Chapter 4
Challenging Tasks and Mathematics Learning

Arthur B. Powell, Inger Christin Borge, Gema Inés Fioriti, Margo Kondratieva, Elena Koublanova, and Neela Sukthankar

In this chapter, we present a view of didactical goals of challenging mathematical problems and the cognitive importance of problem-solving schemas. We distinguish between mathematical tasks, exercises and challenging problems and discuss how challenging problems promote the construction of problem-solving schemas. Similar in purpose to the nine case studies presented in Chapter 5, we offer six diverse examples of challenging mathematics problems from varied cultural and instructional contexts. For each example, we examine issues related to its mathematical, cognitive and didactical aspects. Two examples are research-based and accompanied by analysis and discussion of students' work, while the other examples are informed by considered reflection on their use in practice. In the aggregate, the examples illustrate how challenging mathematics problems are suitable for a range of learners and diverse didactical situations; how such problems can be instruments to stimulate creativity, to encourage collaboration, and support the formation of problem-solving schemas; and, finally, how the use of challenging problems invite educators to study learners' emergent mathematical ideas, reasoning and schemas.

4.1 Introduction

4.1.1 A goal for challenging mathematical problems

In many countries, students have come to experience school mathematics as cold, hard, and unapproachable, a mysterious activity quite distinct from their everyday lives and reserved for people with special talents. After repeated failure in school mathematics and estrangement from the discipline, students often assume a view similar to what a student once expressed to the first author: "mathematics is something that you do, not something that you understand."

A.B. Powell (✉)
Rutgers University, New Jersey, USA
e-mail: powellab@andromeda.rutgers.edu

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Similar views emerge from other students' school experiences. A considerable proportion of such students become excluded from meaningful participation in academic mathematics. This is particularly true of students who are members of socially excluded sectors of their societies, lacking in privileged economic or social capital, to use Bourdieu's (1986) categories. As Zevenbergen (2000) notes, "aspects of pedagogy and curriculum...can exclude students ... [since] patterns of language, work, and power are implicated in the construction of mathematics, it becomes [important] to understand how we can change our practices in order that they become more accessible and equitable for our students" (p. 219).

To contribute toward making mathematics more accessible and equitable or less exclusionary and, thereby, more inclusive, this chapter posits the use of mathematical tasks that have particular characteristics. Even further, in addition to the social function of inclusion, such tasks have important psychological and cognitive consequences. The chapter will explicate how engaging students in solving challenging mathematical problems can lead them to construct effective and important problem-solving schemas. The pedagogical goal is to engage students with different mathematical backgrounds in different settings so that they can further develop their mathematical ideas, reasoning and problem-solving strategies, as well as enjoy being mathematical problem solvers.

4.1.2 Importance of schemas in mathematical problem solving

A paramount goal of mathematics education is to promote among learners effective problem solving. Mathematics teaching strives to enhance students' ability to solve individually and collaboratively problems that they have not previously encountered. To discuss the role of schemas in achieving this goal, we first discuss our understanding of problem solving and then that of schemas.

The meaning of mathematical "problem solving" is neither unique nor universal. Its meaning depends on ontological and epistemological stances, on philosophical views of mathematics and mathematics education. For the purposes of this chapter, we subscribe to how Mayer and Wittrock (1996) define problem solving and its psychological characteristics:

> Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver (Mayer 1992). According to this definition, problem solving has four main characteristics. First, problem solving is cognitive—it occurs within the problem solver's cognitive system and can be inferred indirectly from changes in the problem solver's behavior. Second, problem solving is a process—it involves representing and manipulating knowledge in the problem solver's cognitive system. Third, problem solving is directed—the problem solver's thoughts are motivated by goals. Fourth, problem solving is personal—the individual knowledge and skills of the problem solver help determine the difficulty or ease with which obstacles to solutions can be overcome. (p. 47)

Coupled with these cognitive and other psychological characteristics, problem solving also has social and cultural features. Some features include what an individual or cultural group considers to be a mathematical problem (D'Ambrosio 2001, Powell and Frankenstein 1997), the context in which an individual may prefer to engage in mathematical problem solving, and how problem solvers understand a given problem as well as what they consider to be adequate responses (Lakatos 1976). In instructional settings, students' problem solving activities are strongly influenced by teachers' representational strategies, which are constrained by cultural and social factors (Cai and Lester 2005).

