematical background the bar-diagrams are closer to the perfect shape, the profiles for different notions (curves) still show a difference. They signal a possibility of improvement in performance with the same notions (curves) that present a challenge for the group of freshmen. Thus, despite the performance of students, taking calculus courses, improves the statistical difference between the levels of knowledge in each category remains.







The goal of our discussion was not to provoke a search for a reason or examine how good or poor the freshmen's performance is, but rather to attract the instructors' attention to the following observation. If, during a lecture for this group of students an equation $x + y = 2$ was given as a simple example, then 93% of the audience would not evoke an image of line with negative slope, although at least 84% of the group know what the line with negative slope is! Even if the line is drawn on the board, many mathematically inexperienced students will not make a connection between the equation and the graph unless it is explicitly explained. The explanation may only take minutes, but could make a big difference in the clarity of the example. Systematicity in such explanations leads to students' development of the ability of making necessary connections themselves.

5. Demands of the undergraduate mathematics curriculum: calculus.

Calculus is a major and important component of the introductory undergraduate university level mathematics. More senior courses such as real, complex and functional analysis, differential geometry, integral and partial differential equations, and many applications in physics, biology, economics and business build up their content on the solid ground of differential, integral and vector calculus. In the calculus sequence, the courses focus on general notions such as limit, as well as on the differentiation and integration techniques for finding such quantities as rates of change, areas, volumes etc. Students often find themselves being able to follow the explanations of general ideas but experience difficulties when the ideas are applied to concrete examples. This is indeed a paradoxical situation: the examples which ought to be illustrative are instead confusing. One of the major reasons is a non-flexibility of students' knowledge concerning some basic mathematical examples, e.g. fundamental curves such as parabola, ellipse and hyperbola, but often times even lines and circles, and their algebraic equations.

Criticizing Bloom's taxonomy of educational objectives, where Knowledge and Comprehension are regarded as two distinct levels, Pring (1971) remarks that "it does not make sense to talk about knowledge of terms or symbols in isolation from the working knowledge of this terms and symbols, that is, from the comprehension of them and thus the ability to apply them". The familiarity with terminology, without working knowledge and comprehension, is certainly not the final pedagogical objective. But in the reality of the learning processes this is a clearly observable stage of cognitive development, when some images start to be attached to the terms (words), but they are so fragile and rough, they are so "not a precise idea such as reasoning can take hold of" (Poincare, 1996).

Ironically, many students taking calculus courses have this precise kind of knowledge of the basic algebraic curves. This is a deceiving situation for students themselves as well as for their instructors relying on students' ability to comprehend while they often have just an illusion of knowing.

For instance, when it comes to visualizing 3D surfaces, such as an elliptic or hyperbolic paraboloid given by an algebraic formula, the students know that the task can

be approached by the slicing method, i.e. by identifying the curves occurring as the vertical and horizontal slices of the surface and then mental gluing the curves together. Note that the first task is to recognize the curves algebraically and then imagine their graphs, including the shifting and stretching aspects. If the students are not flexible in doing this part, the rest of the exercise is meaningless for them regardless of how extensive was the explanation. This is where the notion of the family of parabolas, ellipses or hyperbolas becomes essential, and the whole idea of correspondence between the algebraic and geometric transformations. Specifically, let the students analyze the equation $z = a(x - b)^2 + c(y - d)^2$, where a, b, c, d are the parameters of the surface in the (x, y, z) -coordinate space. Students are instructed to fix the value of $y = s$ in order to get a vertical slice of the surface in a plane parallel to the (x, z) coordinate plane. While keeping in mind that for different values of s there will be a different curve, they ought to see algebraically that the curve is always a parabola $z = a(x - b)^2$ shifted at a different height $c(s - d)^2$.

Similarly, the students shall identify the other family of vertical slides, $x = t$, as being a family of parabolas $z = c(y - d)^2$ shifted vertically by $a(t - b)^2$. The horizontal slides of the surface appear to be either a family of ellipses (case $ac > 0$) or a family of hyperbolas (case $ac < 0$), which gives either an elliptic or a hyperbolic paraboloid.

A special remark concerns two different forms of equation of a hyperbola. For example, a hyperbola in the form $u^2 - v^2 = k$ (where $k \neq 0$) never appears in the senior high school books. Therefore, a special effort is required to make a connection with the standard form $y = \frac{1}{x}$, using a 45° rotation of the coordinate system (u, v) such that $x = \frac{u + v}{\sqrt{2}}$ and $y = \frac{u - v}{\sqrt{2}}$. Then we have $1 = xy = \frac{(u + v)(u - v)}{2} = \frac{u^2 - v^2}{2}$.

The task of visualization in 3D space is by itself a difficult one, especially if the solid has a composite description that is typically bounded by several standard surfaces of the second order: cone, sphere, paraboloid etc. When students are instructed how to find a volume of a solid by evaluating a multiple integral, the most difficult part for them is to set up the limits of integration based on the algebraic description of the surface. Oftentimes, the problem is that they cannot visualize the boundaries of the solid and translate this image into the proper algebraic inequalities. Once again, the root of such difficulty lies in non-flexibility of their knowledge of elementary curves and surfaces.

An instructor who systematically fosters and reinforces the connection between algebraic and geometric manipulations, using elementary but fundamental mathematical examples, will see a remarkable difference in the students' performance at all complexity levels encountered in calculus problems.

Appendix. The questionnaire.

1. Draw what comes to mind first when you read the following words.

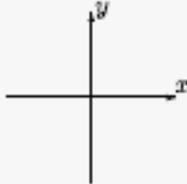
(a) *line*



(b) *circle*



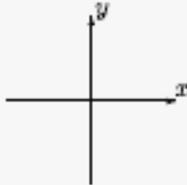
(c) *semicircle*



(d) *ellipse*



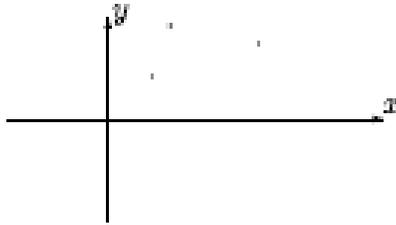
(e) *parabola*



(f) *hyperbola*



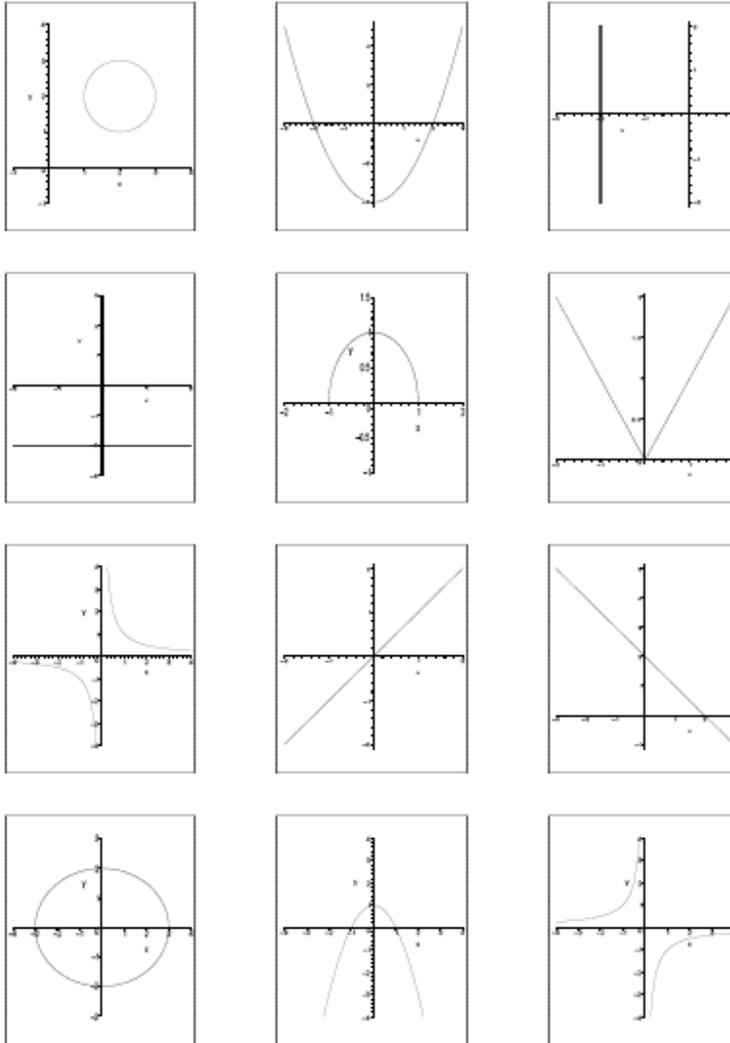
2. Draw an arbitrary graph of a function which passes through three points. How many different graphs can be drawn?



3. Pick the name which you think corresponds to each graph from the following list.
 Pick a formula which you think corresponds to each graph (there might be more than one correct answer).
 Please write the name and the corresponding formula below each image.

Names: horizontal line; line with positive slope; parabola; ellipse; line with negative slope; circle; vertical line; absolute value; hyperbola; semicircle.

Formulae: $(x - 2)^2 + (y - 2)^2 = 1$; $y = -2$; $y = |x|$; $y = x$; $x = -2$;
 $y = \sqrt{x^2}$; $x + y = 2$; $xy = 1$; $y = x^2 - 4$; $y = \sqrt{1 - x^2}$; $y = \frac{1}{x}$;
 $2 + x = 0$; $\frac{x^2}{9} + \frac{y^2}{4} = 1$; $y = 1 - x^2$; $xy = -1$.



References

- Arcavi, A. (2003). The Role of Visual Representations in the Learning of Mathematics. *Educational Studies in Mathematics*, 52(3), 215-241.
- Barry, M., Besteck-Shaw, C., Brown, R.N., & Avar-Spinney, A. (2002). *Mathematical Modeling. Book 4*. Canada: Nelson, Thomson Learning.

- Barry, M., Besteck-Hope, C., Hope, J., Pilmer, C.D., Small, M., Avard-Spinney, A., & Wheadon, L.L.B. (2002). *Mathematical Modeling. Book 3*. Nelson, Thomson Learning.
- Barry, M., Small, M., Avard-Spinney, A., & Wheadon, L.L.B. (2000). *Mathematical Modeling. Book 1*. Canada: Nelson, Thomson Learning.
- Bloom, B.S., & Krathwohl, D.R. (1956). *Taxonomy of Educational Objectives: The Classification of Educational Goals, by a Committee of College and University Examiners*. New York, NY: Longman/Green.
- Furst, E.J. (1981). Bloom's Taxonomy of Educational Objectives for the Cognitive Domain: Philosophical and Educational Issues. *Review of Educational Research*, 51 (4), 441-453.
- Hershkowitz, R., Parzys, B., & van Dormolen, J. (1996). Space and shape. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 161-204). Dordrecht, The Netherlands: Kluwer.
- Hershkowitz, R., Schwarz, B.B., & Dreyfus, T. (2001). Abstraction in Context: Epistemic Actions. *Journal for Research in Mathematics Education*, 32 (2), 195-222.
- Hershkowitz, R., & Vinner, S. (1983). The role of critical and non-critical attributes in the concept image of geometrical concepts. In R. Hershkowitz (Ed.), *Proceedings of the seventh international conference for the Psychology of Mathematics Education* (223-228). Rehovot, Israel: Department of Science Teaching, The Weizmann Institute of Science.
- Poincaré, H. (1996). *Science and Method*. Bristol: Thoemmes Press. (Original work published 1914)
- Polya. G. (2004). *How to solve it. A new aspect of mathematical method. With a new foreword by John H. Conway*. Princeton University Press. (Original work published in 1945)
- Pring, R. (1971). Bloom's taxonomy: A philosophical critique (2). *Cambridge Journal of Education*, 1, 83-91.
- Rosch, E., & Mervis, C.B. (1975). Family resemblance: Studies in the internal structure of categories. *Cognitive Psychology*, 7, 573-605.
- Schwarz, B.B., & Hershkowitz, R. (1999). Prototypes: Brakes or Levers in Learning the Function Concept? The Role of Computer Tools. *Journal for Research in Mathematics Education*, 30 (4), 362-389.

- Sockett, H. (1971). Bloom's taxonomy: A philosophical critique (I). *Cambridge Journal of Education*, 1, 16-25.
- Tall, D. O. (1988). Concept Image and Concept Formation. *Senior Secondary Mathematics Education*, Utrecht, 37-41.
- Tall, D.O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Vinner, S. (1983). Conflicts between definitions and intuitions: the case of the tangent. *Proceedings of the P.M.E.*, 6, Antwerp.
- Vinner, S., & Dreyfus, T. (1989). Images and Definitions for the Concept of Function. *Journal for Research in Mathematics Education*, 20 (4), 356-366.
- Wilhoite, R.L. (1966). Problems of meaning and reference in Bloom's taxonomy: Cognitive domain (Doctoral dissertation, Indiana University, 1965). *Dissertation Abstracts*, 26, 6569-6570.

KOREAN TEACHERS' PERCEPTIONS OF STUDENT SUCCESS IN MATHEMATICS: Concept versus procedure

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Abstract: This article examines the Korean classroom teachers' beliefs about mathematics education in elementary schools. Their perceptions about contributing factors to Korean students' high achievement scores in international comparative studies in the area of mathematics are explored. Elementary classroom teachers were surveyed using the researcher-made questionnaire (Teacher Perception about Mathematics Curriculum) and 141 teachers completed the questionnaire. The data collected was analyzed by a descriptive analysis. The results reveal that the majority of classroom teachers agreed that real life applications, processing skills, using concrete instructional manipulatives, and conceptual knowledge are very important in teaching children mathematics. Most of teachers participating in this study were aware of the fact that Korean students ranked in the top percentile in the international comparative students' mathematics achievement studies. The teachers claimed that Korean students still heavily focus on practice and drill computational skills, private lessons at the after school program and parents' high expectation of their child's education, and active involvement in his/her education generated the high scores in mathematics.

Keywords: conceptual knowledge; constructivist myths; drill and practice; Instructional methods; Korea; Korean Teachers; Teacher Beliefs; Teacher perceptions; Student achievement

Introduction

According to various international comparative studies of students' achievement ;[i.e. the Third International Mathematics and Science Study (TIMSS, 1995 & 1999); the Trends in International Mathematics and Science Study (TIMSS, 2003); the Program for International Student Assessment (PISA, 2003); and the Organization for Economic Co-operation and Development (OECD, 2006)] Korean school students performed at a very high mean score in mathematics. Particularly in 2003, Korean 8th graders ranked 2nd in mathematics among 46 countries participating in TIMSS and their achievement scores had been continuously improving. These results encouraged Korean educators, especially mathematics educators, to reflect on strength and weakness in terms of Korean mathematics education including the national curriculum and instructional methods. The whole educational environment was analyzed, in order to retain and even to improve students' mathematics achievement scores.

The TIMSS 1995 report indicated that Korean mathematics education had serious issues to be addressed. In spite of Korean students' successful achievement, students did not have positive dispositions towards mathematics. This issue has been validated by the PISA 2003 report. According to this report, Korean students' intrinsic interest in mathematics was very low and their self-concept and self-efficacy levels were in the lowest group. Sixty two percent of Korean students participated in the study reported that they did not think they did well in mathematics (Leung, 2002) and Korean students' anxiety in mathematics was very high (5th among 41 countries participating in PISA 2003). Interestingly, the PISA (2003) results showed that students' self-esteem in mathematics does not equate to high scores. This brought a discussion by some researchers in the United States claiming that schools need not be fun to be effective, and schools should work on academics rather than focus on feelings and happiness of students. Students' true self-esteem will be fulfilled by true achievement (in Mathews, 2006). However, Korean educators considered their students' affective characteristics as one of the areas that needed to improve and strived to develop a particular program for nurturing students' affective disposition in mathematics. This educational movement impacted the 7th national mathematics curriculum revision issued in 1998 (Lew, 2004).

Korean schools use a national curriculum. This mathematics curriculum has been developed and revised by a committee consisting of educational leaders among classroom teachers in different grade levels, mathematics educators, and researchers from academic institutes under the authorization of the Ministry of Education and Human Resources Development (MEHRD). The current Korean mathematics curriculum, which is the 7th national curriculum, was revised in 1998 (Lew, 2004) and implemented since 2000 (Paik, 2004). The *Report on Mathematics Education in Korea* presented by the Korean research team at the 10th International Conference of Mathematics Education (ICME-10) in 2004 claims that the main focus of the 7th national mathematics curriculum was that it was "learner centered." This approach actively planned to implement the curriculum in a stepwise and level-reference manner, emphasizing learner's voluntary and positive learning activity, and provoking learner's interests in mathematics (Paik, p. 14). If this direction was clear and effectively implemented in actual classrooms, the PISA 2003 results should be different from what the TIMSS 1995 reported.

In the United States, the National Council of Teachers of Mathematics (NCTM) published the *Curriculum and Evaluation Standards for School Mathematics* (1989). Since then, an agenda for the reform of school mathematics has focused on "mathematics as sense-making," as well as the importance of all students in grades K-12 studying a common core of broadly useful mathematics (Janvier, 1990). These ideas were affirmed in another publication by the NCTM, the *Principles and Standards for School Mathematics* (2000); that suggests learners should be provided with the autonomy to select activities that blend with their interests and prior experiences to build mathematical connections through active learning. The NCTM standards have been based upon a learning theory termed Constructivism, which is supported by cognitive theorists, such as Jean Piaget, Lev Vygotsky, and Jerome Bruner, who advocated that children must construct their own knowledge through interaction with the physical and social environments (DeVries & Kohlberg, 1987, Heddens & Speer, 2006).

