# Basic Geometric Configurations and Teaching Euclidean Geometry


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Mathematicians have always recognized geometry as an important source of meaning in mathematics (Hilbert & Cohn-Vossen, 1952). Mathematics educators also regard geometry as "a natural area of mathematics for the development of students' reasoning and justification skill" (NCTM, 2000). The visual aspect of geometric problems is their distinctive and invaluable property since geometric questions appeal to solvers' natural spatial intuition. While solving them, students need to become aware of geometric properties and their relations, drawing conclusions not immediately evident in the diagram (Henderson & Taimina, 2005), or learning from resolving apparent contradictions (Kondratieva, 2009).

Despite the fact that geometry offers a rich ground for developing mathematical intuition, students often lack the ability to apply their knowledge in a problem-solving situation. Among others, the following reasons were identified. The first relates to the very formal approach often adopted in the teaching of geometry, including the memorization of formal definitions and an emphasis on formal proofs. Such an approach often fails because "its deductivity could not be reinvented by the learner but only imposed" (Freudenthal, 1971:418). The second relates to a rigid, utilitarian approach, which includes learning a list of basic formulas for calculation of lengths, areas, and volumes, with emphasis on procedural knowledge. Such an approach is often unsuccessful since "geometry is not a collection of definitions and formulas, but the ability to see, imagine, and think" (Sharygin & Protasov, 2004:1).

This suggests that more emphasis needs to be placed on *meaningful* and *creative* use of geometric facts and ideas, as well as on making connections between them. "When students connect mathematical ideas, their understanding is deeper and more lasting, and they come to view mathematics as a coherent whole" (NCTM, 2000:64). In addition, students need to be exposed to more challenging problems which would allow them to see the power of geometric statements.

Many mathematics teachers find it difficult to ensure deep knowledge of geometry in students. From my informal conversations with teachers I learned that "*one of the main challenges in studying plane geometry for students is making the connections between various concepts*". In response to these teachers' concerns, I discuss a few geometric proofs as a way of illustrating interconnectivity of geometric knowledge, and their possible exposition in an instructional setting. I pay special attention to *basic geometric configurations* (BGCs) - fundamental geometric facts expressed in drawing. Some BGCs may be given names in order to help students recall their images by association. For instance, names such as "bow-tie" and "Star-Trek" are often used for images showing relations between central and inscribed angles. Basic geometric configurations are the stepping stones to proving or solving geometric problems, and examples given in this paper aim to illustrate this point. These examples also emphasize the following teaching actions which I found to be useful in working with various geometry learners:

- Asking students to explain what relations they observe in a figure and what they think about the role of the auxiliary lines drawn on the original figure.
- Constantly showing connections to already learned geometrical facts and focusing students' attention on BGCs and key ideas used in a particular solution.
- Demonstrating several proofs or solutions of the same problem in order to show connections between geometry, trigonometry and algebra.
- Directing students' attention to the implications, converse and equivalence of statements.
- Helping students summarize their findings in the form of a mathematical statement.

The paper concludes with heuristic principles of problem solving in geometry which are applied throughout this text.

The amount of time and the degree of teachers' assistance required to implement the examples shown in this paper in a particular classroom will significantly depend on the students' level of preparedness. Nevertheless, the BGCs approach is worth trying as it helps the learner to develop geometrical imagination while thinking in small connected steps, and moves away from both too utilitarian and too formal expositions of geometry.

**Basic geometric configurations: the unity of a statement, reason, and diagram**

Many problems in Euclidean geometry are challenging for students because their solutions do not follow an algorithmic process which can be learned step by step, but rather require an insight coming from recognition and combination of several basic geometric configurations (BGCs) such as facts related to isosceles triangles, similar triangles, or the right-angled triangle. In order to be able to recognise such basic facts it is important to know their images along with proofs and reasons why they are true. My first example of a BGC refers to the *isosceles triangle* defined as a triangle with two equal sides. Once students recognize an isosceles triangle with $AC = BC$ (Figure 1) they know that it has two equal angles $C\hat{A}D$ and $C\hat{B}D$. Equity of two angles in a triangle implies equity of corresponding sides as well, presenting equivalency of the two conditions.