An attribute that distinguishes expert mathematical problem solvers from less successful problem solvers is that experts have and use schemas—or abstract knowledge about the underlying, similar mathematical structure of common classes of problems—to form solutions to problems. In general terms a problem schema, as Hayes (1989) characterizes it "is a package of information about the properties of a particular problem type" (p. 11).

The role of schemas in mathematical problem solving has been investigated by psychologists and cognitive scientists, as well as mathematics education researchers. Below is a summary of this research (Schoenfeld 1992):

- Experts can categorize problems into types based on their underlying mathematical structure, sometimes after reading only the first few words of the problem (Hinsley et al. 1977, Schoenfeld and Hermann 1982).
- Schemas suggest to experts what aspects of the problem are likely to be important. This allows experts to focus on important aspects of the problem while they are reading it, and to form sub-goals of what quantities need to be found during the problem-solving process (Chi et al. 1981, Hinsley et al. 1977).
- Schemas are often equipped with techniques (e.g. procedures, equations) that are useful for formulating solutions to classes of problems (Weber 2001).

To illustrate the notion and utility of schemas for problem solving, consider the following problem: Two men start at the same spot. The first man walks 10 miles north and 4 miles east. The second man walks 4 miles west and 4 miles north. How far apart are the two men? In discussing a similar problem, Hayes (1989) notes that when experienced mathematical problem solvers read this statement, it will evoke a "right triangle schema" (problems in which individuals walk in parallel or orthogonal directions to one another can often be solved by constructing an appropriate right triangle and finding the lengths of all of its sides). A technique for solving such problems involves framing the problems in terms of finding the missing length of a right triangle, setting as a sub-goal finding the lengths of two of the sides of the triangle, and using the Pythagorean theorem to deduce the length of the unknown side.

4.1.3 Mathematical tasks, exercises, and challenging problems

In the mathematics and mathematics education literature, no universally accepted definition exists for the mathematical terms "task", "problem", or "exercise" and for the appellation "challenging" when describing a mathematical task or problem. In this chapter, as a starting point, we use Hayes's (1989) sense of what a problem is: "Whenever there is a gap between where you are now [an initial situation] and where you want to be [an adequate response], and you don't know how to find a way [a sequence of actions] to cross that gap, you have a problem" (p. xii).

In other words, "a problem occurs when a problem solver wants to transform a problem situation from the given state into the goal state but lacks an obvious method for accomplishing the transformation" (Mayer and Wittrock 1996, p. 47). For something that may or may not be a problem, to talk about it, we use the generic term "task". To complete a mathematical task, a problem solver needs to apply a sequence of mathematical actions to the initial situation to arrive at an adequate response. Even before applying mathematical actions, the problem solver will have to represent the gap virtually or physically—which is to say, to understand the nature of the problem (Hayes 1989).

The definition provided by Hayes as well as that provided by Mayer and Wittrock suggest grounds to distinguish between two closely related tasks: exercises and problems. Distinguishing these terms cannot be done without consideration of the problem solver. A mathematical task is an exercise to an individual learner if, due to the individual's experience, the learner knows what sequence of mathematical actions should be applied to achieve the task (such as knowing what equation into which to insert givens). In contrast, solving a mathematical problem involves understanding the task, formulating an appropriate sequence of actions or strategy, applying the strategy to produce a solution, and then reflecting on the solution to ensure that it produced an appropriate response.

A mathematical problem may present several plausible actions from which to choose (Schoenfeld 1992, Weber 2005). We call a mathematical problem challenging if the individual is not aware of procedural or algorithmic tools that are critical for solving the problem and, therefore, will have to build or otherwise invent a subset of mathematical actions to solve the problem.

For instance, most proofs in high school geometry are problems, and sometimes difficult ones, since the prover needs to decide which theorems and rules of inference to apply from many alternatives (Weber 2001). However, proofs that require the prover to create new mathematical concepts or derive novel theorems would make these proofs challenging problems. To solve challenging mathematics problems, learners build what are for them new mathematical ideas and go beyond their previous knowledge.