The history of Korean mathematics curriculum clearly shows that it was influenced by the reform movement in the USA. The 1st curriculum (1955-1963) was called “real life centered curriculum” which was influenced by Progressivism in the USA. The 2nd curriculum (1964-1972) was characterized as “mathematics structure centered,” the 3rd one (1973-1981) as “new math oriented,” the 4th curriculum (1982-1988) as “back to basics,” the 5th one (1989-1994), as “problem solving oriented,” the 6th curriculum (1995-1999) as “problem solving and informational society oriented,” and finally the 7th curriculum (2000-present) was characterized as “learner centered” (Paik, 2004, p. 12). This reveals that the sequence of mathematics history in Korea is very similar to the US mathematics history and reform movement. This implies that Korean educators and classroom teachers should be aware of the current mathematics reform movement within the international context. Classroom teachers especially need to explore the current reform movement to help students develop their mathematical knowledge (NCTM, 1989). Teachers’ perceptions are directly related to mathematics education since their role is an essential part of curriculum when curriculum is defined as “all the experiences children have under the guidance of teachers (Caswel & Campbell, 1935, p. 66). Further, there are various studies reporting that teacher beliefs and instructional methods are significant variables in improving students’ achievement (e.g., Rowan, Correnti, & Miller, 2002). Teacher beliefs about mathematics play a crucial role in shaping the teacher’s instructional choices (Shuhua, 2000) as well as correlating with higher students’ achievement (Love & Kruger, 2005). Given research findings and growing research interest in Asian mathematics education after international comparative studies reported Asian students outperformed their western counterparts in mathematics, this article investigates Korean elementary classroom teachers’ perceptions about mathematics education and speculation regarding factors that contribute to Korean students’ high achievement scores.

Purpose

This survey study was conducted to investigate Korean elementary classroom teachers’ perceptions regarding mathematics education. Two research questions guided this study: 1) What are teacher perceptions of Korean mathematics education? 2) What do Korean classroom teachers believe regarding the contributing factors to Korean students’ high achievement scores in the international comparative studies?

Method

Participants

Classroom teachers were randomly selected by convenient sampling from the public elementary schools in the Chullabuk-do provincial school district which is located in the southwestern area of Korea. Participating teachers represented grades 1 through grade 6 (the Korean elementary school includes grade 6 at the elementary level) in 21 elementary schools. Two hundred teachers were selected and 141 of those classroom teachers (101 female, 40 male) completed and returned the questionnaire (70.5% response rate). Among them, 19 were first grade classroom teachers (13.5%), 22 second grade (15.6%), 21 third grade (14.9%), 18 fourth grade (12.8%), 31 fifth grade (22%), and 30 sixth grade (21.3%) classroom teachers. The mean of their teaching experience was 13.38 years. The mean class size was 33.45 students. . The mean teacher age was 36.66 years. One hundred twelve (79.4%) held a bachelor’s degree and

Twenty- three (16.3%) held a master's degree. Six teachers (4.3%) were currently enrolled in a graduate program in pursuit of a master's degree in education.

Instrument

The 26-item 3 part survey instrument entitled "Teachers' Perceptions about Mathematics Education "(TPMC) was developed based on a comprehensive review of the Korean mathematics education and the current mathematics reform movement literature. This process to develop the instrument helped to establish face validity of the questionnaire. The first part contained questions about participants' demographic information, i.e., gender, age, teaching experience, grade level, and class size. The second part had 10 likert scale questions (agree, not sure, disagree) about their instructional pedagogy in mathematics education. For example, teachers were asked if real life application skills are the most important for the children to learn from their instruction in mathematics class. The Third part consisted of two open-ended questions and a forced-answer question (yes, no). The open-ended questions were asking their opinion about their instructional pedagogy and the factors they believe contribute to Korean students' high scores in the international mathematics comparative studies. The forced-answer question asked if the teachers were aware of the fact that Korean school students ranked high in the international mathematics comparative studies, such as the Trends in International Mathematics and Science Study (TIMSS). The survey questionnaire was developed in English first and translated by the researcher into Korean. The Korean version of the questionnaire was reviewed by an associated principal, with a master's degree in mathematics education and a classroom teacher with a particular interest in mathematics education. With the classroom teacher's assistance, the first draft of the questionnaire was given to thirty seven classroom teachers at a public school in a suburban area of Chonju city, Chullabuk-do, Korea. Based on the responses of the teachers, the final draft of the questionnaire was established.

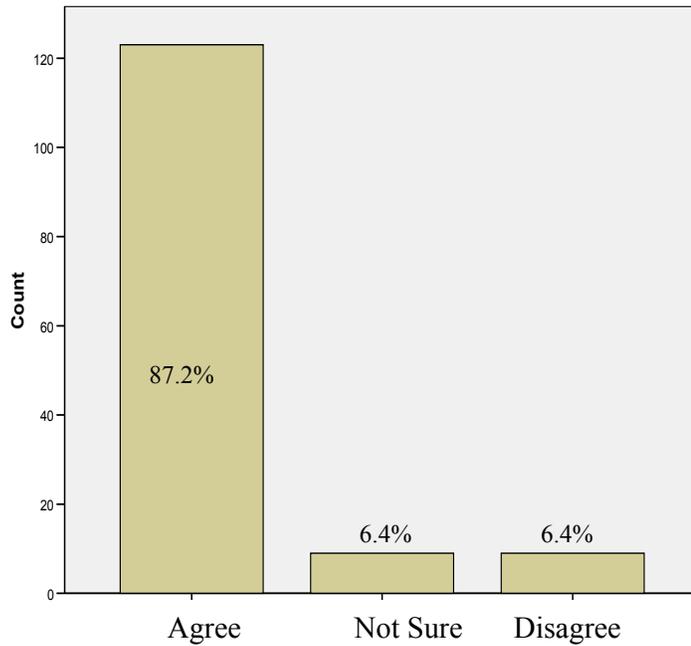
The questionnaire along with a letter explaining the purpose of the study and participant consent form was distributed from late May to late June, 2005. The questionnaire went to 24 elementary schools within Chullabuk-do provincial area with the assistance of the principals and associate principals. The questionnaires, completed anonymously by the classroom teachers, were collected by the principals and associate principals during the period of July and November of 2005. SPSS 14.0 for windows was used for data entry and analysis. A descriptive analysis utilizing frequencies and cross tabulation was employed to analyze the data to examine the purposes of this study.

Results

Teacher's beliefs about the educational pedagogy

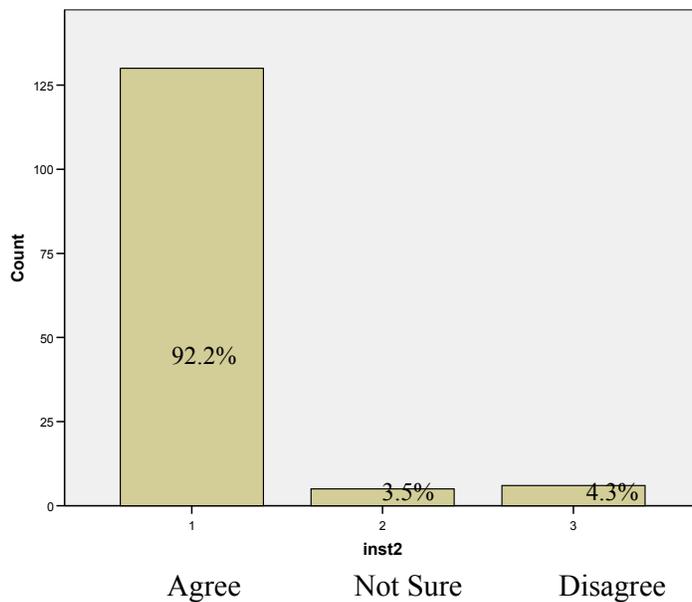
Using the SPSS 14.0 descriptive analysis and frequency of responses, one hundred twenty three teachers (87.2%) agreed that teaching children to apply mathematics knowledge and skills to real life is the most important skill. Nine teachers (6.4%) said they were not sure or disagree with the statement (see Figure 1).

Figure 1. Real life application is the most important in mathematics education.



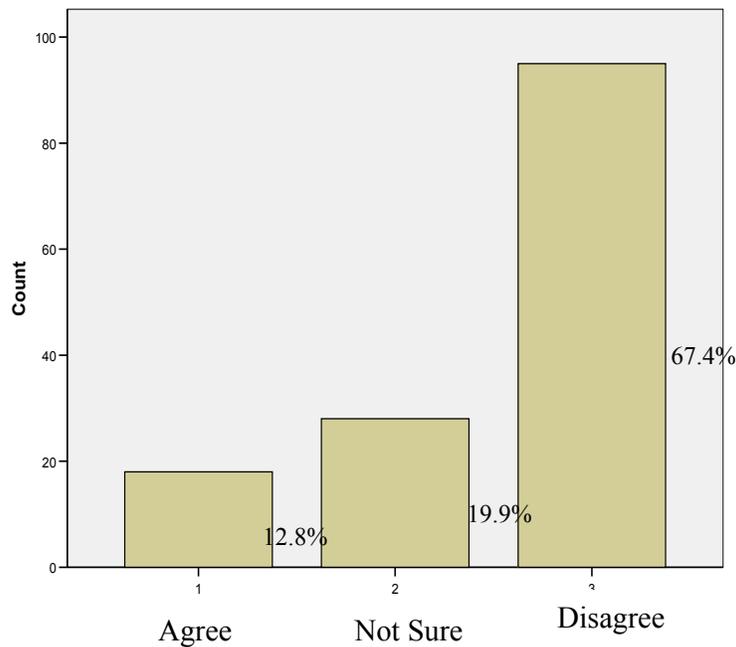
One hundred thirty teachers (92.2%) responded that they agree with the statement, “teaching students to see process while solving problem is the most import.” Five teachers (3.5%) said they were unsure and six teachers (4.2%) disagreed (see Figure 2).

Figure 2. Process is very important in teaching mathematics



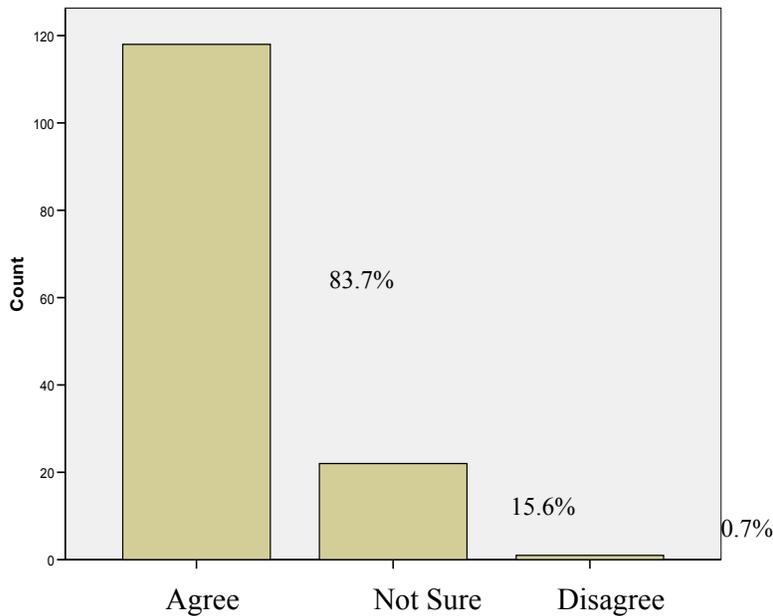
Regarding the statement “The most important thing is for students to memorize algorithms and use them to solve problems in mathematics education,” ninety five teachers (67.4%) answered “Disagree,” twenty eight teachers (19.9%), “Not sure,” and eighteen teachers (12.8%) answered “Agree” (see Figure 3).

Figure 3. It is important to memorize algorithm to solve mathematics problems.



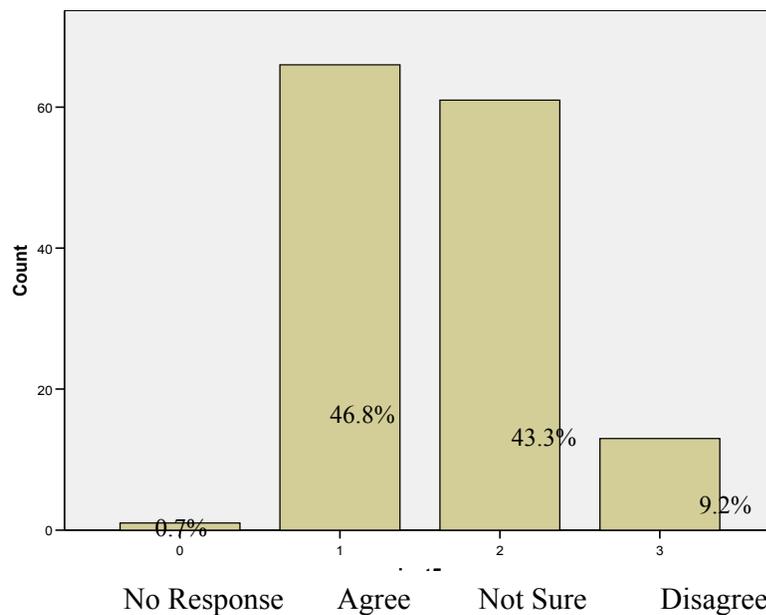
When classroom teachers were asked if various concrete manipulatives should be used to illustrate mathematical concepts for the students, one hundred eighteen teachers (83.7%) replied that they agreed, twenty two teachers (15.6%) were not sure and one teacher (0.7%) replied “Disagree” (see Figure 4).

Figure 4. To teach mathematics, we need to explain concepts using concrete materials.



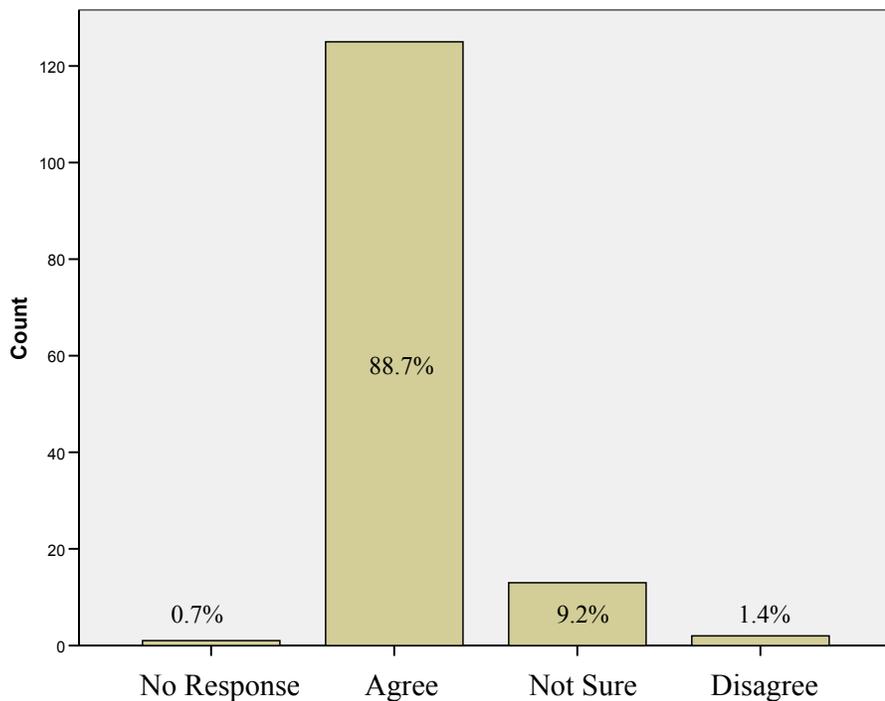
Sixty six teachers (46.8%) did feel confident explaining mathematics concepts to the students using various instructional manipulatives (i.e., small counters & Base-10 blocks). Sixty one teachers (43.3%) were not sure if they were confident or not; and thirteen teachers (9.2%) were not confident in teaching mathematics using different concrete instructional materials. One teacher (0.7%) did not provide an answer (see Figure 5).

Figure 5. I am confident explaining mathematics concepts to the students using manipulatives.



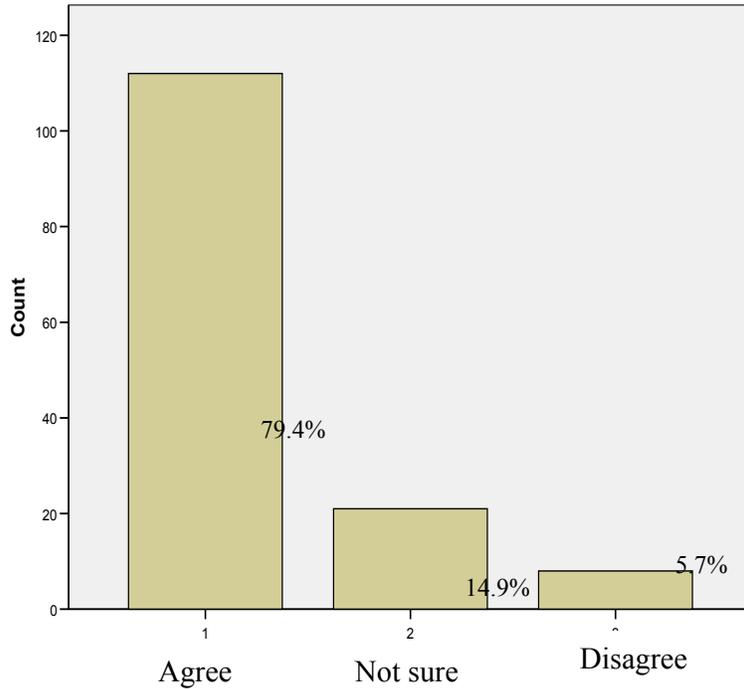
One hundred twenty five teachers (88.7 %) believed that concrete examples should be demonstrated first and then information related to abstract knowledge added to help students understand concepts. Thirteen teachers (9.2%) were not sure about it and two teachers (1.4%) disagreed with this statement. One teacher (0.7%) did not answer to the question (see Figure 6).