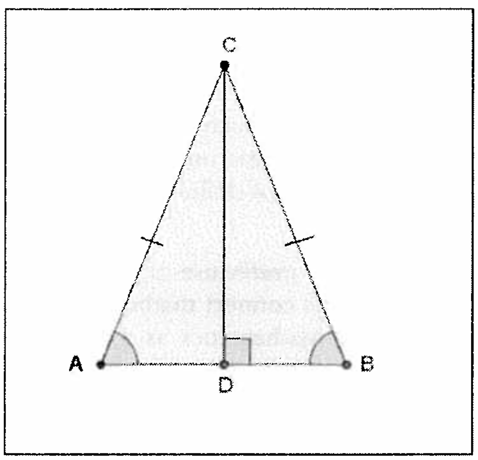


*Figure 1.* Basic geometric configuration "isosceles triangle".

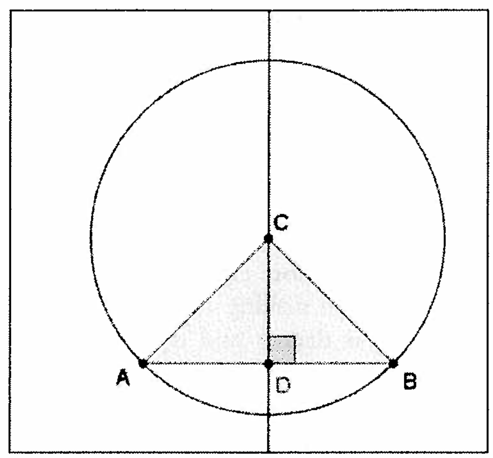


*Figure 2.* Basic geometric configuration "radius bisecting a chord".

This property reflects the axial symmetry of the figure with respect to line *CD*, which is an altitude, median and angle bisector at the same time. The line of symmetry cuts any isosceles triangle into two congruent right-angled triangles. This axial symmetry also implies the equality of the two medians, altitudes and angle bisectors from vertices $A$ and $B$, which could also be added if necessary.

The BGC shown in Figure 1 can be used as the basis for another important BGC, "radius bisecting a chord is perpendicular to this chord" (Figure 2). Indeed, we have an isosceles triangle *ACB*, where two sides are equal to the radius of the circle with center at *C* and passing through *A* and *B*. The line segment *CD* bisects the chord *AB*, thus it is a median. But in an isosceles triangle the median is also an altitude, so *CD* is perpendicular to *AB*.

Viewing a tangent line as an extreme position of a chord, we can claim that "a line tangent to a circle is perpendicular to its radius at the point of tangency", $CH \perp EF$ (see Figure 3). The axial symmetry of an isosceles triangle and a circle remains to be a fundamental part of the BGCs depicted in Figures 2 and 3.

One figure may often serve as a tool, helping to recall and connect several basic geometric facts. For example, one may observe that line segments *AB* and *EF* in Figure 3 are parallel, which implies the similarity of the right-angled triangles *CBD* and *CFH*. Here a teacher may re-emphasize the fact that in order to establish the similarity of two right-angled triangles it is sufficient to identify only one pair of equal acute angles.

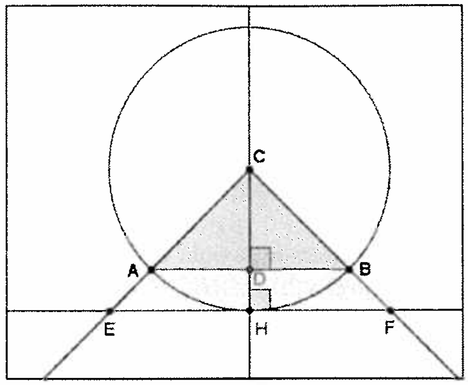


*Figure 3.* Basic geometric configuration "tangent line to a circle".

**Two examples of guided discoveries with the use of basic geometric configurations**

Recognition of an isosceles triangle is the key step in the following guided discovery episode. The dialogue aims at the formation of another basic configuration, "right-angled triangle and circumcircle".