4.1.4 Use of challenging problems to promote schema construction

In mathematics education, challenging mathematical problems have psychological and cognitive importance. Since "problem-solving expertise is dependent upon the acquisition of domain-specific schemas" (Owen and Sweller 1985, p. 274), many researchers argue that an important goal of the mathematics curricula should be to provide students with the opportunities to construct problem-solving schemas (De Corte et al. 1996, Nunokawa 2005, Reed 1999). What is less clear is how this goal should be achieved. Marshall (1996) argues that the issues of how students construct problem-solving schemas and what types of environments or instructional techniques might foster these constructions are open questions in need of research.

Some psychologists and mathematics educators have suggested that students construct schemas by transferring the solution of one problem to another superficially different but structurally analogous problem (Novick and Holyoak 1991, Owen and Sweller 1985). Unfortunately, students often have difficulty seeing the deep structure of problems and transferring the solution of one problem situation to another (Lobato and Siebert 2002, Novick and Holyoak 1991). Accordingly, it is suggested that schema construction can be facilitated by providing students with basic problems to which that schema applies, both to increase the likelihood of successful transfer and to minimize the cognitive load that students use to solve these problems, thus leaving more resources available for learning (Owen and Sweller 1985). Contrary to these findings, discussing a long-term research project on the development of students' mathematical reasoning, Francisco and Maher (2005) report evidence that students often develop a rich understanding of essential ideas in the context of solving complex, challenging problems. In this chapter, in one specific example among others, we will illustrate how students developed a powerful combinatorial schema while solving strands of problems that were challenging (in the sense described earlier in this chapter).

4.2 Categories of challenging mathematics problems

There are many different categories of mathematics problems that are suitable as challenges for a learner or a group of learners. This diversity is also discussed and illustrated in Chapter 5 of this volume, and Chapter 3 has treated the issue of challenging mathematics and the use of information and communication technologies. Whether a specific mathematics problem is a challenge depends on the mathematical experience of an individual learner. Nevertheless, appropriate challenges can be given to mathematically talented students as well as to socially excluded and struggling students, be they children, teenagers, or adults.

Moreover, as will be discussed later in this chapter, there are important pedagogical, psychological and social reasons that all students should be engaged with challenging mathematics problems. In this chapter, we present different types of challenging problems, some of which are about paradoxes, counterintuitive propositions, patterns and sequences, geometry, combinatorics and probability. It goes without saying that the categories of challenging problems that we present are neither comprehensive nor exhaustive: there are many areas of elementary and advanced mathematics that our examples do not include.

Not only is it important to consider the type of problems to use but also to contemplate the physical setting and pedagogical climate in which they are used. For instance, the setting might be formal as a school classroom or informal as an afterschool program or with street kids or adults learners in a public space. The pedagogical approach may include collaborative or cooperative learning with an instructor as a facilitator or involve groups of learners presenting their solutions. The actual mathematical challenge may be selected by students or be a sequence of challenging problems that contribute to students building problem-solving schemas.

4.3 Challenging mathematics problems and schema development

We present six diverse examples of challenging mathematics problems from varied contexts, one in this section, four in Section 4.4, and a final one in Section 4.5. Some of the examples that we present contain several challenging problems. The first and third examples are empirically based, while the remaining four are informed by reflection on practice. Following the presentation of each example, we provide three types of analysis: mathematical, cognitive, and didactical. The two research-based examples are each also accompanied by an analysis of students' work and a discussion.

4.3.1 Strands of challenging mathematical tasks

In this section, based on analysis of Weber et al. (2006), we exemplify how over time students can develop an important and effective combinatorial schema from their work solving a strand of challenging problems. The students' building of problem-solving schemas related to combinatorics occurred within the context of a longitudinal study, now in its 20th year, tracing the mathematical development of students while they solve open-ended but well-defined mathematical problems (Maher 2005).