Figure 6. I need to help children develop abstract knowledge from concrete examples by illustrating the concept using concrete models.



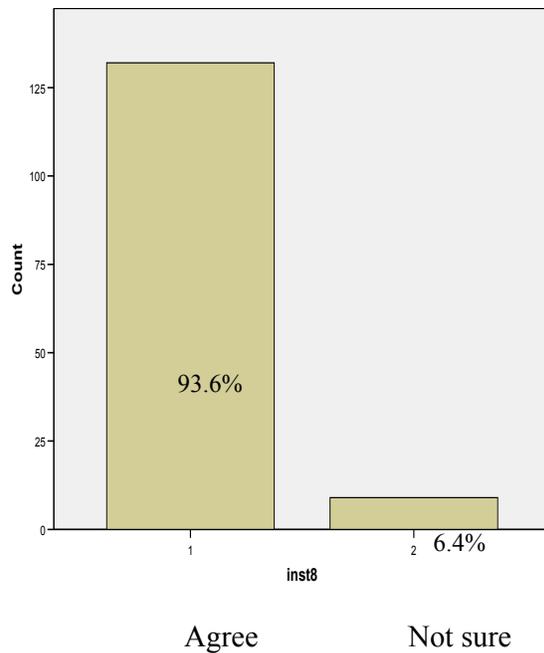
In terms of using concrete objects to introduce a new concept, one hundred twelve teachers (79.4%) said that concrete instructional materials must always be used when students learn new concepts. Twenty one teachers (14.9%) were not sure and eight teachers (5.7%) did not think it was an appropriate way to help students build concept (See Figure 7).

Figure 7. When introducing a new concept, we always need to use concrete objects.



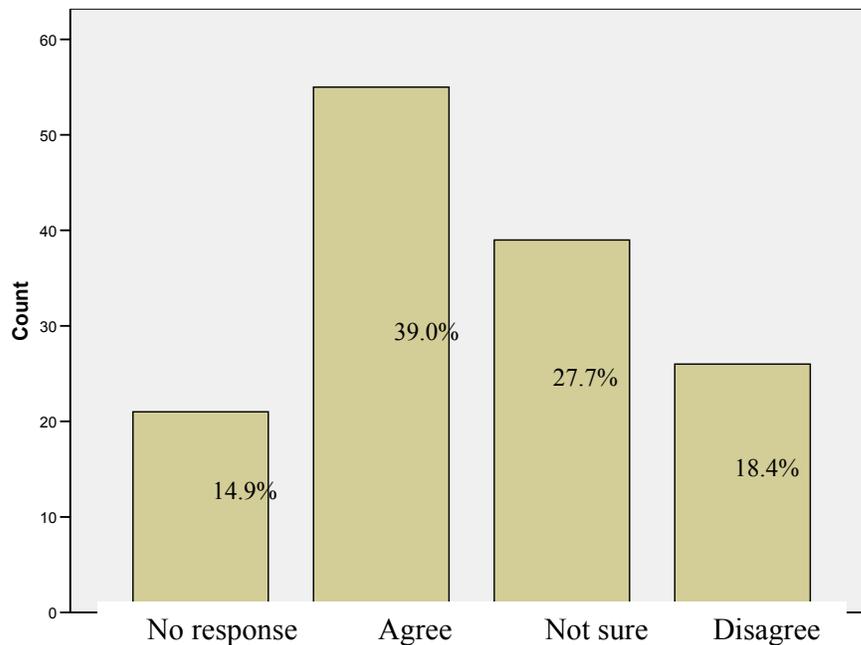
One hundred thirty two teachers (93.6%) thought both conceptual and procedural knowledge are equally important in teaching students mathematics. Only nine teachers (6.4%) were not able to answer either way. There were no teachers who disagreed with this statement (see Figure 8).

Figure 8. In mathematics education, conceptual knowledge is very important.



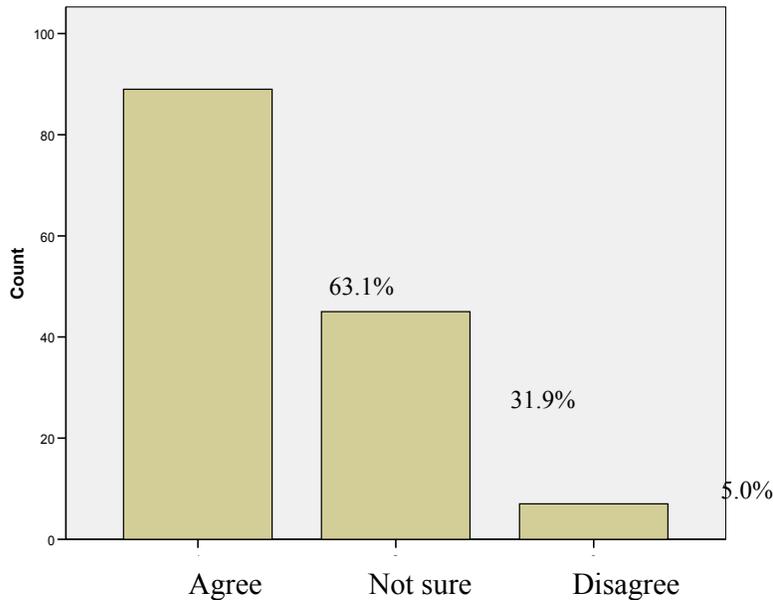
Teachers were also asked if algorithm was very important for mathematics education. Fifty five teachers (39%) agreed, thirty nine teachers (27.7%) were not sure, and twenty six teachers (18.4%) disagreed with the idea. Twenty -one teachers (14.9%) did not answer the question (see Figure 9).

Figure 9. In mathematics education, procedural knowledge is very important.



Eighty-nine teachers (63.1%) felt the students learn mathematics well through their instructional methods. Forty five teachers (31.9%) did not know if their instructional methods were effective and seven teachers (5.0%) replied they did not feel their instructional methods help students learn mathematics (see Figure 10).

Figure 10. I feel my students learn mathematics well through my instructional methods.



Teachers' educational pedagogy

In answer to a question regarding what is the most important thing they need to teach students in mathematics education, sixty one (26.6%) of total (229) responses indicated that the concept is the most important for the students to acquire. Fifty-nine responses (25.8%) indicated that understanding principles is the most important, and twenty responses (8.7%) indicated that understanding process was most important. Eighteen (7.9%) responded that helping students have fun with mathematics to increase interest in it, and seventeen (7.4%) said that students should develop problem solving skills. Sixteen teachers (6.9%) said that students should build logical thinking skills and fifteen (6.6%) said that real life application is very important in mathematics education. Other responses (between 0.4 - 3.9%) included that students' basic computational skills, using concrete manipulatives in teaching mathematics, allowing students to be self-motivated, helping students construct algorithms on their own, investing skills, cooperative learning skills, and memorizing facts were most important (see Table 1).

Table 1. What is the most important thing you need to teach mathematics in the elementary classroom?

	Items Teachers Think the Most Important in Teaching Math	No. of Response	Percent (%)
1	Understanding concepts	61	26.6
2	Understanding principles	59	25.8
3	Understanding process	20	8.7
4	Fun math and student's interest	18	7.9
5	Developing problem solving skills	17	7.4
6	Building logical thinking skills	16	6.9
7	Real life application	15	6.6
8	Basic computational skills	9	3.9
9	Using concrete manipulative	7	3.1
10	Student's self-motivation	2	0.9
	Construct algorithm	2	0.9
11	Investigation skills	1	0.4
	Cooperative learning	1	0.4
	Memorizing facts	1	0.4
Total Responses from Teachers		229	

Teachers' beliefs about Korean students' high score in mathematics

When asked if teachers were aware that Korean students achieved high scores in the international comparative studies in the area of mathematics, the majority of 141 teachers (93.6%) indicated they knew. Seven teachers (5.0%) were not sure and two teachers (1.4%) did not respond to the question.

The last open-ended question examined the teachers' speculations about why Korean students rank high in the international comparative studies in mathematics. Forty-three responses from teachers (22.1%) indicated that Korean students practice computational skills through repeatedly solving various mathematics problems. Twenty seven responses from teachers (13.8%) said private lessons at the after school program made students' gain high achievement scores because many Korean students take private lessons or tutoring sessions for mathematics. These private programs teach students mathematics at a higher grade level than the students are taught in school. These students who receive these special lessons demonstrate higher mathematical academic skills than those who did not attended private programs. Twenty four responses from teachers (12.3%) claimed parents' high expectation of their child's education resulted in students putting more effort into getting a higher grade in mathematics. Fourteen responses from teachers (7.2%) alleged that students think mathematics is very important for their success in school and focus on the study of it. Twelve of the responses from teachers (6.2%) indicated that parents' active involvement in their child's education attributed to Korean students' high achievement scores. Ten responses (5.1%) stated that the zeal of education and competitive college entrance exams generated students' high scores. Other

responses (between 0.5% - 4.6%) included that students' hard work; that success results from mathematics skills taught in early childhood settings; that well developed mathematics curriculum; and that students were inherently smart and test-wise. Teacher and parents' perception about math as an important subject, as well as the teacher's hard work were also cited (see Table 2).

Table 2. Why do you think Korean students achieved high scores in the international mathematics assessment comparative studies?

	Factors contributing to students' high achievement	No. of Response	Percent (%)
1	Focus on practice and drill in solving problems	43	22.1
2	Private lessons at the after school programs	27	13.8
3	Parent's high expectation on child's education	24	12.3
4	Think math is very important and focus on it	14	7.2
5	Parent active involvement in child's education	12	6.2
6	Zeal of education in the society	10	5.1
	Preparing for college entrance exam	10	5.1
7	Student's hard work	9	4.6
8	Math taught in early childhood setting	8	4.1
9	Well developed math curriculum	7	3.6
10	Students are smart	6	3.1
11	Students are test-wise	5	2.6
	Various competitive math contests	5	2.6
12	Curriculum is difficult in content	3	1.5
13	Understanding principles	2	1.0
	Teacher & parent think math is important	2	1.0
	Teacher's hard work	2	1.0
14	TIMSS does not assess creativity	1	0.5
	Korean nationalism	1	0.5
	Gifted education	1	0.5
	Individual Excellency/superior	1	0.5
	Test result is only from upper academic level students	1	0.5
	Competitive society	1	0.5
Total Responses from Teachers		195	

Discussions and Conclusions

The findings of this study indicate that Korean elementary classroom teachers' educational pedagogy is based on Constructivism, which proposes that children construct their own knowledge of mathematics. The majority of the teachers (87.5%: mean for all questions) thought real life application and understanding the process of problem solving aided learning. They believe that use of concrete materials to explain mathematical concepts and connection between conceptual understanding and abstract knowledge are important, as well as recognizing that conceptual knowledge is very important in mathematics education. In the Constructivist classrooms, students learn through action, discovery-oriented activities and guided questions and discussions (DeVries & Kohlberg, 1987).

When teachers were asked to provide their speculation regarding the contributing factors to Korean students' high mathematics achievements in international comparative studies, the top three responses were: (1) Korean mathematics education still focuses on practice and drills computational skills; (2) private lessons in after school programs are common; (3) parents' high expectations for their child's education influence children's performance.

This study has revealed that Korean elementary classroom teachers were well aware of the current mathematics reform movement based on Constructivism. They used Constructivist theory to influence their educational pedagogy. However, interestingly, these teachers identified that the first factor contributing to students' high mathematics achievement is emphasizing computational skills in mathematics education. This implies that Korean classroom teachers use traditional instructional methods in their actual classrooms that focus on computational skills even though the majority of Korean teachers' educational pedagogy in this study was founded Constructivist approach. A study conducted by Shuhua (2000) reported that teachers' pedagogical beliefs about mathematics play a significant role in shaping their instructional practice, but Korean elementary teachers did not seem to practice Constructivist instructional methods in the classrooms, even though they believed that Constructivist-based teaching is very important. Kutz (1991) indicated that, in actuality, classroom teachers tend to be neither traditionalist nor Constructivist in the sense that they teach in ways that they were taught and in ways that seem to work. The decision about how to teach is based on one's own teacher education, learning theory, tradition, socialization into the school system, past schooling, and student reactions to teaching practice. As a result, many classroom teachers blend the learning theories of the traditionalist and Constructivist literature, but more closely follow those practices characterized by the traditionalist learning theories. A traditionalist approach is based on the behaviorist theory, where the classroom is dominated by teacher talk (Goodlad, 1984) and the teachers rely heavily on textbooks, drills, and worksheets (Ben-Peretz, 1990). Teachers try to discover whether students know the right answers (Brooks & Brooks, 1993). The instructional emphasis lies in the outward production of responses. These descriptors explain why Korean elementary classroom teachers are using a traditionalist approach that emphasizes practices and drills in their actual classrooms, in spite their educational pedagogy was based on Constructivism.

The second factor, claimed by Korean elementary school teachers, was that private lessons students received in after-school programs influence student success. This obviously influences high achievement scores in mathematics competition because the tutors or instructors in the private programs could not help focusing on speed and accuracy to prepare students to solve problems quickly. Parents who pay for the private lessons expect success in their child's

mathematics scores on the exams. Because of this, students are trained to be test-wise by mastering algorithms. In school, teachers have students who already know the answers even before the concept are explained because these students have already mastered algorithms through the private tutoring. This issue might generate Korean elementary school teachers' reluctance to incorporate the Constructivist way of teaching using concrete objects to teach concepts. Sherman and Richardson (1995) studied elementary school teachers' beliefs and practices related to teaching mathematics with manipulatives. They reported that teachers tended to choose traditionalist approach due to concerns about discipline and classroom management issues. If teachers have students who represent a wide range of mathematics abilities, teachers spend more time controlling the class than practicing their effective instructional methods, especially since Korea was reported to have the highest student-to-teacher ratio (approximately 33 students per class) in elementary classrooms among the 40 countries in the Trends in International Mathematics and Science Study (TIMSS) in 2003. The mean class size of the teachers who participated in this study was 33.45 students.

The third contributing factor indicated by the Korean elementary teachers was parents' high expectation of child's education. One of the explanations discussed in other research studies in terms of this factor is the Confucian Heritage Culture (CHC) referred by Biggs' study "*Western misconceptions of the Confucian-Heritage Learning Culture* (1996, p. 46). As in other East Asian countries, Koreans share a common cultural value underlying this CHC. The values under CHC include a strong emphasis on the importance of education, high expectation for students to achieve, attribution of achievement more to effort than to innate ability, and a serious attitude towards study (Park, 2004, p. 91). Koreans place a very high value on academic credentials and on securing a good education for their children. Parents' self-esteem was intimately tied to the academic success or failure of their children. Another explanation for Korean parents' high educational expectation centers on the extremely competitive national college entrance examination. Mathematics is one of the four areas that are assessed on the college entrance examination. Because of this reason, students must be successful in mathematics and schools tend to place a relatively high importance on the subject of mathematics (Park, 2004). This fiercely competitive nature of the Korean educational system has made students' academic success, especially for mathematics, an all-consuming enterprise for most families, requiring much time, energy, money, and sacrifice, with the mother assigned to this task full time (Kim, 1996). Most Korean children from the elementary and even from the preschool level had to attend after-school private tutoring sessions as Korean elementary teachers said in the early section of this study. This often precipitated a soaring financial burden for the whole family. Due to this financial sacrifice of their parents and family members, Korean parents expect their children to achieve academic success by excelling in school. The child brings honor to the family while preparing for future educational and occupational success that would improve the family's social status and ensure financial support for the parents as well as the individual and his/her family (Serafica, 1990). With this high value placed on education and the family's sacrifice for education, parents and students consider education very seriously and put forth their efforts in doing well in mathematics. This resulted in Korean students getting more effective instruction and practice in mathematics.

The results of this study projected some common factors that were discussed in the report done by Park (2004). She listed the factors contributing to Korean students' high achievement in mathematics as: 1) College examination and selection; 2) Korean number system; 3) Attitudes of students towards tests; 4) Pragmatism and repetitive learning; 5) Competence of mathematics

teachers; and 6) Competence cycle. Issues about the college examination and selection and attitudes of students towards test are very closely related to what the Korean elementary classroom teachers suggested in this study. Another report done by Fuchs and Wobmann (2004) examined the PISA data regarding the accounts for international differences in student performance and concluded that student characteristics, family backgrounds, home inputs, resources and teachers, and institutions all contribute significantly to differences in students' educational achievement. The issues reported by these reports share the same baseline and are intertwined among contributing factors to mathematics education, but used different terms to categorize the factors. This study attempted to investigate factors that attribute to Korean students' high achievement scores in mathematics education, but this research showed that it would be very hard to find single or distinctive factors since all the factors contribute in an interactive way with each other.