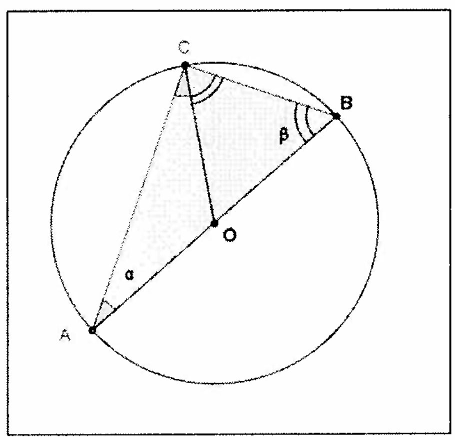


*Figure 4.* BGC "right-angled triangle and circumcircle": Proof of Theorem 1.

Students Anne and Cecile are given the diagram shown in Figure 4. The dialogue that follows resembles teacher-student interactions occurring during our training sessions for regional mathematics competitions. While the amount of teacher's assistance may vary depending on students' mathematical background and ability, focusing students' attention on BGCs is central to the approach endorsed in this paper.

*Teacher*: What do you see?
*Anne:* I see a triangle inscribed in a circle. The centre O of the circle lies on the side of the triangle. The centre O is the midpoint of the side AB. The centre is also connected to the third vertex of the triangle. $AO = OB = OC$ and they are all equal to the radius of the circle.
*Teacher*: Good. What else can you say about this triangle?
*Anne:* $\Delta ABC$ looks pretty arbitrary to me, but $\Delta BOC$ and $\Delta COA$ are isosceles.
*Teacher*: Look more carefully. Can you find the size of angle $A\hat{C}B$ ?
*Anne*: All I can see is $C\hat{A}O = A\hat{C}O = \alpha$ and $C\hat{B}O = O\hat{C}B = \beta$, so $A\hat{C}B = \alpha + \beta$. It could be anything.
*Teacher*: Recall what you know about the sum of all angles in a triangle. Can you use it here?
*Anne*: In $\Delta ACB$ we have $\alpha + (\alpha + \beta) + \beta = 180^o$. So $\alpha + \beta = 90^0$. Aha! $\Delta ACB$ is right-angled!

Here we see an example of a student using proof as a *discovery* of new fact. Alternatively, the student could measure the angle using a protractor and find out that it is $90^o$. The teacher would then prompt her to prove her statement in order to *explain why* it is true in general (see more on different roles of proof in de Villiers (1999)).

*Teacher*: Very good! Now let us formulate a mathematical statement. Let's call it **Theorem 1:** *If point C lies on the circumference of the circle with diameter AB then the angle* $A\hat{C}B$ *is the right angle.* The proof of this theorem is depicted in Figure 4. Notice that the line OC is drawn in order to reveal isosceles triangles. Now, do you think that a converse of this theorem is also true?
*Anne*: Let me think first what the converse statement would be. I need to "reverse" the if-then implication in the original statement. Let us call it **Theorem 2:** *If* $\Delta ABC$ *is a right-angled triangle with right angle at vertex C then point C lies on the circumference of the circle with diameter AB.*
*Teacher*: For now it is just a conjecture. Can you prove it? Look at the diagram.
*Anne:* Let O be the midpoint of the hypotenuse *AB*. Then we need to show that the vertices *A, B* and *C* lie on a circle with centre at *O*. Let $AB = c$. We have $AO = BO = c/2$. It remains to show that $CO = c/2$.
*Teacher*: Sounds like a plan. Remember that your triangle is right-angled. How can you use this information?