The problems are challenging in the sense that students often initially are not aware of procedural or algorithmic tools to solve the problems but are asked to develop them in the problem-solving context. The strand of problems presented here are used in an environment in which collaboration and justification are encouraged, and teachers and researchers do not provide explicit guidance on how problems should be solved or whether the solution that students develop is correct or not, that judgment being left to the students.

One aspect of this study was that students worked on strands of challenging tasks—or sequences of related tasks that may differ superficially but designed to pertain to identified mathematical concepts. The use of a strand of challenging problems allows teachers and researchers to trace the development of students' reasoning about a particular mathematical idea over long periods of time (Maher and Martino 1996).

This study in which the challenging mathematics problems were used has an important distinguishing feature. Most studies examining schema construction or transfer take place over a short period of time in conceptual domains in which students have limited experience (Lobato and Siebert 2002). However, meaningful mathematical schemas are likely constructed over significant stretches of time after students become accustomed to the domain being studied.

Hence, Anderson et al. (1996) argue such studies seek evidence of schema usage and transfer in places where one is least likely to find it. We are not aware of long-term studies in mathematics education that address schema acquisition. Hence, the longitudinal and empirical nature of the study that Maher (2005) describes has the potential to offer unique research findings in an important area.

The following set of mathematical challenges is an example of the problems in a strand of combinatorial tasks. Working of the problems in the strand allowed students to develop mathematical ideas and reasoning strategies within a particular domain.

To provide a comprehensive sense of the possibility that students can develop problem-solving schemas within a specific mathematical domain, we detail a case of five students from a research project at Rutgers University (Weber et al. 2006).

First, we present three problems—challenging for the particular group of students in the study—a brief mathematical analysis of the problems, and indicate cognitive, mathematical structures that learners can build from engaging with these problems. Next, we will provide results and a discussion of how a group of five students solved the three problems. Following this presentation, in the next section, we present other examples of mathematics problems, challenging for the context in which they have been used.

4.3.2 Examples from a strand of challenging mathematical tasks

The Four-Topping Pizza Problem

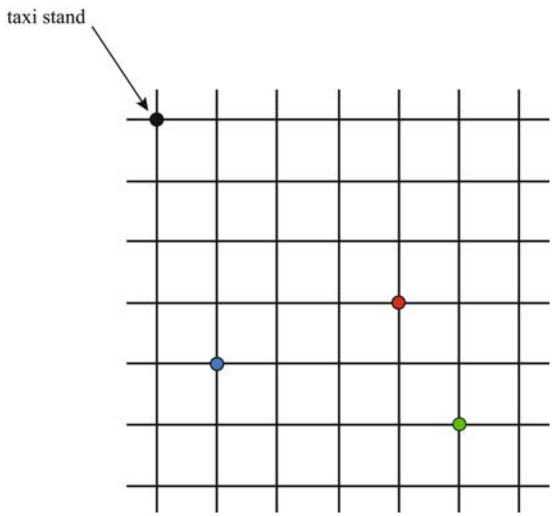
A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom and pepperoni. (No halves!) How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

A Towers Five-Tall Problem

Your group has two colors of Unifix cubes. Work together and make as many different towers five cubes tall as possible, each with three red and two yellow cubes. See if you and your partner can plan a good way to find all the towers four cubes tall.

The Taxicab Problem
A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass the time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.



4.3.3 Mathematical analysis

The answer to the first task is $\sum_{r=0}^{4}\binom{4}{r} = 2^4$. The second has as an answer $\binom{5}{2} = \frac{5!}{2!3!} = 10$ or, equivalently, $\binom{5}{3}$. The answer to the third task is $\binom{5}{1} = \frac{5!}{1!4!} = 5$, $\binom{7}{4} = \frac{7!}{4!3!} = 35$ and $\binom{10}{5} = \frac{10!}{5!5!} = 252$ for the three pickup points. However, these problems all have the same underlying mathematical structure that can be associated as a "Pascal's triangle schema." The students in the research project had not studied combinatorics, were not familiar with the standard notation for permutations or combinations, and yet, as we will show, they correctly solved the three problems by other means.