References

- Ben-Peretz, M. (1990). *The teacher-curriculum encounter: Freeing teachers from the Tyranny of texts*. New York, NY: State University of New York Press.
- Biggs, J. B. (1996). Western misconceptions of the Confucian-Heritage Learning Culture, in Watkins, D. A., & Biggs, J. D. (ed.). *The Chinese Learner*. Hong Kong: Comparative Education Research Centre.
- Brooks, J. G., & Brooks, M. G. (1993). *In search of understanding the case for Constructivist classrooms*. Alexandria, VA: Association for Supervision and Curriculum Development.
- Caswell, H. L., Campbell, D. S. (1935). *Curriculum Development*. New York: American Book.
- DeVries, R., & Kohlberg, L. (1987). *Constructivist early education: Overview and comparison with other programs*. Washington, DC: National Association for the Education of Young Children.
- Fuchs, T., & WoBmann, L. (2004). What accounts for international differences in student performance? A re-examination using PISA data. Retrieved September 6, 2006 from <http://www.oecd.org/dataoecd/29/47/33680685.pdf>.
- Goodlad, J. (1984). *A place called school*. New York, NY: McGraw-Hill.
- Heddens, J. W., & Speer, W. R. (2006). *Today's mathematics: Concepts, classroom methods, and instructional activities*. Danvers, MA: John Wiley & Sons, Inc.
- Janvier, C. (1990). Contextualization and mathematics for all. In T. J. Cooney, & C. R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990s* (pp. 183-193). Reston, VA: National Council of Teachers of Mathematics.
- Kim, B.-L.C. (1996). Korean families. In M. McGoldrick, J. Giordano, & J.K. Pearce (Eds.), *Ethnicity and family therapy* (pp. 281-294). New York: Guilford Press.
- Kutz, R. E. (1991). *Teaching elementary mathematics*. Boston: Allyn and Bacon.
- Leung, F. (2002). Behind the high achievement of East Asian students. *Educational Research and Evaluation*, 8(1), 87-108.
- Lew, H. (2004, July). Mathematics Education in Korea after TIMSS. In H. Lew (Chair), *Development and characteristics of Korean elementary mathematics textbooks*. The national presentation at the meeting of the 10th International Conference of Mathematics Education, Copenhagen, Denmark.

- Love, A., & Kruger, A. C. (2005). Teacher beliefs and student achievement in urban schools serving African American students. *Journal of Educational Research*, 99(2), 87-98.
- National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: The Council.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: The Council.
- Mathews, J. (2006, October 18). For math students, self-esteem might not equal high scores: U.S. lags behind countries that don't emphasize self-regard. *Washington Post*, pp. A02.
- Paik, S. (2004, July). Mathematics Curriculum in Korea. In H. Lew (Chair), *The Report on mathematics education in Korea*. The national presentation at the meeting of the 10th International Conference of Mathematics Education, Copenhagen, Denmark.
- Organisation for Economic Co-operation and Development (OECD) (2006). Education at a glance: OECD indicators. Retrieved September 5, 2006 from <http://www.oecd.org/dataoecd/32/0/37393408.pdf>.
- Park, K. (2004, July). Factors contributing to Korean students' high achievement in mathematics. In H. Lew (Chair), *The Report on mathematics education in Korea*. The national presentation at the meeting of the 10th International Conference of Mathematics Education, Copenhagen, Denmark.
- Program for International Student Assessment (PISA). PISA 2003 summary. Retrieved September 5, 2006 from <http://www.pisa.oecd.org/dataoecd/1/60/34002216.pdf>
- Serafica, F. C. (1990). Counseling Asian American parents: A cultural-developmental framework. In F. C. Serafica, A. I. Schwebel, R. K. Russel, P. D. Issac, & L. J. Meyers (Eds.), *Mental Health of Ethnic Minorities* (pp. 222-244). New York: Praeger.
- Sherman, H., & Richardson, L. (1995). Elementary school teachers' beliefs and practices related to teaching mathematics with manipulatives. *Educational Research Quarterly*, 18(4), 27-36.
- Shuhua, A. (2000). Mathematics teachers' beliefs and their impact on the practice of teaching in China. (ERIC Document Reproduction Service No. ED 442 669).
- Trends in International Mathematics and Science Study (TIMSS), *TIMSS 2003 International Reports*. Retrieved September 5, 2006, from http://timss.bc.edu/timss2003i/intl_reports.html
- Trends in International Mathematics and Science Study (TIMSS), *TIMSS 1999 International Reports*. Retrieved September 5, 2006, from <http://timss.bc.edu/timss1999.html>
- Third International Mathematics and Science Study (TIMSS), *TIMSS 1995 International Reports*. Retrieved September 5, 2006, from <http://isc.bc.edu/timss1995i/Highlights.html>.

Chung

HOW TO INCREASE MATHEMATICAL CREATIVITY- AN EXPERIMENT

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Abstract: Creativity is an integral part of mathematics. In this article I examine the increase in awareness of creativity in mathematics using Fröbel's blocks in a college classroom. A majority of students found the introduction of the "gifts" of the founder of Kindergarten to a college geometry classroom enhancing their interest in mathematics. They judged the wooden blocks helpful in their understanding of geometry. The students showed increased awareness of creativity in mathematics.

Keywords: creativity; Fröbel's blocks; geometry; teaching of geometry; reflective practice

1. Introduction

Many students dislike classes in mathematics. They give a wide variety of reasons for this and among the most mentioned ones are that mathematics is hard, mathematics is boring and mostly irrelevant. Part of this problem stems from misconceptions about mathematics. It is described as inflexible and formulaic as opposed to fun and creative. As a teacher of mathematics it is my duty to counteract those prejudices and create a fertile learning environment. I continually seek to inspire students and convince them that mathematics in all its forms is worthwhile.

In this paper I describe an experiment aimed at revealing the creative process in mathematics. Creativity enters mathematics in many different ways. Three important ways are abstraction, connection, and research. The creativity of abstraction concerns the creation of models that reflect the real world and can be solved with mathematical tools known to the individual. The creativity of connection is the realization that known mathematical tools can be applied to new problems, allowing problems to be viewed in a new way. Connections are also made when mathematical and other knowledge come together to understand and solve problems from a variety of areas. Finally, the creativity of researching is the discovery of new mathematical tools that fit unsolved problems and add to the available tools for other users of mathematics.

The class chosen for this experiment was an undergraduate college class in Euclidean geometry populated by aspiring teachers, and the tool for creative development was Friedrich

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Fröbel's "gifts", based on their simplicity as well as scientific and geometric connections. The students were presented with the gifts and Fröbel's instructions to examine a possible increase in awareness of creativity in mathematics.

1.1. Euclidean Geometry

Euclidean Geometry is the study of plane and space objects. Its main concern is the relations and interactions of points, lines and polygons. Euclid was the first to systematize the study of geometry in his work *Elements*. From basic premises (axioms) and common notions he developed successively more complicated facts. The proof of each new proposition or theorem was based only on the axioms, common notions and previously proven propositions and followed a logical path.

However, high school geometry is mostly presented without proofs, or at most a limited exposition to the logical structure that lies at the foundation of Euclid's *Elements*. As a result, knowledge of basic notations and logical foundations are often lacking in students. Also, three-dimensional geometry often is neglected or omitted. Lastly, the students supposedly discover many facts with tools such as the Geometer's Sketchpad or other applications. While this element of discovery helps the students understand and retain some of the information, it obscures the structures of geometry.

Creativity in the creation of proofs is a fundamental part of mathematics. In geometry there are often several ways of proving a theorem and sometimes the proofs are far from obvious. In addition, students often lack the understanding of why certain statements have to be proven. The most common example is that the interior angle sum in a triangle in the Euclidean plane is 180° . The students all know this fact but when it comes to proving it they often do not know why a proof is required or how to attempt it. Some students lack the creativity to think of situations where the angle sum might be different or to (re-)create a valid proof of the fact. The first case can be remedied by drawing triangles on the outside of a sphere where one can easily draw a triangle with angle sum close to 360° . The other situation requires more work.

1.2. Friedrich Fröbel

Friedrich Fröbel was a German educator, scientist and naturalist who first introduced the concept of "Kindergarten". He was among the first to realize that learning starts at birth and gave the first task of teaching the children to their mother. He developed the idea of Kindergarten as an aid to early learning and developing inquisitive habits. His main work, *The Education of Men* (*Menschenziehung*) (Fröbel, 1826/1905) was published in 1826. One of his many ideas is the use of gifts he designed for the students.



Source: <http://www.constructiontoys.com/>

Gift three consists of eight 1 inch cubes, gift four consists of eight 1□2□½ inch blocks, gift five consists of cubes, triangular prisms from cutting a cube into half or fourth, and gift six consists of 1□1□½, 1□½□½ and 1□2□½ inch blocks

These gifts are sets of wooden blocks and are used to create objects of beauty, nature and the physical environment. We used gifts two through six in class. A note on the word “gift”: it is now the standard English word for describing Fröbel’s “Gaben” which could be translated from the German as gift. They were not intended to be presents but merely a tool given to the student. The structures of the gifts can be linked to his studies of crystals under Professor Weiss in Berlin (Rubin, 1989). It can be viewed as an early precursor to the use of manipulatives to aid students in learning. The gifts are rooted in Fröbel’s work on crystals and exhibit many of the properties that crystallographers use to describe crystalline structures (Rubin, 1989).

Another of Fröbel’s ideas is that of guided discovery, which he promotes throughout his works. The teacher’s guidance can take several forms. He can steer students towards certain activities. He can limit the choices put before a student and he can guide the student through questions and prompts. It was his intention to teach young children through play and to instill in them the scientific method at an early age. Guided discovery would at a later age take the form of discussion of the results.

Many current ideas in education were previously presented by Fröbel in his landmark book “*The Education of Men*” and other works. He put a great emphasis on student learning that occurred when the students were actively involved and had hands-on material. This was one of the main ideas of the gifts. Students create objects by manipulating blocks and other tangible objects. He also emphasized the importance of outdoor activity and the first Kindergarten had a plot of land for each child to tend, hence children’s garden (Kindergarten).

Fröbel saw a teacher more as a guide than a lecturer. He believed that discovery learning is much more fruitful for the children than being taught concepts without a hands-on activity. However, Fröbel stressed that the guidance by the teacher is of the utmost importance and the gifts again reflect that principle. While his first experiment had about 100 blocks the gifts finally came to their current shape as part of the idea of guidance by limiting the choices given to the children.

2. Literature Review

Recent research into the life and work of Friedrich Fröbel focuses on two basic ideas; his influence in the historic context and how his ideas can be applied to the modern school. In the light of increased standardized testing, William Jeynes (Jeynes, 2006) made his case for a Fröbelian approach for schooling in kindergarten and first grade. Jeynes suggests that “a kindergarten curriculum dedicated to developing mind, the spirit and the body” (Jeynes, 2006, p. 1941) should be developed and we agree with his assertion that it can be found in Fröbel’s work. John Manning (Manning, 2005) makes a similar point in his call to re-examine Fröbel’s life and gifts. He thinks that the ideas can be used as a supplement to testing rather than in its place.

Many in today's education world dismiss Fröbel as a Romantic educator whose child centered view cannot possibly work in the modern school system, especially since his curriculum does not produce immediately measurable results but is based on the education of the whole person, mind, spirit and body. Like so many others, Reese laments that Fröbel was "alternatively obtuse and highly prescriptive" (Reese, 2001, p.15). He admits that "Fröbel's followers substantially revised the ... gifts". I believe that in order to understand Fröbel one has to go back to the original documents and learn from the idea and manifestation of Kindergarten. "Fröbel was searching for the unity of things, for order" (Reese, 2001, p.3) and this is reflected in his work in general and his gifts in particular. For more on Fröbel read the exceptional book "A Child's Work" by Joachim Liebschner (Liebschner, 2001).

In contrast, other researchers point out the historical importance of Fröbel to different school systems. For example Meike Baader (Baader, 2004) investigates Fröbel's influence on the American system in conjunction with educational theory while Brehony and Valkanova (Brehony & Valkanova, 2006) investigate the influence on the Russian system.

Use of some of Fröbel's ideas in the modern classroom has been suggested before. Geretschlaeger (Geretschlaeger, 1995) has used the ideas of paper folding or origami in his geometry classroom. The activity of paper folding is one of the "occupations" that Fröbel suggested. Occupations are materials and instructions given to the students just like the gifts. But unlike the gifts the occupations are altered in the process. I have used the gifts before but in an introductory course in modern geometry with a focus on abstraction and connections between seemingly unconnected objects and ideas (Brunkalla, 2006).

The research into creativity is, on the other hand, very voluminous. A good overview of mathematics and creativity can be found in Treffinger et al. (Treffinger, Young, Shelby & Shepardson, 2002). Most research is centered on children from Pre-Kindergarten through grade nine. Few publications deal with creativity in highly accomplished mathematicians. Moreover, there is a curious lack of research in the area of creativity in college mathematics. The most basic problem is that there is no universally accepted definition of mathematical creativity (Haylock, 1997) and no single test or assessment of it. Many researchers agree on certain qualities of creativity but show some divergence on others. Significantly, most researchers link mathematical creativity to mathematical ability. Often a positive attitude towards mathematics is linked to creativity while a negative attitude would imply less mathematical creativity (Mann, 2005).

Another focal point of mathematical creativity is the ability to solve problems (Silver, 1997). Many attempts have been made to formalize the problem solving process. Most notable among them is Polya, who studied creativity in the 1930's and 40's. His approach to problem solving is at the heart of almost every introductory mathematics textbook on the market today. See for example (Stewart, 2003). Most textbooks use Polya's strategies or strategies based on his work, but do not give him credit for it.

3. Basic ideas and Purpose of the Study

This experiment had several goals. First, it was to increase the student's awareness of the creative process as it occurs in mathematics. Second, to establish a link between creativity and mathematics and also link mathematics to the real world. Third, the students were to evaluate thought processes and creative processes in themselves and others. Fröbel's ideas were introduced and linked to current trends in education, such as manipulatives and the teacher as a guide. Lastly, the connection between the gifts, crystallography and architecture was explored.

The gifts can help students understand that not everything is what it seems to be. Especially, the second gift brings that aspect of geometry to the forefront. The rotation cube that will look like a circular cylinder when spun fast enough gives students at least a brief pause to examine objects more closely. Creativity, although its measurement is difficult, is integral to learning mathematics. I hope that introducing Fröbel's gifts to the students will increase their awareness of the link between mathematics and creativity as well as increase their use of creative (although mathematically correct) ways of looking at geometric facts and theorems.

Mathematics by most people is viewed as a rigid, formulaic subject without any bearing on real life. While it is correct that part of mathematics consists of rules, logical structures and formulas, most of mathematics centers about the ability to develop tools that are applicable to a wide variety of problems. Thus mathematics includes the ability to abstract real world situations, choose the proper mathematical tool for the solution and to interpret abstract results in the light of reality. Most of these abilities are included in Froebel's considerations and teachings.

Creativity in the mathematical process has been studied in young children and early school age children as well as in highly accomplished mathematicians. However, there is a curious lack of concentration on the population between these two extremes. Mathematical creativity in college students has been all but ignored and this experiment is an attempt to close the gap. As far as I know, it is unique in its use of Fröbel's ideas and gifts in a college classroom.

4. Set-up

In a Euclidean geometry class taught at Walsh University in the spring of 2007, 22 students were presented with gifts three through six. The instructions varied with the gifts. Students were asked to form small groups of three to four students. The instructions with gift three were simply to create as many objects of nature, beauty or the environment as possible. Gift four had more specific instructions as some objects such as numbers and letters were excluded. The instructions for gift five were to repeat the process from the last gift with more attention to the process of developing an object. Finally the instructions for gift six told students to each create exactly one object with the gift and the other students had to describe the process of building or creating. Students were also asked to describe their own thought process when creating their object with gift six. After each session students were asked to summarize their experiences with the gift and the instructions. Classroom observations by the teacher were made at the beginning of the class and after each session.

Next, students were required to write a paper including their observations and experiences with Fröbel's gifts, including descriptions of objects and the process of their

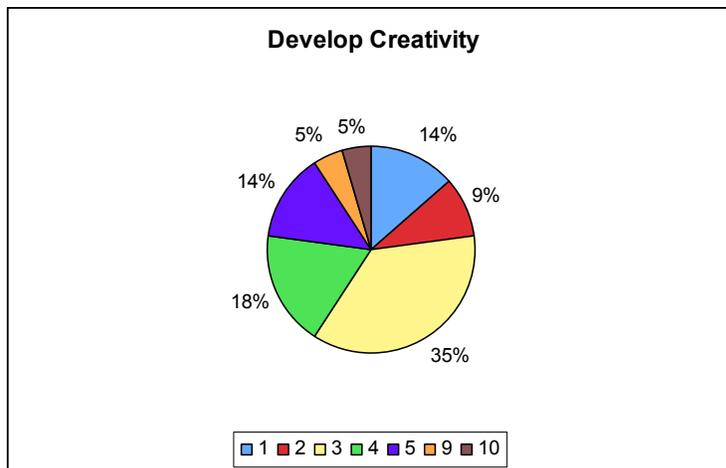
creations. They were asked to provide verbal descriptions as well as perspective drawings of some of their objects. Reading for this paper included a section on Fröbel's life and work. Another part of the paper was a description of the process of creating objects with Fröbel's gifts from observation of others and from the student's own perspective.

Finally, data were collected by asking students to complete a short survey and comments on the class as a whole and specifically the introduction of Fröbel's work. The surveys were anonymous and subjected to standard statistical procedures for small samples.

Since Fröbel's ideas and writings concern mostly kindergarten children and young pupils in elementary school, the concepts have to be adapted to fit into a college classroom. Some of the students noticed the differences in instructions given by Fröbel from the instructions given in class. It should be noted that in college you see your students three times a week for one hour, whereas K-4 teachers will typically see their pupils every day for longer periods of time. Also, the experiment was restricted to a one-semester course.

5. Results

All students rated their experience with Fröbel's gifts and his ideas as very positive and interesting. 91% of the students acknowledged the importance of creativity in learning mathematics, although some students qualified this as being restricted to geometry. Students rated Fröbel's gifts as helpful in developing creativity as a 3.68 on a scale from one to ten with one being the most and ten being the least.



The chart shows the individual answers given by students to the question "How helpful do you think that block play is in developing ones creativity?" 58% of students rated it a three or better.

Students are a little less sure of the importance of creativity in the development of mathematical skills. They rate the importance only as a 4.05. The correlation of 0.56 between the development in creativity and the development of mathematical skills shows that students who recognize Fröbel's gifts as important mostly acknowledge the idea that creativity contributes to mathematical skill. The gifts are seen as helpful in understanding geometry. Although, students

rate the helpfulness of the gifts in this task only as a 4.86, this still shows a positive attitude towards the manipulatives.