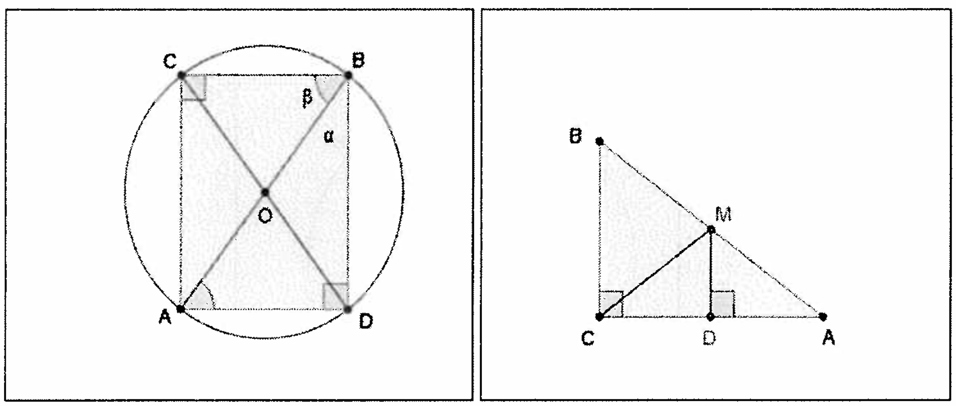


*Figure 5.* BGC "right-angled triangle and circumcircle": Proof of Theorem 2.

Seeing that the students were getting stuck, the teacher shows them the drawings in Figure 5 and asks if the images are helpful to achieve their goal. After looking at the left image (Figure 5) for several minutes, Anne had her aha-moment.
*Anne*: I see a rectangle which consists of the initial right-angled triangle $\Delta ABC$ and its rotation $\Delta ABD$. Now, the diagonals of the rectangle meet at point O and they divide each other in halves so that all the segments *OA, OB, OC* and *OD* have the same length. This proves that all four points lie on a circle with center *O* and diameter *AB*, which is the hypotenuse of $\Delta ABC$.
*Teacher*: This is good! But can you justify that the quadrilateral *ACBD* is a rectangle?
*Anne*: This is because angles $\hat{C}$ and $\hat{D}$ are the right angles of a right-angled triangle, and each of angles $\hat{B}$ and $\hat{A}$ is just the sum of the acute angles in the right-angled triangle $\Delta ABC$, which also gives $90^0$.

Cecile selected the right image in Figure 5 and her reasoning required a little more assistance from the teacher.
*Cecile:* Line *MD* is perpendicular to the side *CA*, and thus *MD* is parallel to the side *CB*. This means that $\Delta AMD$ and $\Delta ABC$ are similar right-angled triangles.
*Teacher*: Can you tell what the coefficient of similarity is?
*Cecile:* Since point M is the mid-point of the hypotenuse AB, $\Delta ABC$ is twice the size of $\Delta AMD$, so $CA = 2DA$ and $CB = 2MD$.

*Teacher*: Now, what can you say about $\Delta CMA$?
*Cecile*: I want to show that it is isosceles… And it is isosceles because $\Delta MDA$ and $\Delta MDC$ are two congruent right-angled triangles, since they share one leg *MD* and the second pair of legs is equal as well ($DA = DC$ since $CA = 2DA$). Thus their hypotenuses *MA* and *MC* are equal, which proves that *A, B* and *C* lie on a circle with centre at *M* and diameter *AB*.
*Teacher*: Very good. We have two proofs of this theorem, but there are many more. I can show you another one which uses algebra and trigonometry so you can recall some formulas and practice you algebra skills as well. My proof is as follows (referring to Figure 4):

If $C\hat{A}O = \alpha$ then $AC = c\cos\alpha$. And now $CO$ can be found from $\Delta AOC$ by the Cosine Law. Recall that $AO = c/2$. I will let you to work it out.

*Anne:* I have $CO = \sqrt{\frac{c^2}{4} + c^2\cos^2\alpha - 2\frac{c}{2}(c\cos\alpha)\cos\alpha} = \frac{c}{2}$. How nice that the terms with $\alpha$ cancel each other! This proves that $CO = AO$.
*Teacher*: Now, can you summarize and describe the result and Figure 5 in your own words?
*Anne*: I will call this figure "Hypotenuse of a right-angled triangle is a diameter of its circumcircle".

These examples illustrate how the students were able to internalize basic geometric configurations during their reasoning on diagrams with the guidance of a teacher.

**Pythagorean Theorem as a basic geometric configuration**

Most secondary school students know Pythagoras' Theorem: $a^2 + b^2 = c^2$ where $a$ and $b$ are legs, and $c$ is the hypotenuse, of a right-angled triangle. Unfortunately, only a few of them are able to prove this fundamental statement. The majority of students simply give an example such as the famous 3-4-5 right-angled triangle, $3^2 + 4^2 = 5^2$, accompanied by the picture of a right-angled triangle with squares attached to its sides. I would not regard this image as a basic configuration as long as it lacks an explanation of the geometric facts. What follows are two examples of a proof of Pythagoras' Theorem which generate the basic geometric configurations depicted in Figure 6 and Figure 7 respectively.