4.3.4 Cognitive analysis

Based on teaching and research experiences, the mathematical ideas and reasoning strategies that students are likely to develop or engage include the following:

1. counting without omission or repetition;
2. symmetry;
3. powers of 2;
4. Pascal's triangle;
5. counting the number of distinct subsets, combinations $\binom{n}{r} = {}_nC_r$;
6. reasoning by controlling variables (determining which independent variable to change and manipulating this independent variable to determine changes in the dependent variable);
7. reasoning about isomorphism (see Table 4.1).

Table 4.1 Taxonomy of isomorphisms among three mathematical tasks

	Taxicab	Towers	Pizzas
Objects	East and south vectors	Red and blue Unifix cubes	Different toppings
Actions	Go east or south	Affix red or blue Unifix cube	Add a topping or no topping
Products	Different shortest taxicab routes	Different Towers	Different Pizzas

4.3.5 Students' work on problems from a strand of challenging tasks

Weber et al. (2006) prepared the work excerpted in this section for the ICMI Study 16. The students whose work is analyzed are participants in the long-term study described by Maher (2005). Here, Weber et al. (2006) examine how a group of five students (Ankur, Jeff, Brian, Michael and Romina) solved the three problems presented above when they were in 10th and 12th grades.

4.3.5.1 How many pizzas are there with four different toppings?

In a 10th grade session, Ankur, Jeff, Brian and Romina used case-based reasoning and various counting strategies to obtain the correct answer—fifteen pizzas with toppings plus one pizza with only cheese and tomato sauce. Michael developed a binary representation to create each of the pizzas. Each of the pizzas was represented using a four-digit binary number, where each topping was associated with a place in that number, where a one signified that the topping was present on the pizza and a 0 signified that the topping was absent. For instance, with the four toppings—pepperoni, sausage, onion and mushroom—the binary number 0010 would refer to a pizza with only onions. Michael was able to use this notation to explain why 16 pizzas could be formed when there were four toppings available and convinced his group that there would be 32 pizzas if there were five toppings available (the other group members believed that there would be 31, not 32 pizzas).

At the end of the session, the researcher asked the group if this problem reminded them of any other problems. Brian responded "towers"—referring to the problem of forming four-tall towers from red and yellow cubes. However, Ankur noted the problems were "similar, but not exactly the same", since more than one yellow could appear in an acceptable tower, but you couldn't list mushroom more than once on the toppings of the pizza. All of the students at this time accepted Ankur's explanation. The following week, Michael represented the towers problem using binary notation—the *n*th digit in the notation refers to the *n*th cube in the tower, with a 0 signifying a yellow cube and a 1 a red cube. For example, 0010 would represent a four-tall tower in which the third block was red but the other three were yellow. Hence, using this binary notation, Michael was able to show his group a correspondence between the towers and the pizzas. (For an elaborated analysis of Michael's binary representation and how he used it to indicate an isomorphism between the towers and pizza problems, see Kiczek et al. 2001.)

There are two things worth noting about these problem-solving episodes. First, when students were initially comparing the pizza and towers problems to one another, they did not seem to see the deep structure between the problems. In fact, Ankur argued the problems differed significantly. The connections between the problems were not immediately perceived but were only constructed by Michael after reflection. Secondly, the notational system that Michael developed while working on the pizza problem was critical for the construction of his correspondence.

4.3.5.2 Linking the pizza problem, the towers problem, and Pascal's triangle

One month later, students were invited to further explore the relationship between pizza problems and tower problems. They were asked to determine how many five-tall towers could be formed with three yellow blocks and two red blocks. Using Michael's binary representation, they translated this problem to determine how many five-digit binary numbers with three 0s and two 1s could be formed. By controlling for where the first one in this sequence occurred, the students were able to deduce that 10 such towers could be formed. Note that the methods Michael developed to cope with the previous pizza problems were now a scheme that the students used to make sense of a new pizza problem (see Uptegrove 2004). After obtaining their solution, a researcher introduced students to Pascal's triangle, explained how the nth row of Pascal's triangle were the coefficients of the expression $(a + b)^n$, and that the terms in Pascal's triangle were often represented using combinatorial notation. For instance, the fourth row—1, 4, 6, 4, 1—can be written as $\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$. She then asked the students to try to understand what these coefficients might mean in terms of what they've just done. After thinking about these problems, the students were able to make these links. They noticed the 10 that appears in the fifth row in Pascal's triangle corresponding to the expression $\binom{5}{2}$ also corresponded to five-tall towers with two red blocks (and three yellow blocks). Further investigations led these students to describe the relationship between Pascal's triangle and the pizza problem—namely, that the $\binom{n}{i}$ entry in Pascal's triangle corresponds to the number of pizzas that could be formed with i toppings if there were n to choose from. These students could also explain why $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ and $\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$ (Pascal's Identity) were true by using the towers problem and the pizza problem.