Question	Mean	Standard Deviation
1. Did working with Fröbel blocks make this geometry class more interesting? (Y/N)	1.00	0.00
2. Do you think that creativity is important in the study of math? (Y/N)	0.91	0.29
3. Would you consider using Fröbel blocks with your children or with your students? (Y/N)	0.91	0.29
4. How helpful was studying Fröbel to your understanding of Geometry? (1-10)	4.86	1.98
5. Is block play an important activity? (Y/N)	0.95	0.21
6. How helpful do you think that block play is in developing ones creativity? (1-10)	3.68	2.18
7. How helpful do you think that creativity is in developing ones math skills? (1-10)	4.05	2.08
8. What was your comfort level with playing with blocks? (1-10)	2.77	2.11
9. Rate how much you enjoy math. (1-10)	2.55	1.59
10. Rate how much you would enjoy playing with Legos. (1-10)	2.86	2.67
11. As a child did you ever play with wooded block? (Y/N)	0.90	0.29
12. Did this class enhance your understanding of how math is found in the world? (1-10)	4.14	2.26
13. Would more reading material help you in this class? (Y/N)	0.23	0.42
14. How difficult would you rate understanding Fröbel's concept? (1-10)	4.95	2.29
15. Do you feel that you understand Fröbel's method? (1-10)	3.73	1.96

Questions were either Yes-No questions indicated by (Y/N) or on a scale of one to ten indicated by (1-10). The scale was set up so that one was the most and 10 the least, to ensure that students read the instructions correctly. For the Yes-No questions yes was coded as a 1 and no coded as a 0, so that a mean of 0.91 indicates that 91% of the students answered yes to that particular question.

Most of the work submitted lacks in the use of patterns or objects of beauty created with the Fröbel gifts. Students exhibited a strong tendency towards real world objects and neglected the chance to create patterns with the given materials. The work was often centered on thematic groups like animals or football and objects were then created to fit within the chosen category. Even when the creation is passed from one student to another in a group the theme was more than likely to remain constant. I also noticed limited variation from gift to gift. Students tried to recreate the same or similar objects that they created before when presented with a new gift.

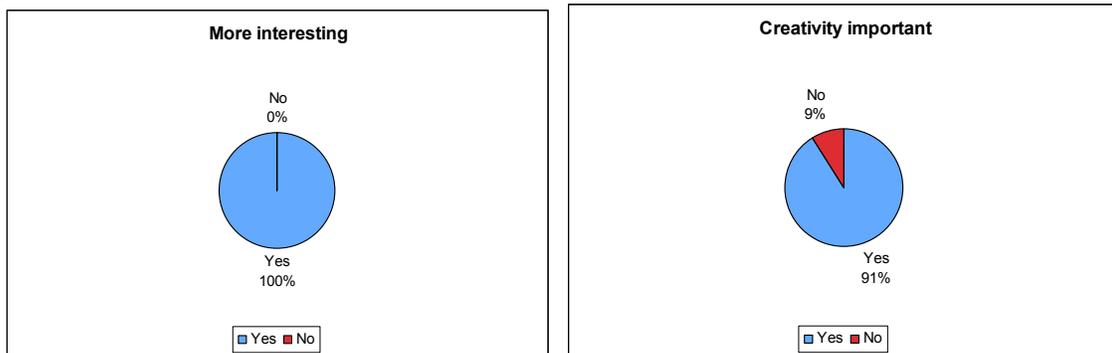
As mentioned above, most students agree with the importance of creativity but do not reflect this in their papers or their work. Similarly, students showed a general lack of abstraction skills. At first they would rather abandon a project than to figure out how to make it work using

abstractions or generalizations about objects. However, as more gifts were introduced, the students became increasingly at ease with the wooden blocks and thus their skills in working with them improved. Some students were able to create very interesting objects with the higher numbered gifts.

At another level, some more general problems were detected. There were difficulties with reading and understanding instructions and a lack of verbal expression skills regarding mathematical ideas and objects. Students have a hard time writing about mathematics and expressing mathematical ideas in written form. They were required to describe the creation of one of their objects without drawing a picture and most students could not give a complete description of the necessary actions and relations of the blocks to allow recreation of their particular object. On the other hand, both class and homework showed an increase in reflective skills and the ability to observe and self-observe.

One student observed that using Fröbel's gifts and its resulting "strategies help develop abstract thinking" which is part of mathematical development since mathematics is the language of abstraction. Part of the problem of developing a good understanding of mathematics is to develop abstraction abilities. On the other hand, students also need to learn to apply abstract mathematics to the real world. Both aspects of the link between mathematics and the real world are important. The students ranked this only as a 4.14.

Understanding of the importance of creativity in mathematics and learning in general went very well. The goal was "to bring out students' creativity and Fröbel's gifts is an excellent way to do that" as one student noted. Further underscored was the importance of creativity by the observation that "creativity allows us to see some of the things we normally would miss in mathematics". It cannot be denied that some students regarded the experiment with a lot of skepticism because "math is all based on logic not creativity".



This chart shows the amount of students judging Fröbel's addition to the geometry class as interesting and the amount of students who judge creativity in mathematics important.

The gifts were praised by students as a tool of understanding geometry and "after working with them it made it easier to understand some aspects" of the class. Also mentioned was the idea that having manipulatives in a college classroom was stimulating to their thinking and raised their interest in the class. Hand-on activities were clearly a surprise for the students in this mathematics class. While most students said that they were comfortable with the gifts, it was

obvious that, especially in the beginning, they had some concern about playing with toys. Overall the students rated their own comfort as a 2.77.

Most students believe that they have a good understanding of Fröbel's ideas and they rated themselves as a 2.97 in terms of understanding. They rate the difficulty of Fröbel's concepts a 4.95. However, the answers to these two questions have a slightly negative correlation coefficient of -0.18, which leads me to believe that few students understand Fröbel's concepts in their entirety.

It was most important for the students to realize that the gifts and the instructions that Fröbel presented together with the gifts were not taken out of thin air but have a firm grounding in the fact that Froebel was a crystallographer who studied nature and its building blocks closely. This gives the experiment with the blocks a new direction and infuses meaning into the seemingly useless limitations and rules that come with the block play.

The awareness of the importance of creativity in mathematics grew noticeably. In the survey 91% of the students agreed that creativity is part of mathematics. Where many students had very little to say about creativity and mathematics at the beginning of the class they admitted to the importance and power of it in the final survey.

6. Conclusions

Overall I think that the students in the class learned many things about creativity and its importance in mathematics. They were exposed to concepts that have been all but forgotten and had a chance to reevaluate some positions they took regarding mathematics. Students have shown a new or renewed appreciation for the mathematical process and the links of mathematics to the real world. Most students regarded the experiment as a success in so far as they were more interested in the class and the material and the gifts actually helped them understand mathematical creativity and geometry better.

It is still not well understood what the triggers for mathematical ability are and how development of mathematical thinking can be furthered, I think that reintroducing Froebel's ideas into the early kindergarten and elementary school curriculum will most definitely help to increase mathematical awareness and creativity. While research into mathematical creativity and creativity in general is taking important steps that will hopefully yield a clear definition and methods of measurement for creativity much is still to be done in that area.

7. Limitations and further research

The class used for this study was a small sample of college students. All students had signed up for a mathematics class and are thus not representative of the whole student body. The study is mainly based on a survey and is limited to observation from the classroom and self-reports in its other data gathering.

In the future I would like to expand the use of the gifts in the classroom to other courses and have access to enough of the gifts so that each student can have their own set for experimentation. I would like to use tools that more accurately measure mathematical creativity. The next experiment will contain a pre- and post-test about attitudes towards mathematics and creativity and about the perception of a connection or disconnect between the two.

References

1. Baader, M. (2004) Froebel and the Rise of Educational Theory in the United States. *Studies in Philosophy and Education*, **23**, 427-444
2. Brehony, K. and Valkanova, Y. (2006) The Gifts and 'Contributions': Friedrich Froebel and Russian Education (1850-1929). *History of Education*, **35**, 189-207
3. Brunkalla, K. (2006) Friedrich Fröbel's Ideas in a College Classroom, *2nd Biennial International Froebel Society Conference*, Dublin, Ireland, June 2006
4. Fröbel, F. (1905) *The Education of Men* (W. N. Hailman, Trans.) New York: D. Appleton and Company (Original published in 1826)
5. Geretschläger, R. (1995) Euclidean Constructions and Geometry of Origami. *Mathematics Magazine*, **68**, 357-371
6. Haylock, D. (1997) Recognising Mathematical Creativity in Schoolchildren. *Zentralblatt für Didaktik in Mathematik*², **3**, 68-74
7. Jeynes, W. (2006) Standardized Tests and Froebel's Original Kindergarten Model. *Teachers College Record*, **108**, 1937-1959
8. Liebschner, J. (2001) *A Child's Work Freedom and Guidance in Froebel's Educational Theory and Practice*. Cambridge: Lutterworth Press
9. Mann, E. (2005) *Mathematical Creativity and School Mathematics: Indicators of Mathematical Creativity in Middle School Students*. PhD thesis, University of Connecticut
10. Manning, J. (2005) Rediscovering Froebel: A Call to Re-examine his Life & Gifts. *Early Childhood Education Journal*, **32**, 371-376
11. Reese, W. (2001) The Origins of Progressive Education; *History of Education Quarterly*, **41**, 1-24
12. Rubin, J. (1989) The Froebel-Wright Kindergarten Connection: A New Perspective. *Journal of the Society for Architectural Historians*, **XLVIII**, 24-37
13. Silver, E. (1997) Fostering Creativity through Instruction Rich in Mathematical Problem Solving and Problem Posing. *Zentralblatt für Didaktik in Mathematik*, **3**, 75-80
14. Stewart, J. (2003) *Calculus*. Belmont: Thomson Learning, Inc.
15. Treffinger, D., Young, G., Shelby, E. and Shepardson, C. (2002) *Assessing Creativity: A Guide for Educators*. Storrs: The National Research Center on the Gifted and Talented

² Editorial Note: ZDM just released a double issue on interdisciplinarity which contains a section focused on creativity. The interested reader can follow this link
<http://www.springerlink.com/content/g71m25052028/?p=eb8fe5ea1dae4145af89a82f6bbd5f3b&pi=0>

Catch me if you can!

Steve Humble¹

The National Centre for Excellence in the Teaching of Mathematics, UK

Learning mathematics outside the classroom is not enrichment, it is at the core of empowering an individuals understanding of the subject.

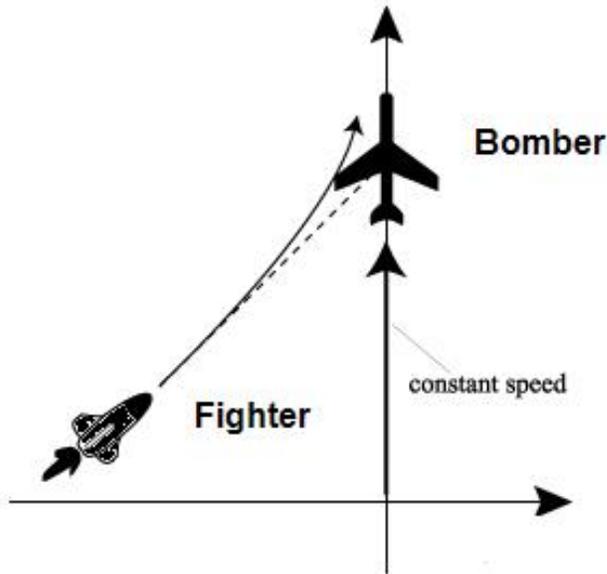
The three activities described in this article can all be used outside the classroom in a maths lesson. Teaching mathematical concepts in this way engages and reinforces learning. It puts the ideas learnt into a setting and allows time for those ideas to be developed without any of the maths hang-ups which can occur in the classroom. By taking maths beyond the classroom, we can more clearly illustrate the connections between the real world and what they are studying in school. In so doing students and teachers alike are enthused by the wealth of resources they have all around them in their own environments.

From a very young age we all play “catch me if you can!”, Tag being the most well known version, where one person chases others. When the player catches another they say “Tag, caught you, your on”. The pursuer then becomes the pursued. The 1968 classic car chase movie Bullitt, had Steve McQueen driving his Mustang GT 390 at speed through the hilly streets of San Francisco(1). This is a wonderful example of a movie car chase, but I am sure you could name others. These movies show students real life cases of pursuit. Another example is a fighter plane in battle following on a pursuit curve to shoot down a bomber aircraft. The fighter will continually point its guns and plane towards the target bomber it is trying to shoot down. As the fighter moves in, closing the gap between itself and its prey, the velocity vector will always be pointing towards the bomber.

¹ **Steve Humble** (aka Dr Maths) is a regular contributor to The Montana Mathematics Enthusiast. He works for The National Centre for Excellence in the Teaching of Mathematics in the North East of England (<http://www.ncetm.org.uk>). He believes that the fundamentals of mathematics can be taught via practical experiments. He is the author of the book *The Experimenter's A to Z of Mathematics*, which develops an experimenter's investigative approach to mathematical ideas. Always having had great fun playing with maths, he enjoys teaching this to others.

For more information on Dr Maths go to
<http://www.ima.org.uk/Education/DrMaths/DrMaths.htm>

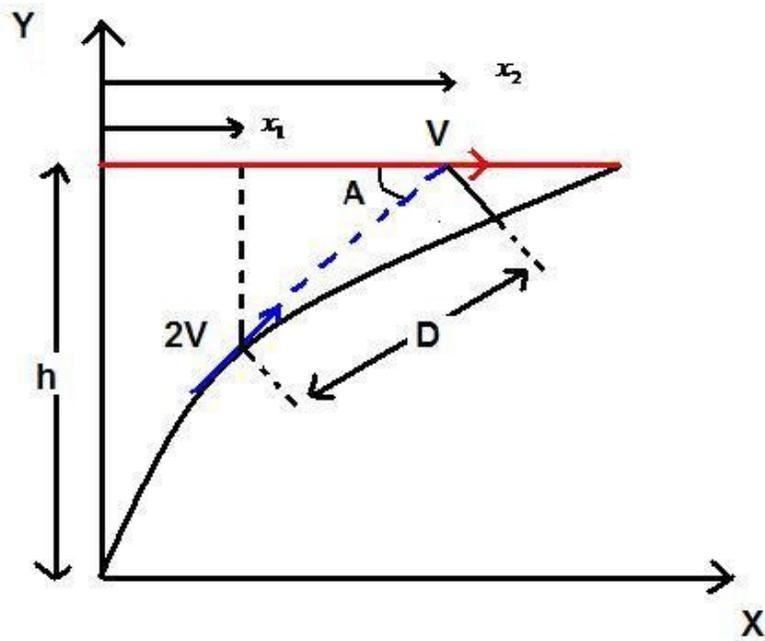
Email: drmaths@hotmail.co.uk



[Fighter and bomber]

The first mathematician to work on the idea of pursuit analysis was the French mathematician Pierre Bouguer in 1732. One way in which his analysis may be illustrated is using the analogy of a cat and mouse chase, with the mouse moving towards its hole in the wall in a straight line at constant speed. The velocity vector of the cat always moves directly towards the mouse. There are two possible areas of mathematics to investigate: how long does it take to catch the mouse? and what is the curve of pursuit the cat follows? Many books (2) tend to look at the latter question which younger students often find difficult. Alternatively by looking to find the point at which the cat and mouse meet, makes the problem more accessible to a wider range of age and ability.

Here is a method to find the rendezvous point, when the cat moves twice as fast as the mouse.



[Cat and mouse]

h is the initial distance between the cat and the mouse.

D is the distance between the cat and the mouse.

x_1 is the horizontal distance moved by the cat.

x_2 is the horizontal distance moved by the mouse.

v is the speed of the mouse, at some general point.

A is the angle between the cat and the mouse, at some general point.

The following three equations define a general point on the pursuit of the mouse.

Humble

$$\frac{dx_1}{dt} = 2v \cos A$$

$$\frac{dx_2}{dt} = v$$

$$\frac{dD}{dt} = v \cos A - 2v$$

Note that these show that the distance D is reducing.

$$\text{Using } \cos A = \frac{1}{2v} \frac{dx_1}{dt} \text{ gives } \frac{dD}{dt} = \frac{v}{2v} \frac{dx_1}{dt} - 2v$$

$$\text{and } \frac{dD}{dt} = \frac{1}{2} \frac{dx_1}{dt} - 2 \frac{dx_2}{dt}$$

$$\text{Integrating this equation gives } D = \frac{x_1}{2} - 2x_2 + c$$

Using the initial conditions when $t = 0$, $D = h$, $x_1 = 0$, $x_2 = 0$ gives $c = h$

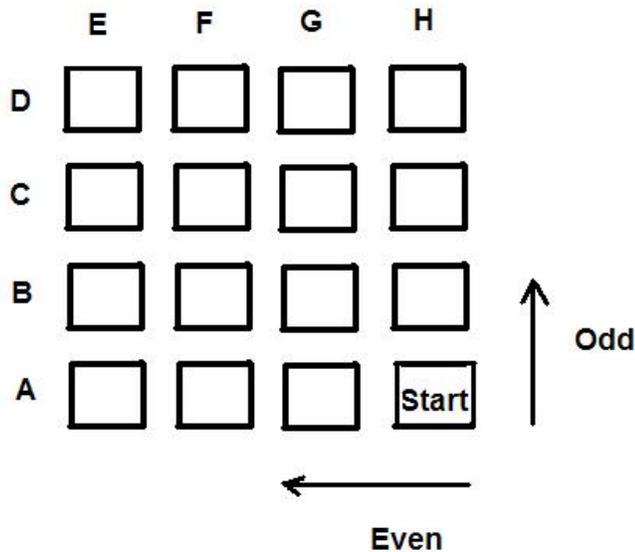
$$\text{Hence } D = \frac{x_1}{2} - 2x_2 + h$$

$$\text{When } t = T, D = 0, x_1 = x_2 = x \Rightarrow 0 = \frac{x}{2} - 2x + h$$

Solving this equation gives the point at which the mouse is caught as $x = \frac{2}{3}h$.