*Proof 1:* Let *ABC* be a right-angled triangle and *CD* an altitude dropped onto the hypotenuse *AB* (see Figure 6). Observe that angles $D\hat{A}C$ and $D\hat{C}B$ are both equal $90^0 - D\hat{B}C$. Thus triangles *ABC, ACD* and *CBD* are similar triangles. Let $BC = a, AC = b, AB = c, BD = f, AD = g$. From the similarity of $\Delta ABC$ and $\Delta CBD$ we have $\frac{a}{c} = \frac{f}{a}$ and thus $a^2 = cf$. The similarity of $\Delta ABC$ and $\Delta ACD$ implies $b^2 = cg$. Thus we have $a^2 + b^2 = cf + cg = c(f + g) = c^2$.

Another proof of Pythagoras' Theorem again uses similar right-angled triangles as well as a few geometrical facts already discussed in this article. Students can be given Figure 7 and asked to find familiar BGCs, list possible relations, and in particular justify the similarity of $\Delta CDB$ and $\Delta CBE$. Teachers may lead the students to the following way of reasoning and conclusion.

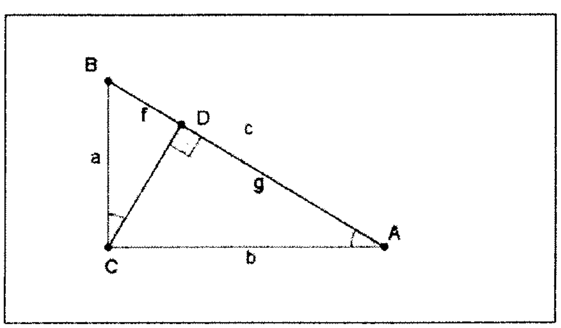


*Figure 6.* Similar right-angled triangles in the first proof of Pythagoras' Theorem.

*Proof 2:* Let $\Delta ABC$ be a right-angled triangle with $C\hat{A}B=\alpha$, $C\hat{B}A=90^0-\alpha$, $AB=c$, $AC=b$, and $BC=a$. Points D and E lie on the extension of the side CA such that $AD=AE=AB=c$. Segments DB and EB are drawn. The "right-angled triangle and circumcircle" configuration from Theorem 1 helps one to see that $\Delta DBE$ is right-angled. $\Delta BAD$ is *isosceles* with $B\hat{A}D=180^o-\alpha$. Thus, $A\hat{B}D=A\hat{D}B=\alpha/2$. In addition:

$$C\hat{B}E=D\hat{B}E-C\hat{B}A-A\hat{B}D=90^0-(90^0-\alpha)-\alpha/2=\alpha/2$$

Now, triangles $\Delta DCB$ and $\Delta BCE$ are similar right-angled triangles because they both have an angle $\alpha/2$. The similarity implies that $CD:BC=BC:CE$. Expressing these segments in terms of the side lengths of the original right-angled triangle gives $CD=b+c$, $CE=c-b$ and $BC=a$. We thus obtain the relation $\frac{b+c}{a}=\frac{a}{c-b}$, which simplifies to $c^2-b^2=a^2$ and thus completes the proof.