4.3.5.3 Solving the taxicab problem

Two years later, Michael, Romina, Jeff and Brian (now in 12th grade) were given a version of the taxicab problem. In essence, they were asked how many ways that a taxi could take a shortest route along a grid to go four blocks down, one block right; three blocks down, four blocks right; and five blocks down, five blocks right. This qualified as a challenging problem for the students. The solution to this problem more or less requires the application and use of combinatorial techniques, yet the students solving this problem had not used such techniques before to solve novel problems. The initial stages of the students' activity were exploratory in nature. They worked to make sense of the problem, posed some initial conjectures that turned out to be incorrect (for example, the distance from the starting point to the endpoint would tell you the number of shortest routes), and tried to answer the question by explicitly drawing and counting the routes.

Romina asks if it would be possible to "do towers" to the problem. Michael and Romina note that the distance to one of the points is 10 and wonder if the total number of shortest routes to that point is 2^{10}. Later, the students attempted to solve the problem by finding the number of shortest routes to corners close to the point of origin (e.g. there are two shortest routes to go one down, one right; three shortest routes to go two down, one right). They produced a table like the following:

1	1	1	1	1
1	1	3	4	5
1	1	6	10	12
1	4	10	15	
1				

In the table, the *m*th by *n*th cell represents the number of shortest paths to go *m* units to the east, *n* units south.

Romina notices that the fourth diagonal of this table is the sequence 1, 4, 6, 4, 1 and declares, "it's Pascal's triangle", where the diagonals in the table correspond to the rows of Pascal's triangle. Jeff notes that the 12 and the 15 in the next diagonal would not be correct if this was the case and asks Brian to re-evaluate the number of routes it takes to go four over and two down. When Brian announces that he found 15 routes, Michael comments, "it means that it is the triangle." A little later, Romina writes a 20 in the box for three right, three down while Brian worked on re-computing this value. At this point, Michael asks his colleagues how they knew it was 20. Jeff responds that if they can show the triangle works, they don't need to verify that it's 20.

To understand why Pascal's triangle would provide the number of shortest routes to any points on the grid, Romina announces that she will try and relate the triangle back to the towers and focuses on the 1 2 1 diagonal. She notes that all of the points on this diagonal are two away from the starting point and this also forms the second row of Pascal's triangle. Further, she notes a connection between the middle entry in that column—with towers, the middle entry would refer to a two-tall tower with one yellow and one red block; with taxicabs, this refers to a trip with one across and one down. Likewise, the entry two down, one right, would refer to a tower that was three-tall, with two yellow and one red blocks, or the taxicab location three away, with two down and one across. The students filled in the rest of their grid in accordance with Pascal's triangle. For instance, when they filled in the cell for five down, two over, they reasoned that the number of routes would correspond to the fifth entry of the seventh row of Pascal's triangle (not counting the beginning 1) since it would be "five of one thing and two of another thing." At a researcher's request, Michael also explains the connection between Pascal's triangle and the pizza by using his binary number notation. For the taxicab geometry problem, a 0 would indicate going down and a 1 would indicate going across. Hence, using the example of going two down and one across, one would need to find the number of binary strings that have two 0s and one 1. In their work relating Pascal's triangle to the pizza problem, the group had already established that this would be the first entry (ignoring the first 1) of the third row of Pascal's triangle. Finally, the group was able to use these constructions to answer the given questions, for instance, the number of shortest routes to the point that was five right and five down would correspond to the fifth entry of the tenth row of Pascal's triangle.