Students who have not yet learnt Calculus could still tackle this problem and solve the algebra for various cat and mouse speeds. For example, if the cat moves k times as fast as the mouse then the equation describing the point at which the mouse is caught can be written as

$$0 = \frac{x}{k} - kx + h \text{ and solved for various values of } k \text{ and } h.$$



[Grid catchers]

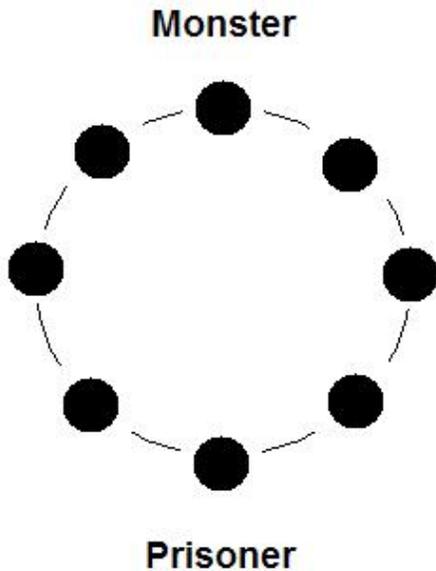
In the next game the pursued has to traverse a 4 by 4 grid to escape the grid catchers. This game is played with a four sided dice or you can use a normal dice, throwing it again if you get five or six on the first throw. Alternatively use a spinner numbered one to four or throw two dice numbered $\{0,0,0,2,2,2\}$ and $\{1,1,1,2,2,2\}$.

Standing in the bottom right hand corner, throw a dice and if you get an odd number move straight up the column the number of squares indicated. An even number indicates a move to the left. If you are still on the grid after the first move throw again, and repeat until you escape the grid.

Before the game starts everyone else has to make a prediction about which point you will exit the grid. They are called the catchers and have to pick A to H marked on the diagram above and stand by this point to catch you as you exit the grid.

Activities to try:

- a) Most likely exit point.
- b) At which points will you never come off the grid?
- c) Least likely exit points (other than those found in (b))
- d) If you had to pick 3 places to stand to catch, which would you pick?
- e) Calculate the probabilities of where you will come off the grid
- f) What happens if you use a dice numbered 1 to 6?



[Monster and Prisoner]

In 1965 Rufus Isaacs(3) created a pursuit-and-evasion game which he called The Princess and Monster Problem. The chase takes place in a pitch black circular tunnel with neither pursuer nor the evader being able to see each other. They both move at the same speeds on a stepping stone type grid around a circular path. In 1972 D Wilson (4) solved the problem mathematically to find the most useful game strategies when played on a discrete interval.

A variation is to play this game with 8 discrete points marked evenly around the circle with the 2 players starting an even number of points apart. I call it Monster and Prisoner. With the Monster and Prisoner game you throw a coin to decide which way you move. In one "move", each player moves one step left (Heads) or right (Tails), each with the probability of a halve . As they always start an even number of steps apart, throughout the game they will always be an even number of steps apart.

Let $E(2)$ and $E(4)$ be the mean number of moves starting at 2 or 4 apart respectively, until they meet on the same stepping stone. These are the only possibilities on an 8-position circle. You can consider one move as the following equations

$$E(4) = 1 + \frac{1}{2}E(4) + \frac{1}{2}E(2) \Rightarrow E(4) = 2 + E(2) \text{ and}$$

$$E(2) = 1 + \frac{1}{4}E(4) + \frac{1}{2}E(2) \Rightarrow 2E(2) = 4 + E(4)$$

Solving these simultaneous equations gives $E(4) = 8$ and $E(2) = 6$.

Therefore on average the game last 6 or 8 moves depending on your starting position. Students can play this version of the game and make predictions about how long it will take to get caught. Possible extensions ideas are to play the game with more or less stepping stones and find the mean number of moves until they are caught.

The following BASIC code allows you to simulate the Monster and Prisoner game for an 8 point stepping stone circle

```
10 T = 1
20 S = 0
30 M = 0: E = 4: I = 0
50 I = I + 1
55 REM ***Monster movement***
60 X=RND
70 IF X > .5 THEN M = M + 1: IF M = 8 THEN M = 0
80 IF X < .5 THEN M = M - 1: IF M = -1 THEN M = 7
90 Y=RND
95 REM ***Prisoner movement***
100 IF Y > .5 THEN E = E + 1: IF E = 8 THEN E = 0
110 IF Y < .5 THEN E = E - 1: IF E = -1 THEN E = 7
115 REM ***Check to see if caught***
120 IF M = E THEN PRINT M, E, I ELSE GOTO 50
130 S = I + S
140 PRINT "Average"; S / T ; "after "; T ; " turns"
150 T = T + 1
160 IF T < 10000 THEN GOTO 30
```

References

- (1) Bullitt, had Steve McQueen <http://www.youtube.com/watch?v=GMc2RdFuOxI>
- (2) Paul J. Nahin, Chases and Escapes. [Princeton University Press]
- (3) R. Isaacs, Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization, John Wiley & Sons, New York (1965).
- (4) D. Wilson, Isaacs' Princess and Monster Game on a Circle by (Journal of Optimization Theory and Applications, 9 1972 (no 4). pp.265-288]

Acknowledgements

I wish to thank Dr John Haigh for his help and suggestions.

Humble

A TRAILER, A SHOTGUN, AND A THEOREM OF PYTHAGORAS

William H. Kazez¹
University of Georgia

Counselor: Please tell the Court your name.

Expert Witness: My name is Will Kazez

Counselor: No, no, no! Your name is...

This is not a good start. I am not naturally a nervous person. I have survived teaching calculus to a large class that included the entire freshman football team of the University of Pennsylvania, but I've never been an Expert Witness. Even though I'm confident of the mathematics, I'm not sure I like the idea of being cross-examined. But still, I'm just rehearsing my testimony with the lawyer, and even if I've got my own name a little wrong, what's the worry? At any rate, lawyers do not like being interrupted.

Counselor: No, no, no! Your name is DOCTOR William H. Kazez.

Expert Witness: O.K. My name is DOCTOR William H. Kazez.

Counselor: And how are you currently employed, Dr.~Kazez?

Expert Witness: I am a Lecturer in the Department of Mathematics at Cornell University.

Counselor: And tell the Court, Dr. Kazez, are you familiar with the theorem of Pythagoras?

Expert Witness: Well your Honor, I don't mean to brag, but yes, I am familiar with the theorem of Pythagoras.

Now this is good! I have rehearsed the last line in my mind many times, and I say bring it on. I'm ready for any cross-examination by any lawyer or judge. Let them take their best shot. But first, maybe I should tell you about the case?

My next-door neighbor at the time was Barry Strom, Director of Cornell's Legal Services. He had a client come to him for help with a problem in elementary geometry. The client was living in a trailer that he kept parked next to one of the boundary lines of his property. One night, his neighbor approached him, with shotgun in hand, and told him to move the trailer, because, in the neighbor's humble opinion, it was parked over the boundary line. We mathematicians like to

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pride ourselves on our ability to craft concise, persuasive arguments, but compared to the guy holding the shotgun, we are amateurs.

As a first step in resolving the conflict, Barry obtained a copy of his client's deed. The deed described the property as a triangle. No problem. It gave the lengths of the sides of the triangle as 90', 110', and 180', so it was a long skinny triangle, but that's alright. Finally, the shape of the land was described as a right triangle! Now this is a problem. Even if you aren't sure how to spell Pythagoras, you know someone really messed up this deed. It was at this point that Barry decided it would strengthen his case to have me explain, in court, the problem with the deed.

You are probably thinking that this has to be the easiest bit of expert testimony ever given. What could go wrong? Why would anyone be nervous? Let me ask you a few questions. First, what if the land the trailer was on wasn't level? On the one hand it was located near Ithaca, N.Y. so it probably wasn't, on the other hand, who parks a trailer on ground that isn't level? No stalling now, this is a cross-examination. If the distances of 90', 110', and 180' are measured between points of different heights, could it be that the triangle they span is a right triangle? Well, no. Pythagoras' theorem still would rule it out.

I'll ask you another question, but first we'll take a time out for a quick experiment. Take a straight piece of wire and fold it in half to form an acute angle. Walk over to the nearest corner in your room, and hold the wire so that the vertex touches the corner, and half of the wire lies on each of the walls forming the corner of your room. Look at the floor, and you'll see that the acute angle of the wire projects to a right angle. Unfold the wire a bit to form an obtuse angle, repeat the experiment, and you'll see that any angle can be projected to a right angle.

What is the meaning of the lengths of edges of a piece of property as described in a deed? Are these lengths the actual distances in 3-space, or do they refer to distances between projections of the corners of the property onto a horizontal plane? How would I know? Do I look like a surveyor? No I don't, but still the question presents itself: Is it possible that the property really is a right triangle that merely projects to a horizontal skinny triangle? Worrying about such things kept me up late the night before our day in court.

Unfortunately, they do not let Expert Witnesses into the courtroom before they testify, but since you are probably wondering what sort of special room they keep us in while we wait, I'll describe that instead. Maybe you think it looks like one of those First Class Medallion Level waiting lounges at airports you've never been in? Well it doesn't. It is a fairly ordinary looking room located right outside the courtroom. There were only two other Expert Witnesses present. One was dressed in an expensive looking suit, and he was huddled close to the other, who was dressed in an orange jumpsuit with numbers on it. After two hours of waiting in a silent room, any action seems like high drama, but it was dramatic when the courtroom doors burst open.

A whole roomful of people piled out with Barry in the lead yelling, "It's over!" Say what? Perhaps my reputation had preceded itself? Just the threat of a mathematician on the witness stand was enough for the opposing legal team to crumble? No, Barry explained, the judge threw the case out, saying that what was needed was a surveyor, not a mathematician.

BOOK X OF *THE ELEMENTS*: ORDERING IRRATIONALS

Jade Roskam¹, The University of Montana

Abstract: Book X from The Elements contains more than three times the number of propositions in any of the other Books of Euclid. With length as a factor, anyone attempting to understand Euclidean geometry may be hoping for a manageable subject matter, something comparable to Book VII's investigation of number theory. They are instead faced with a dizzying array of new terminology aimed at the understanding of irrational magnitudes without a numerical analogue to aid understanding. The true beauty of Book X is seen in its systematic examination and labeling of irrational lines. This paper investigates the early theory of irrationals, the methodical presentation and interaction of these magnitudes presented in The Elements, and the application of Euclidean theory today.

Keywords: Book X; Euclid; Euclid's Elements; Geometry; History of mathematics; rationals and irrationals; Irrational numbers

1. BACKGROUND

Book X of Euclid's *The Elements* is aimed at understanding rational and irrational lines using the ideas of commensurable and incommensurable lengths and squares. Unfortunately, a lack of documentation of the early study of incommensurables leads to speculation on its exact origin and discoverer. Wilbur R. Knorr in a 1998 article from The American Mathematical Monthly dates original knowledge, but not necessarily understanding, of irrational quantities to the Old Babylonian Dynasty Mesopotamians. The mathematical tables of these peoples, dating back to 1800-1500 BC, supposedly demonstrate knowledge of the fact that some values cannot be expressed as ratios of whole numbers. However, many sources disagree with Knorr's article and attribute original knowledge of irrational magnitudes to the school of Pythagoras around 430 BC (Fett, 2006; Greenburg, 2008; Robson, 2007; Posamentier, 2002). Given the most well-known accomplishment to come from the Pythagoreans, the Pythagorean theorem, it seems inevitable that this group of people would discover irrational values in the form of diagonals of right triangles. Take for example the length of the hypotenuse of an isosceles triangle with side lengths 1. This gives one of the most studied irrational quantities, $\sqrt{2}$. Prior to this inexorable discovery, the Pythagoreans viewed numbers as whole number ratios and therefore could not incorporate irrational quantities into their theory of numbers. Irrationals, considered to be an unfortunate discovery and the result of a cosmic error, were treated as mere magnitudes inexpressible in numerical form (Fett, 2006; Greenburg, 2008). These ideas were continued during the writing of The Elements, and would remain until the Islamic mathematician al-Karaji

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translated Euclidean terminology into irrational square roots of whole numbers approximately 13 centuries after Euclid wrote (Berggren, 2007).

The Pythagoreans attitude toward irrationals stunted any studying of the magnitudes beyond the incommensurability of a square's side to its diagonal. Fortunately, the superstition surrounding irrationals did not reach Plato's camp. Theodorus, a student of Plato, and one of Theodorus' own students, Theaetetus, took it upon themselves to study irrational magnitudes at length and put forth the first known theory of irrational lines (Knorr, 1975). Theodorus is cited as the first to produce varying classes of incommensurable lines through arithmetic methods argues Knorr (1975). However, Theodorus' discoveries were limited to specific cases, like lines cut in extreme-and-mean ratio, and he was unable to generalize his findings. It was his student, Theaetetus, who is generally considered as the first to put forth an organized, rigorous theory of irrationals, a work that started intuitively with his master but one that Theodorus ultimately could not prove (Knorr, 1975; 1983). The assembled findings of Theodorus and Theaetetus were published by Plato in a dialogue titled after the younger mathematician. Unfortunately much of Theaetetus has been lost over time and the little that is known about Theaetetus' early theory of irrationals comes from Eudemus, a student of Aristotle. Eudemus lived between the times of Plato and Euclid and is credited as having passed the early theory of irrational lines to Euclid's generation to be examined in full force in Book X of The Elements (Knorr, 1975; Euclid, 2006). If it was not for Plato's Theaetetus and the accounts from Eudemus, we may very well have attributed the entirety of the ideas of commensurable and incommensurable magnitudes to Euclid (Knorr, 1983).

Theaetetus is the one credited with having classified square roots as those commensurable in length versus those incommensurable (Knorr, 1983; Euclid, 2006). The three main classes of irrational magnitudes are the medial, binomial, and apotome. The medial line is defined as the side of a square whose area is equal to that of an irrational rectangle. The binomial and apotome oppose one another, as the binomial is formed by the addition of two lines commensurable in square only and the apotome is defined as the difference between two lines commensurable in square only. Each class of magnitude will be discussed in more detail later. Theaetetus is also said to have tied each class of magnitude with a unique mean: the medial is tied to the geometric mean, the binomial to the arithmetic, and the apotome to the harmonic mean (Euclid, 2006). However, these terms may just have been a replacement by Eudemus to tie irrational lines to Euclidean means, rather than the original correlations Theaetetus may have used (Knorr, 1983). The history behind the advancement of irrationality theory cannot exclude Euclid from its discussion. It was Euclid who generalized the idea of commensurable and incommensurable to squares, and also ordered the binomial and apotome irrational lines into six distinct classes each (Knorr, 1983). Most of the post-Euclidean advancement of the theory of irrational lines is found in propositions 111-114 of Book X which are generally considered to have been additions due to the lack of contiguity between these and the previous properties of irrationals presented. It is important to note that Book X details a theory of irrational *magnitudes* and not a theory of irrational *numbers* (Grattan-Guinness, 1996). Theaetetus' original theory of irrationals may have included numbers, but Euclidean theory deals solely with irrational lines and geometric lengths. The six classes of binomial and apotome are now more easily understood using algebra as the ordering of irrational magnitudes is explained through solutions of a general quadratic formula. The basis of this development is somewhat controversial. Knorr (1975) attributes some of the

“geometric algebra” to Theodorus. Most sources believe this understanding of geometry through algebra originated in the 8th Century through the vast advances made by many Islamic mathematicians in the area of algebra (Gratten-Guinness, 1996; Berggren, 2007). Some now argue that much, if not most, of The Elements is actually algebra disguised as geometry (Gratten-Guinness, 1996). However, as will be discussed later, this idea is a hindrance to understanding Euclidean theory. While using the solutions to a general quadratic is a good way to help understand how each order of binomial and apotome is derived, it inherently ignores all irrationals that are not in the form of a square root and treats irrationals as values rather than magnitudes (Burnyeat, 1978).

2. EUCLID ON IRRATIONALS

At the start of Book X Euclid (2006) provides definitions for commensurability and rationality. For commensurability Euclid states that “magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure” and that “Straight lines are commensurable in square when the squares on them are measured by the same area, and incommensurable in square when the squares on them cannot possibly have any area as a common measure” (p. 693). Euclid (2006) then moves to rationality which he defines as:

Let then the assigned straight line be called rational, and those straight lines which are commensurable with it, whether in length and in square, or in square only, rational, but those that are incommensurable with it irrational....And then let the square on the assigned straight line be called rational, and those areas which are commensurable with it rational, but those which are incommensurable with it irrational, and the straight lines which produce them irrational (p. 693). To simplify, given a rational length (or number), all lengths (numbers) that have common measure with the rational and/or with the square of the rational are also rational. Those lengths that do not have a common measure with the given line are irrational. Squaring a rational length produces a rational area, and those areas that are commensurable with the rational area are rational and those incommensurable with the rational area are irrational. If an area is irrational, the length that was squared to create the irrational area is also irrational.

In total, there are 13 distinct irrational straight lines. In addition to the medial, Euclid sets up six orders of binomials and six orders of apotomes. The Elements also defines a subgroup of irrational lines that can be constructed from the thirteen distinct irrationals which include first and second order bimedial lines, first and second order apotome of a medial line, major, and minor, the first four of which will be discussed briefly.