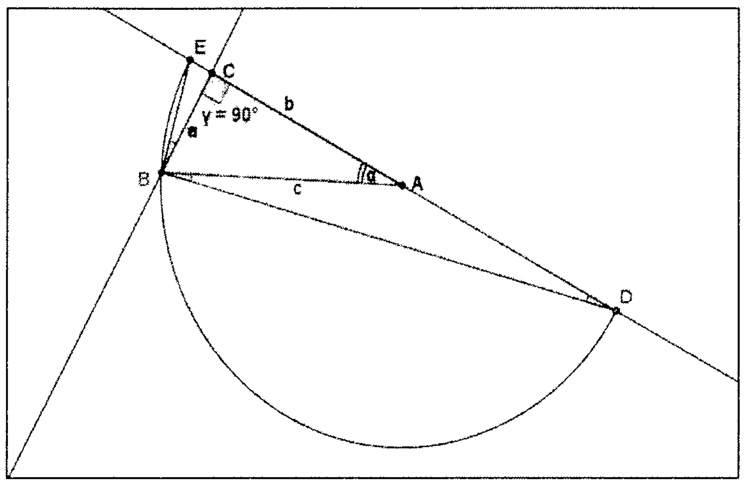


*Figure 7.* A "right-angled triangle and circumcircle" configuration in a proof of Pythagoras' Theorem.

Note that Figure 7 can also be used in a geometric proof of the following trigonometric identity:

$$\cot(\alpha/2)=\frac{b+c}{a}=\frac{c\cos\alpha+c}{c\sin\alpha}=\frac{\cos\alpha+1}{\sin\alpha}=\cot\alpha+cosec\alpha.$$

Thus another mathematical connection can be established as a corollary to the second proof.

**Concluding comments**

A large class of problems in Euclidean geometry has the following structure: they describe a certain configuration and then require showing that either some segments are equal, or specific angles have certain measurements, or some relation holds. This situation puzzles an immature solver because the configuration may seem arbitrary, it may allow a lot of freedom but nevertheless predicts a concrete relation. Often a resolution comes from drawing an auxiliary line and recognition of a BGC which embraces the explanation.

Problems with short solutions are very useful for learning to problem-solve in geometry. However, if only those problems are discussed, students may be misled into believing that all problems in geometry have a short solution based on an insight or a clever geometric construction. To avoid this misconception, problems which require several-steps solutions must be discussed as well. Although they present a bigger challenge, their presence in the learning process is important in activating learned knowledge and making connections between several simple and basic facts.

In conclusion, I summarize several important heuristics for solving geometric problems applied during the above discussion. Teachers may find them useful for both planning lessons in geometry and for scaffolding students during their study of this subject.

- Do not be afraid of geometric problems. Make as many observations about angles and segments as you can. Some of them may be useful for your solution. Even if they do not directly lead to the solution they help you to explore the "unknown territory".
- Introduce notation and denote angles and segments' lengths by letters. You can manipulate with the letters even if their values are unknown. The goal is to find new algebraic relations between different parts of the diagram.
- Recall basic geometric configurations and facts along with their proofs. The proofs may contain mathematical ideas suitable for the problem you are solving.
- Keep solving problems and reflect on your own and others' solutions. This will help you to understand and remember fundamental mathematical facts, recognize basic geometric configurations, and learn how several ideas work together.
- Even if you have a problem solved, look for an alternative solution and try to apply and interpret newly learned material to the problems you have considered in the past. This will help to establish connections between mathematical ideas and will thus strengthen your knowledge and skills as a problem solver.

**References**

**De Villiers, M.** (1999). *Rethinking proof with Geometer's Sketchpad.* USA: Key Curriculum Press.

**Freudenthal, H.** (1971) Geometry between the devil and the deep sea. *Educational Studies in Mathematics,* 3, 413-435.

**Henderson, D.W., & Taimina, D.** (2005). *Experiencing geometry.* Pearson Education Inc. New Jersey.

**Hilbert D., & and Cohn-Vossen, S.** (1952). *Geometry and the Imagination.* (Trans. by P. Nemenyi.) New York: Chelsea.

**Kondratieva, M.** (2009). Geometric sophisms and understanding of mathematical proofs. In F. Lin, F. Hsieh, G. Hanna, & M. de Villiers (Eds.). *Proceedings of the ICMI Study 19: Proof and Proving in Mathematics Education, vol. 2,* 3-8.

**National Council of Teachers of Mathematics** (NCTM) (2000). *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

**Sharygin, I. F., & Protasov, V.Yu.** (2004). Does the school of the 21st century need geometry? *Proceedings of The10-th International Congress on Mathematical Education.* Technical University of Denmark, Lyngby, Denmark. Retrieved from: http://www.icme10.dk/proceedings/pages/regular_pdf/RL_Sharygin_&_Protasov.pdf

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