4.3.5.4 Discussion—strand and schema

In the first two excerpts above, we illustrated how students constructed a powerful problem-solving schema for solving combinatorial problems. We then illustrated how students applied that schema to solve the challenging taxicab geometry problem. The application of this schema not only allowed them to construct the solution to the problem, but it also provided them with a deep understanding of their solution and enriched the schema that they had constructed. In this section, we will discuss four aspects of our problem-solving environment that enabled students to make these constructions.

First, students were asked to work on *challenging* problems. If students were asked to work on problems for which they had already had strategies, they may have attempted to see whether various techniques that they had learned would be applicable to the problem. As the students needed to *develop* techniques to make progress on these problems, this was not an option for these students. A particularly important precursor toward developing the schema that these students constructed was the development of useful ways of representing the problem. Michael's binary representation of the towers and the pizza problem, in particular, paved the way for students to see the deep structure that these problems shared. One general finding from the longitudinal study was that students developed powerful representations in response to addressing challenging problems (Davis and Maher 1997, Maher 2005).

Secondly, students were asked to work on *strands* of challenging problems that were superficially different but shared the same mathematical structure. This provided students with the environments in which schemas could be constructed. Researchers also fostered this construction by encouraging students to think about how the problems they were solving might be related to problems that they had solved in the past. However, we believe that having students work on strands of challenging tasks is a necessary but not sufficient condition for schema construction and usage. Students also need time to explore the task and benefit from heuristics that guide their explorations in productive directions.

Thirdly, students were given *sufficient time* to explore the problems and were also given the opportunities to revisit the problems that they explored. The students did not instantly see the connections between the towers and pizza problems, nor did they see how the taxicab problem was related to either of these problems. It is especially noteworthy that students initially believed that the towers and pizza problems were similar, but also differed significantly, and that Romina's initial suggestion to relate the taxicab problem to the towers was not immediately pursued. Further, as the students revisited problems, their representations of the problems became increasingly more sophisticated, enabling them to see links between the problem being solved and previous problems on which they had worked. As Uptegrove (2004) illustrates, many of the connections students made could be traced back to problem-solving sessions on which they worked months or years before.

Finally, as Powell (2003) emphasizes, the heuristics that students used in their problem solving enabled them to relate the problem situation to their schema. Among the heuristics that the students used were the following: solve a difficult problem by solving easier ones (before finding the number of shortest routes to a location ten blocks away, find the number of shortest routes to a location two blocks away); generate data and look for patterns; and see if there is an analogy between this problem and a familiar one (Powell 2003). Without the use of these heuristics, the links to an existing schema may not have been made. However, the disposition to use such heuristics was likely developed during the students' years of solving challenging problems (Powell 2003, Uptegrove 2004). Moreover, these students' co-constructed schemas through a process that Powell (2006, p. 33) terms *socially emergent cognition*.

4.4 Other examples and contexts for challenging mathematics problems

In this section, we present four other examples of challenging mathematics problems and describe the context in which each has been used. In the fifth section, we present another category of challenging tasks: paradoxes. As noted earlier, the second example in this section is empirically based, while the remaining three are informed by reflection on practice.

4.4.1 Example: Number producer

The Context

The problem we will discuss as an example of a challenging mathematical task is called the Number Producer, and we consider two different settings where it has been used. In the first setting, the participants were students taking part in an entrance interview for the University of Oxford, UK. The student, while alone with the interviewer, was given problems on a piece of paper and had paper available for calculations. The Number Producer was given as one of the problems the student should attempt to solve in front of the interviewer, to provide information on his or her potential as a mathematics student, and hence on whether to offer this student a place. The student was given some time to think about the problems before the discussion with the interviewer started.

(The Number Producer was suggested as a problem by Juliette White of the Open University, UK, and has its origin with Smullyan (1982)).

In the second setting, the participants were third- and fourth-year mathematics students at a teacher training college in Vestfold, Norway. They were presented with the Number Producer problem in class where it was talked through. They were then given the problem as an assignment to hand in after five days. Some worked in groups, others individually. Some asked for and received hints and clarification via e-mail. After the solutions had been handed in, there was a discussion of the process.