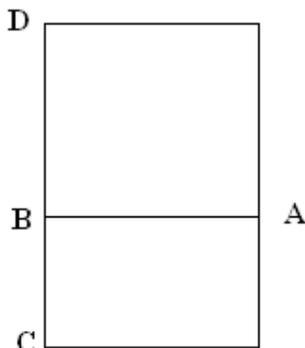
According to Euclid (2006), a medial is formed when a rectangle contained by two rational straight lines commensurable in square only is irrational and the side of the square equal to it is irrational. The side of the square is called the medial (X. 21)¹.

Book X. Proposition 21

In the diagram below, lines AB , BC are assumed to be rational lengths that are commensurable in square only. That

is to say, the square on AB and the square on BC have the ratio of a whole number to a whole number, but lengths AB, BC do not have a common measure. Now construct the square AD such that $AB = BD$.

¹Note that for ease, I will denote propositions from The Elements by (Book. Proposition Number). For instance, Proposition 47 from Book I will be cited as (I. 47).



Then the square AD is rational since $AD = AB^2$ and AB is rational. We know that AB and BC are incommensurable in length, which implies that BD, BC are also incommensurable in length. Note that

$$\frac{BD}{BC} = \frac{BD * AB}{BC * AB} = \frac{AD}{AC}$$

Since BD, BC are incommensurable, this implies that AD, AC are also incommensurable. But we know that AD is rational, so AC must be irrational. Since AC is an irrational area, a square with equal area will also be irrational and, by definition, will have a side of irrational length. This irrational side length is known as a medial.

Binomials on the other hand are formed when two rational straight lines commensurable in square only are added together, making the whole irrational. The following is adapted from The Elements (X. 36):

Let lines x, y be rational and commensurable in square only, meaning that nothing measures both x and y , but x^2 and y^2 have a common measure. It is proposed that their sum, $x + y$, will be irrational and, as per The Elements, called a binomial.

(i) Since x is commensurable in square only with y , x and y are incommensurable in length. Therefore, since

$$\frac{x}{y} = \frac{x^2}{x * y}$$

(ii) It follows that x^2 and $x*y$ are also incommensurable. But since x^2 and y^2 have a common measure, a , then

$$\begin{aligned} n * a &= x^2 \\ m * a &= y^2 \end{aligned}$$

where m, n are integers.

By substitution,

$$x^2 + y^2 = n * a + m * a = a * (n + m)$$

(iii) So a measures x^2 and a measures $(x^2 + y^2)$, which implies that x^2 and $x^2 + y^2$ are commensurable.

(iv) It is obvious that $x*y$ is commensurable with $2*(x*y)$.

(v) Since (iii) x^2 and $x^2 + y^2$ are commensurable, (iv) $x*y$ and $2*(x*y)$ are commensurable, but (ii) x^2 and $x*y$ are incommensurable, it follows that $x^2 + y^2$ and $2*(x*y)$ are incommensurable. From this, we see that

$(x^2 + y^2 + 2*(x*y))$ must be incommensurable with $(x^2 + y^2)$. Rearranging the first term, $(x + y)^2$ and $(x^2 + y^2)$ are incommensurable. Since x, y are rational, then x^2, y^2 are also rational and it follows that $x^2 + y^2$ is rational. This implies that $(x + y)^2$ is irrational, and therefore $(x + y)$ is irrational.

Euclid defines an apotome in proposition 73 of Book X as the remainder of two rational straight lines, the less subtracted from the greater, which are commensurable in square only. It is, in essence, the counterpart of the binomial. Euclid's proof that the apotome is irrational follows the same logical steps as those used to prove the irrationality of the binomial. We start with the same basic assumption, that lines x, y are rational and commensurable in square only. It is proposed that the apotome, $x - y$, is irrational. Steps (i) through (v) are identical to the proof of Proposition 36. For the apotome, note that

$$x^2 + y^2 = 2 * x * y + (x - y)^2$$

Since (v) $x^2 + y^2$ and $2*(x*y)$ are incommensurable, it follows that $x^2 + y^2$ and $(x - y)^2$ are also incommensurable.

But since x, y are rational by construction, $x^2 + y^2$ must be rational. This implies that $(x - y)^2$ is irrational, from which it follows that $x - y$ is irrational. Thus we have proven that if a rational straight line is subtracted from a rational straight line, and the two are commensurable in square only, the remainder will be irrational.

It was stated earlier that Theaetetus tied the three known types of irrationals at the time to unique means: the medial with the geometric, the binomial with the arithmetic, and the apotome with the harmonic. The first two of these pairings follow somewhat simply. The medial is tied to the geometric mean, which can be found using the following general formula.

$$G(x_1, x_2, \dots, x_n) = \sqrt[n]{(x_1 * x_2 * \dots * x_n)}$$

A medial is defined as the length of the side of a square whose area is equal to that of an irrational rectangle formed by two rational lines commensurable in square only. Using our above diagram, the square on the medial, we will call it MN for simplicity, is equal to the area of rectangle AC . Algebraically,

$$MN^2 = AC = AB * BC$$

$$MN = \sqrt{AB * BC}$$

Medials can therefore be represented symbolically as the geometric mean

$$\sqrt{x * y}$$

for two given rational magnitudes x and y commensurable in square only.

As stated earlier, the binomial is defined as the sum of two rational straight lines commensurable in square only and is closely related to the arithmetic mean, of which the following is the general formula.

$$A(x_1, x_2, \dots, x_n) = \frac{1}{n} * (x_1 + x_2 + \dots + x_n)$$

It is obvious how the representation of the binomial $(x + y)$ is closely linked to this formula. However, the coupling of the apotome and the harmonic mean is more complex. To explain, the harmonic mean of two numbers, x and y , is

$$\frac{2 * x * y}{x + y}$$

If you consider the propositions X.112-4, you can see that if a rational area has a binomial for one of its sides, the other side will be an apotome commensurable with the binomial and of corresponding order. Using our knowledge of the general form of an apotome and a binomial, we can see that this area would be

$$(x + y) * (x - y) = x^2 - y^2$$

With $x*y$ representing a medial area and the above equation for the given rational area, we see that

$$\frac{2 * x * y}{x^2 - y^2} * (x - y)$$

with $(x - y)$ representing the basic form of an apotome. Again, this seems like a stretch given the ease with which the binomial and medial are tied to their respective means. It should be noted that this relationship between the apotome and the harmonic mean is explained in the commentary by Woepcke in an Arabic translation of Book X of The Elements (Euclid, 2006). Whether this was Theaetetus' original reasoning for pairing the apotome and harmonic mean is unknown. Again, these algebraic explanations are not the original work of Euclid, but theories imposed upon his work by later mathematicians. This is important to note because Euclidean theory pertained solely to irrational magnitudes and not to irrational numbers. Since most of Theaetetus' original theory is lost, it cannot be determined conclusively whether the Platonic mathematician described the above relationships. The ties between three types of Euclidean quantities and three Aristotelian quadrivium is seen elsewhere in The Elements. According to Ivor Gratten-Guinness' 1996 article, the three types of quantities Euclid addresses, number,

magnitude, and ratio, correlate to arithmetic, geometry and harmonics, respectively. These relationships certainly follow more readily than the irrational magnitudes to the corresponding means, and the latter associations may have been formed in response to the former.

3. ORDERING IRRATIONALS

A class is defined as a set of objects connected in the mind due to similar features and common properties (Forder, 1927). All magnitudes in the class of binomials are of the form $(x + y)$ where x, y are lines commensurable in square only. All binomials share common features, which will be discussed later. The same is true of apotomes. All are of the form $(x - y)$ where x, y are lines commensurable in square only, and they share common properties. These represent two of the three classes proposed in Theaetetus' early theory of irrationals. Within each of these classes, Euclid defines six orders, or sub-classes, of each. Each member of a sub-class contains all the properties common to the class as a whole, but has different properties from members of other sub-classes (Forder, 1927). Theaetetus is credited with ideas of the medial, binomial, and apotome, but he makes no reference to the six orders of binomials and apotomes listed in The Elements. Therefore it was up to Euclid's discretion on how to best order the sub-classes. The difficulty in Euclidean theory of irrationals lies in the overlap of properties between sub-classes. As will be discussed in detail, the six orders of each class are paired into three groups, with one from each pair representing commensurability and one from each pair representing incommensurability. The struggle arises from what is most important in the class, the commensurability or the pairing with another sub-class. Despite the algebraic understanding of Euclid's irrationals making the pairing of sub-classes easier to follow, Euclid chose to first break each class of irrational line into commensurable versus incommensurable, and then pair the members in each.

Euclid defines each of the six orders of binomial and apotome in Definitions II and III, respectively, of Book X and the introduction to Book X provides an algebraic understanding of how each type is derived. To clarify the definitions given by Euclid, we will represent a binomial using the general form $(x + y)$ with x being the greater segment and y being the lesser segment and an apotome using the general form $(x - y)$ with x being the whole and y being the annex (or what is subtracted from the whole).

Consider the general quadratic formula

$$x^2 \pm 2 * a * x * p \pm b * p = 0$$

where p is a rational straight line and a, b are coefficients. Only positive roots of this equation will be considered as x must be a straight line. Those roots include

$$x_1 = p * (a + \sqrt{a^2 - b})$$

$$x_1^* = p * (a - \sqrt{a^2 - b})$$

$$x_2 = p * (\sqrt{a^2 + b} + a)$$

$$x_2^* = p * (\sqrt{a^2 + b} - a)$$

First, consider the expressions for x_1 and x_1^* . Suppose a, b do not contain any surds. That is to say, they are either integers or of the form $\frac{m}{n}$, where m, n are integers. If this is the case, either

$$(i) b = \frac{m^2}{n^2} * a^2$$

Or

$$(ii) b \neq \frac{m^2}{n^2} * a^2$$

If (i), then x_1 is a first binomial and x_1^* is a first apotome. Euclid defines the first order in Definitions II, for the binomial, and III, for the apotome, in Book X as:

Given a straight line and a binomial/apotome...the square on the greater term/whole [x] is greater than the square on the lesser/annex [y] by the square on a straight line *commensurable* (emphasis added) in length with the greater/whole...the greater term/whole commensurable in length with the rational straight line set out then the entire segment is known as a first binomial/first apotome (p. 784, 860). This wordy definition is translated by Charles Hutton in his 1795 two volume edition of *A Mathematical and Philosophical Dictionary* to more comprehensible terminology: the larger term, x , is commensurable with a rational and is thus a rational itself and $x^2 - y^2 = z^2$, where z is commensurable in length with x , so z must also be rational. Using this new definition and numerical examples provided by Islamic mathematician al-Karaji in the 10th Century, we can understand better what Euclid was representing geometrically (Berggren, 2007). For instance, $3+\sqrt{5}$ would be considered a first binomial and $3-\sqrt{5}$ would be considered a first apotome. The greater term (3) is rational and

$$3^2 - (\sqrt{5})^2 = 9 - 5 = 4 = 2^2$$

If (ii), then x_1 is a fourth binomial and x_1^* is a fourth apotome.

The fourth order of binomial and apotome are defined as opposing the first order. Euclid's definition for the fourth order of each class of irrational states:

If the square on the greater term/whole [x] be greater than the square on the lesser/annex [y] by the square on the straight line *incommensurable* (emphasis added) in length with the greater/whole, then if the greater term/whole be commensurable in length with the rational straight line set out then the entire segment is called a fourth binomial/fourth apotome (p. 784, 860). Much like the first binomial and first apotome, the greater term, x , will be rational. However, unlike the first order, the square root of the difference of the squares of the two terms, z , will be incommensurable with x , meaning that z will not have a rational ratio to x . Take, for example $4-\sqrt{3}$. The greater term (4) is a rational number, and

$$\frac{\sqrt{4^2 - (\sqrt{3})^2}}{4} = \frac{\sqrt{16 - 3}}{4} = \frac{\sqrt{13}}{4}$$

which is not a rational ratio.

Now look at the possibilities for the x_2, x_2^* expressions. If we stick to our supposition that a, b do not contain surds, then either

$$(i) b = \frac{m^2}{n^2 - m^2} * a^2$$

Or

$$(ii) b \neq \frac{m^2}{n^2 - m^2} * a^2$$

If (i), then x_2 is a second binomial and x_2^* is a second apotome.

Like the first order, the second order of binomials/apotomes has the square root of the difference of squares of the two terms (z) commensurable with the greater term/whole (x). The difference between the first and second order is that the lesser term/annex (y) is the segment that is commensurable with the rational straight line set out. This indicates that the lesser term, y , is rational, and that the ratio of the square root of the difference of the squares of the two terms, z , and the greater term, x , is a rational ratio expressible in whole numbers. Again, a wordy definition easily explained with actual values, like $\sqrt{18} \pm 4$. The lesser term (4) is a rational number and

$$\frac{\sqrt{(\sqrt{18})^2 - 4^2}}{\sqrt{18}} = \frac{\sqrt{18 - 16}}{\sqrt{18}} = \frac{\sqrt{2}}{\sqrt{18}} = \frac{\sqrt{1}}{\sqrt{9}} = \frac{1}{3}$$

which is a rational ratio expressed in whole numbers.

If (ii), then x_2 is a fifth binomial and x_2^* is a fifth apotome.

The fifth order of binomial and apotome is a combination of the second and third order definitions. Like the second order, the lesser of the two terms (y) is commensurable with the rational straight line set out. However, the square root of the difference between the two terms (z) is incommensurable in length with the greater term/whole (x). This means that again the lesser term is rational and that the ratio of the square root of the difference of the squares of two terms, z , and the greater term, x , is not a rational ratio. For instance, $\sqrt{6} \pm 2$. 2 is a rational number and

$$\frac{\sqrt{(\sqrt{6})^2 - 2^2}}{\sqrt{6}} = \frac{\sqrt{6 - 4}}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{\sqrt{1}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

which is not a rational ratio.

To obtain the final two orders of binomial and apotome, we must consider the case where

$$a = \sqrt{\frac{m}{n}}$$

where m, n are integers. To abbreviate, let $\lambda = \frac{m}{n}$. Therefore

$$\begin{aligned}
x_1 &= p * (\sqrt{\lambda} + \sqrt{\lambda - b}) \\
x_1^* &= p * (\sqrt{\lambda} - \sqrt{\lambda - b}) \\
x_2 &= p * (\sqrt{\lambda + b} + \sqrt{\lambda}) \\
x_2^* &= p * (\sqrt{\lambda + b} - \sqrt{\lambda})
\end{aligned}$$

“If $\sqrt{\lambda - b}$ in x_1, x_1^* is not surd but of the form $(\frac{m}{n})$, and if $\sqrt{\lambda + b}$ in x_2, x_2^* is not surd but of the form $(\frac{m}{n})$, the roots are comprised among the forms already shown” (X. Introduction). To explain, in our original equations for $x_1, x_1^*, x_2,$ and x_2^* , a was assumed to be rational (containing no surds) and $\sqrt{a^2 \pm b}$ would then be irrational. In our new equations, we define a as being irrational. The above quote states that if $\sqrt{\lambda \pm b}$ is rational (containing no surds) then we will again obtain the 1st, 2nd, 4th, and 5th order binomials or apotomes. The original x_1, x_1^* and the new x_2, x_2^* are taking a rational magnitude plus (binomial) or minus (apotome) an irrational to obtain the 1st and 4th orders. The original x_2, x_2^* and newly formed x_1, x_1^* start with an irrational magnitude and add (binomial) or subtract (apotome) a rational magnitude, forming the 2nd and 5th orders of binomial and apotome. Therefore, the only case that needs to be investigated is the case where an irrational magnitude is added or subtracted from another irrational magnitude.

If $\sqrt{\lambda - b}$ in x_1, x_1^* is surd, then either

$$(i) b = \frac{m^2}{n^2} * \lambda$$

Or

$$(ii) b \neq \frac{m^2}{n^2} * \lambda$$

If (i), then x_1 is a third binomial and x_1^* is a third apotome.

In the case of the third order of each type of irrational, we again have a connection to the language describing the first order. The square on the greater term/whole (x) is greater than the square on the lesser term/annex (y) by the square on a straight line commensurable with the greater/whole. However, in this order neither of the terms, x or y , are commensurable with the rational straight line set out. In terms of real numbers, both x and y must be irrational and ratio of the square root of the difference of the squares of the terms, z , and the greater term, x , is a rational ratio expressible in whole numbers. To explain by example, look at the third order binomial ($\sqrt{24+\sqrt{18}}$) or the third order apotome ($\sqrt{24-\sqrt{18}}$). Both terms are irrational and

$$\frac{\sqrt{(\sqrt{24})^2 - (\sqrt{18})^2}}{\sqrt{24}} = \frac{\sqrt{24 - 18}}{\sqrt{24}} = \frac{\sqrt{6}}{\sqrt{24}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$$

with $\frac{1}{2}$ being a rational ratio expressible in whole numbers.

If (ii), then x_1 is a sixth binomial and x_1^* is a sixth apotome.

Much like the third, in a sixth order of binomial and apotome neither the lesser term (y) nor the greater term (x) are commensurable in length with the rational straight line set out, but the square on the greater/whole is greater than the square on the lesser/annex by the square on a straight line incommensurable in length with the greater term/whole. This translates to the sixth apotome having the form of two irrational terms with the ratio of the square root of the difference of the squares of the two terms, z , to the greater term, x , being an irrational ratio. Look at the sixth order binomial/apotome $\sqrt{6\pm\sqrt{2}}$. Both terms are irrational and

$$\frac{\sqrt{(\sqrt{6})^2 - (\sqrt{2})^2}}{\sqrt{6}} = \frac{\sqrt{6-2}}{\sqrt{6}} = \frac{\sqrt{4}}{\sqrt{6}} = \frac{2}{\sqrt{6}}$$

which is not a rational ratio.

The attached table summarizes the six orders of binomial and apotome.

Euclid also designates two orders of bimedral lines and two orders of apotome of medial straight lines. Bimedral lines are the sum of two medial lines which are commensurable in square only. Proposition 37 demonstrates how to construct a first bimedral line (two medial lines commensurable in square only and containing a rational rectangle can be added together). Constructing a second bimedral line is discussed in proposition 38, where by all the same conditions as the first bimedral apply, except that the two medial lines form a medial rectangle instead of a rational rectangle. An apotome of a medial is defined as the difference between two medial lines, the lesser of which is commensurable with the whole in square only. If a rational rectangle is contained with the square of the whole, then the remainder is a first apotome of a medial straight line (X. 74). If a medial rectangle is contained with the square of the whole, the remainder is known as a second apotome of a medial straight line (X. 75). An obvious connection can be drawn between bimedral lines and apotome of medial lines. All four types are constructed by manipulating two medial lines, with the first orders of each referring to a contained rational rectangle and the second orders of each having a medial rectangle contained by the two medial lines. The name apotome of a medial is fitting in an obvious way: the line is formed by the difference of two medials ($x - y$). What is confusing is the naming of the bimedral. With the connection between the bimedral and apotome of a medial mentioned above and the definition of the bimedral as the sum of two medial lines ($x + y$), it is interesting that Euclid did not use the more obvious title of binomial of a medial line. It is possible that the original terminology was binomial of a medial line and through translation was shortened to bimedral, but this is mere speculation.

4. PROPERTIES AND INTERACTIONS

One of the most fascinating things about the three main types of irrational lines is studying the ways that they interact with each other. One example of this is the algebraic representation of binomials and apotomes. Binomials can be understood as a process of addition represented by ($x + y$). The opposite is true of the apotome which is represented as ($x - y$). These in turn have a product of ($x^2 - y^2$). It is obvious that there are numerous relations that these lines hold with each

other, and yet for all of their similarities, each of the categories of irrational lines are mutually exclusive. Euclid goes so far as to say, “The apotome and the irrational straight lines following it are neither the same with the medial straight line nor with one another” (X. 111). However, these lines are not just mutually exclusive categories, but are also unique in their division into parts. Proposition 42 demonstrates that if AB is a binomial, then there is only one point C between A and B such that AC, BC are rational and commensurable in square only. This proves that for a given binomial, there is only one way to separate its length into greater and lesser segments. The same is proven of a first bimedial (X. 43) and a second bimedial (X. 44). Likewise, from a given apotome, only one length can be subtracted such that both segments are rational and commensurable in square only (X. 79). Again, Euclid goes on to prove in propositions 80 and 81 the uniqueness of first and second apotome of a medial lines.

We must first look at the major properties of medials, binomials, and apotomes before we can delve into the interactions between these lines.

Common to all of the types of irrational lines is that fact that lines commensurable with the given length are of the same type and order where applicable (X. 23(medial), 66(binomial), 67(bimedial), 103(apotome), 104(apotome of a medial)). Unique to medials are the ideas that rectangles contained by medial lines commensurable in length is medial (X. 24), that rectangles contained by medial lines commensurable in square only are either rational or medial areas (X. 25), and that the difference between two medial areas will never be a rational area (X. 26). In maybe the most important proposition of book X, Euclid proves that an infinite number of unique irrational lines arise from a medial line (115). Interestingly, he chose to make this the last proposition in the book, possibly with the hopes that future students would use this property of medials to further investigate the theory of irrationals, possibly coming up with new unknown forms of irrational lines or a new classification system. There are a few propositions that deal with only binomials or apotomes, but these are usually taken in sets with the ensuing propositions using a bimedial or apotome of a medial, and thus will be discussed later. However, propositions 48-53 do deal strictly with binomials, in that each describes how to find a binomial of particular order. Propositions 85-90 perform the same action for orders of apotome.

Why Euclid chose to classify apotomes and binomials such that the first and fourth, second and fifth, and third and sixth orders were paired is not explained. We can note that the first three orders deal with commensurable lengths between the differences in squares explained above and the greater segments while the last three orders have greater terms being incommensurable with the difference in squares. We can also note each of the pairings are based on which term (greater, lesser, or neither) are rational. From this, it is plausible to assume Euclid’s ordering is first based on the commensurability of given aspects of the line, and second based on which part of the given line is rational, leading to the two fold classification system seen today. Whether this was Euclid’s reasoning or not, it does appear that he was not particularly concerned with functional order throughout the elements. For example, the first time a reader is introduced to a line cut in extreme-and-mean ratio is in the beginning of Book II. Yet it is not until Book X that the properties of such a line (with greater length is an apotome and lesser length a first apotome) are explained and not until Book XIII that this type of line is applied, which will be discussed in more detail later. It is also interesting to note Euclid devotes books VII, VIII, and IX to investigating numbers and number theory, but certain properties of numbers appear in many of

the other ten books, including addressing ratios of numbers in Book V well before putting forth a theory of numbers (Grattan-Guinness, 1996).

Despite what may or may not be a flawed ordering system, the vast majority of Book X is devoted to exploring the interactions between the classes of irrational lines. It should be noted that for each property of binomial lines, the same property is proven just thirty-seven short propositions later for apotome lines. Starting with propositions 54 and 91, Euclid proves that if a rectangle is formed by a rational line and a first order binomial or apotome, the “side” or diagonal of that rectangle will be a binomial or apotome. As I mentioned before, the propositions describing the interactions of irrational lines often come in sets. Just as 54 and 91 prove the above statements, propositions 55, 56 and 92, 93 prove a similar situation occurs with bimedials and apotome of medial lines. An area formed by a rational and a second order binomial has for its side a first order bimedral (X. 55). For a rational and a third order binomial, the second bimedral is the diagonal (X. 56). Switching “apotome” for binomial and “apotome of a medial” for bimedral, we have the statements of propositions 92 and 93. We learn that if a rectangle is formed with rational length and area equal to a binomial squared, the width of the rectangle will be a first order binomial in proposition 60. The likewise is true of apotomes (X. 97). Propositions 61-62 and 98-99 are devoted to proving a similar statement: that if a straight line AB is a first bimedral or apotome of a medial (or second order for proposition 99), and a rational straight line CD is the side of rectangle CE such that the area of CE is equal to the square on AB , then the other side of rectangle CE , side CF , is a second binomial or apotome (third order for 99). Finally, as stated previously, propositions 112-113 prove that if a rational area has a binomial for one of its sides, the other side will be an apotome commensurable with the binomial and of corresponding order, with 114 proving that if a binomial and apotome that are commensurable and of the same order form the length and width of a rectangle, the diagonal will be rational.

5. MODERN IMPLICATIONS

Euclid’s dialogue on irrational lines is not restricted to Book X. Indeed he puts forth an important application of the apotome in Book XIII. Proposition 6 states that if a line is cut in extreme-and-mean ratio (first introduced in II. 11), then the greater segment will be an apotome and the lesser segment a first apotome. This one proposition has enormous implications for the theory of irrational magnitudes. The golden ratio, one of the most applicable and well-studied areas of math, is created by a line cut in extreme-and mean ratio. This is an important topic to understand due to the vast number of properties held by objects that contain this ratio. One example is the logarithmic spiral which is formed through the construction of both golden rectangles, whose sides, when taken in proportion, equal the golden ratio, and the golden triangle, whose angles are 72° , 72° , 36° . Logarithmic spirals are seen throughout nature. Ram horns, elephant tusks, nautilus shells, pine cones, sun flowers and many other living things grow in accordance with the golden ratio (Fett, 2006). This proportion is said to be the most aesthetically pleasing to look at, which is why many great paintings and sculptures contain the golden ratio. The Parthenon in Athens, which not only houses sculptures containing the golden ratio but in fact can be inscribed in a golden rectangle, and five of Leonardo da Vinci’s works,

including two of his most famous “Madonna on the Rocks” and “Mona Lisa”, are also said to contain the golden ratio (Fett, 2006). Many plastic surgeons still use the golden ratio several times over to construct what is believed to be a universal standard of beauty (Fett, 2006).

Each of the five Platonic solids, the only existing solids to have identical and equilateral faces and convex vertices, incorporates the golden ratio in its construction. The tetrahedron, octahedron and icosahedron are based on equilateral triangles, while the cube and dodecahedron are based on the square and pentagon. These shapes are discussed in detail in Book XIII of The Elements after the introduction of the line cut in extreme-and-mean ratio. Of particular interest are the dodecahedron and the icosahedron. Exodus of Cnidus, who lived after Theaetetus, is credited with having first discovered the irrationality of a line divided in extreme and mean ratio after working with the problem of inscribing a regular pentagon in a given circle (Knorr, 1983). The pentagon is actually formed by three golden triangles, and the ratio of the shorter side to the longer is equal to the golden ratio. This implies the construction of the icosahedron is dependent upon the golden ratio. The golden ratio is also present in the calculation of surface area and volume of the dodecahedron as well as the volume of the icosahedron (Fett, 2006). Since Theaetetus is credited with first discovering the icosahedron, Book XIII, along with Book X, is firmly based in the Athenian’s work. In fact, M. F. Burnyeat quotes B. L. Van der Walden in his 1978 journal article as saying “The author of Book XIII knew the results of Book X, but...moreover, the theory of Book X was developed with a view to its applications in Book XIII. This makes inevitable the conclusion that the two books are due to the same author...Theaetetus”.

The golden ratio is also seen in the comparison of sequential values in the also well-studied Fibonacci sequence. The Fibonacci numbers are defined by the recursive formula

$$f_{n+1} = f_n + f_{n-1}$$

for $f_n > 2$ with $f_1 = 1$ and $f_2 = 1$ (Fett, 2006; Posamentier, 2002; Rosen, 2005). When comparing the n^{th} Fibonacci number with the $(n-1)^{\text{th}}$, the ratio will approach the golden ratio as n increases. Not surprisingly given its relationship with the golden ratio, the Fibonacci sequence is often found in the growth of natural objects. For example, the number of spirals in plants that grow in a phyllotaxis pattern will always be a Fibonacci number (Rosen, 2005). Another sequence related to the both the Fibonacci numbers and the golden ratio is the Lucas numbers. These are defined using the same recursive formula as the Fibonacci numbers and still begins the sequence with 1, but $\ell_2 = 3$ instead of 1 (Posamentier, 2002; Rosen, 2005). Interestingly enough, the same relationship exists between Lucas numbers and the golden ratio as the Fibonacci numbers and the golden ratio (Posamentier, 2002).

6. CONCLUSIONS

It is unfortunate that so little is known about the early theory of irrationals or who advanced our understanding to what it is today. What is also regrettable is the lack of progress we have made since the days of Euclid. On the positive, we can be thankful for the meticulous systematic presentation of irrational magnitudes and their properties and interactions demonstrated in Book

X of The Elements. While this chapter of Euclidean geometry has not been developed to the degree that most of his work has, the multitude of Book X leaves us with plenty of information on irrationals.

We know that there are 13 types of irrational lines, with each category being mutually exclusive. We also know that for every irrational line, there is only one way to divide the line to meet the criteria of its category and order, proving the uniqueness of each. Maybe most importantly, we know from proposition 115 that there are infinitely many irrational lines. Euclid provides us with the application of this theory in Book XIII, showing how the pentagon and icosahedron utilize irrational magnitudes in the construction of each. This application of the extreme-and-mean ratio has led to significant discoveries in the area of aesthetics, art, and music through the apotome known as the golden ratio.

Perhaps this information was enough to satisfy mathematicians throughout history. Or possibly our Euclidean understanding of irrationals is complete. We highly doubt the latter, but believe so much time is spent simply trying to understand the already burdensome theory of irrational lines that little is left for the advancement of the theory. Many mathematicians have devoted time to aiding future students in understanding the classification of commensurable and incommensurable magnitudes. Hopefully this will eventually lead to a more readily comprehensible theory, a base step from which a more innovative, improved theory of irrational lines can be developed.

N. Sirotic and R. Zazkis (2005) conducted a research project to find out how much we retain of the Euclidean theory of irrational lengths. They asked a group of college students studying to be secondary teachers if it was possible to locate $\sqrt{5}$ on a number line. The results were somewhat frightening. Less than 20% of the participants used a geometric construction to find $\sqrt{5}$ on the number line, most of those having used the Pythagorean Theorem, approximately 65% used some sort of decimal approximation in varying degrees of exactness, and an abysmal 15% either did not answer, or worse, argued it was not possible to find exactly where $\sqrt{5}$ falls on a number line. Most of those who argued it was impossible reasoned that since $\sqrt{5}$ was irrational, the decimal approximation is infinite and non-repeating and that was why it cannot be accurately positioned. However, these same participants believed a repeating infinite decimal, like $\frac{1}{3}$, could be placed in its exact position, but could not explain why whether the decimal repeated or not made a difference. This means that 80% of those future secondary teachers could not think past our understanding of decimal approximations to use a well-known and highly practiced idea (the Pythagorean Theorem) to find where $\sqrt{5}$ falls on a number line. To these people, it seems the number line is really a *rational* number line and irrational numbers cannot be placed exactly since “because [the decimal] never ends we can never know the exact value” (Sirotic, 2005). This is an unfortunate side effect of Theodorus’ and al’Karaji’s work to aid students in understanding irrationals. The original understanding of irrational lines using geometry is lost to the more easily comprehensible geometric algebra presented in almost all current editions of The Elements. While the use of algebra is integral to helping students understand this dense topic, a return to irrational magnitudes’ geometric roots appears to be just as important for students to gain a true understanding of Book X.

ACKNOWLEDGMENTS

Special thanks to Steve Williams for initial assistance in researching this vast subject and to Dr. Bharath Sriraman of the University of Montana for his valuable input on various drafts and for challenging me with such a complex topic.

REFERENCES

- Burnyeat, M. F. (1978) The Philosophical Sense of Theaetetus' Mathematics. ISIS: The Official Journal of the History of Science Society. 69(4), 489-513
- Euclid (2006) The Elements. (Sir T. L. Heath, Trans.) New York: Barnes & Noble Publishing, Inc. (Original work written around 300 BC).
- Fett, B. (2006) An In-Depth Investigation of the Divine Ratio. The Montana Mathematics Enthusiast. 3(2), 157-175.
- Forder, H. G. (1927) The Foundations of Euclidean Geometry. London: Cambridge University Press.
- Grattan-Guinness, I. (1996) Numbers, Magnitudes, Ratios, and Proportions in Euclid's *Elements*: How Did He Handle Them? Historia Mathematica. 23, 355-375.
- Greenburg, M. J. (2008) Euclidean and Non-Euclidean Geometries (4th ed.). New York: W. H. Freeman and Company.
- Hutton, C. (1795) A Mathematical and Philosophical Dictionary. London
- Imhausen, A., Robson, E., Dauben, J., Plofker, K., and Berggren, J. L. (2007) The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook. (V. Katz, eds.) Princeton: Princeton University Press.
- Knorr, W. R. (1975) The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry. Dordrecht, Holland: D. Reidal Publishing Company.
- Knorr, W. (1983) Euclidean Theory of Irrational Lines. Bulletin of the American Mathematical Society. 9(1), 41-69.
- Knorr, W. (1998) 'Rational Diameters' and the Discovery of Incommensurability. The American Mathematical Monthly. 105(5), 421-429
- Kyburg, H. Jr. (1997) Quantities, Magnitudes, and Numbers. Philosophy of Science. 64, 377-410.
- Posamentier, A. S. (2002) Advanced Euclidean Geometry. New York: Key College Publishing.

Rosen, K. H. (2005) Elementary Number Theory and Its Applications (5th ed.). Boston: Pearson Addison Wesley.

Sirotic, N. and Zazkis, R. (2007). Irrational numbers on the number line – where are they? International Journal of Mathematical Education in Science and Technology. 38(4), 477-488.

	Definition of Binomial (Apotome)		Algebraic Interpretation of the Roots of General Quadratic Formula
Order	Square on the greater segment (whole) greater than the square on the lesser segment (annex) by the square on a straight line _____ in length with the greater term (whole)	_____ be commensurable in length with the rational straight line set out Rational Term	$x_1 = p \cdot (a + \sqrt{a^2 - b})$ $x_1^* = p \cdot (a - \sqrt{a^2 - b})$ $x_2 = p \cdot (\sqrt{a^2 + b} + a)$ $x_2^* = p \cdot (\sqrt{a^2 + b} - a)$
First	commensurable	greater segment (whole) greater term is rational	If a, b do not contain surds and $b = \frac{m^2}{n^2} \cdot a^2$ x_1 is a binomial and x_1^* is an apotome
Second	commensurable	lesser term (annex) lesser term is rational	If a, b do not contain surds and $b = \frac{m^2}{n^2 - m^2} \cdot a^2$ x_2 is a binomial and x_2^* is an apotome
Third	commensurable	neither segment (neither) neither term is rational	If $\sqrt{(a^2 - b)}$ is surd, $a = \sqrt{\frac{m}{n}}$ $b = \frac{m^2}{n^2} \cdot \lambda$ x_1 is a binomial and x_1^* is an apotome
Fourth	incommensurable	greater segment (whole) greater term is rational	If a, b do not contain surds and $b \neq \frac{m^2}{n^2} \cdot a^2$ x_1 is a binomial and x_1^* is an apotome
Fifth	incommensurable	lesser term (annex) lesser term is rational	If a, b do not contain surds and $b \neq \frac{m^2}{n^2 - m^2} \cdot a^2$ x_2 is a binomial and x_2^* is an apotome
Sixth	incommensurable	neither segment (neither) neither term is rational	If $\sqrt{(a^2 - b)}$ is surd, $a = \sqrt{\frac{m}{n}}$ $b \neq \frac{m^2}{n^2} \cdot \lambda$ x_1 is a binomial and x_1^* is an apotome