The Number Producer

In this problem, a number means a positive integer written in decimal notation with all its digits non-zero. If A and B are numbers, by AB we mean the number formed when the digits of A are followed by the digits of B, and not the product of A and B. For any number X, the number $X2X$ is called the associate of the number X.

There exists a machine. When you put a number into the machine, after a while a number comes out of the machine. However, the machine does not accept all such numbers, only some. Those numbers accepted by the machine are called acceptable. We say that a number X produces a number Y if X is acceptable and when X is put into the machine, Y comes out of the machine.

The machine obeys three rules:

R1. For any number X, the number $2X$ is acceptable, and $2X$ produces X.

R2. If a number X is acceptable and produces Y, then $3X$ is acceptable and produces the associate of Y.

R3. If you cannot decide that a number is acceptable from R1 and R2, then it is not.

Questions:

1. What is the associate of 594?
2. For each of the numbers listed below, find whether or not it is acceptable. If it is acceptable, find the number it produces.
   (a) 27482
   (b) 435
   (c) 25
   (d) 325
   (e) 3325
   (f) 33325
   (g) 345
   (h) 333
   (i) 32586
3. Can you describe the numbers that are acceptable?
4. Can you think of a number that produces itself?
5. Can you think of a number that produces its associate?

Mathematical Analysis

We note that the presentation of the problem is as it was given to the Norwegian participants (translated). In the interview setting, questions 2 and 3 were grouped together as one question, as were questions 4 and 5.

Those who know about functions might think "function" instead of "machine" when they read through the Number Producer. The first question is posed in order to build up the mathematical action that whenever one sees the symbol AB, one should think concatenation, and not multiplication, of the numbers A and B, and also posed to assist students to grasp the definition of the associate of a number. Hence, one should find that the associate of 594 is 5942594.

Questions 2 and 3 help the problem solver understand how the machine works: the numbers in (b), (g) and (h) are examples of not acceptable numbers, whereas the others are acceptable. Also note the order of the given numbers in question 2 (a) is acceptable (and produces 7482); (b) is not acceptable; (c–f) should help the problem solver to see and create a pattern (with answers 5, 525, 5252525, and 525252525252525, respectively); followed by (g) and (h) which are not acceptable; and finally we have (i) as a "check" for the understanding (which produces 5862586). From this, the problem solver might have created the algorithmic tool for answering question 3: the acceptable numbers are of the form "(possibly 3s)2(a number)".

Once the problem solver has been able to do these questions, he or she can try to solve the final two questions, using what he or she now knows. (The answers to questions 4 and 5 are "yes, 323 produces itself", and "yes, there is also a number that produces its associate...," which we leave for the reader to find!)

Cognitive Analysis

For a challenge to have a positive effect on learning, it should not be too difficult, but "just out of reach". That is, it should be within the zone of proximal development, as it is referred to in Vygotskian terms (Vygotsky 1978, defined in this Study Volume in Section 6.2.2.3 and also discussed in Sections 3.1 and 7.3.2). For a particular group of learners, an appropriate challenge has to have the possibility of being mastered. In the Number Producer, there are mathematical and cognitive challenges, where the definitions and the notation must be understood and accepted. For example, it might help to think of the numbers as an alphabet in this problem. In any case, a learner also has to accept the rules the machine obeys, which in turn creates new mathematics for the learner.

In the first setting (the interview), all the students accepted the challenge immediately and some quickly started talking while others thought for a few minutes. The interviewer started asking questions to see whether the student had understood. Through various degrees of hints, they all managed to answer the questions given in the Number Producer. This particular setting forced the students to be extremely focused. They all said it was an interesting problem, and managed to give most answers very quickly. Through the discussion the interviewer could follow the process the students went through to understand how the machine worked, hence learning new mathematics. It was obvious that none of them had seen this problem before. Seeing that they could talk through the problems with the interviewer rather than just presenting the answers they could come up with gave a positive feel to a stressful situation. Being able to discuss a mathematical challenge in such a setting is a valuable experience.

In the second setting (that of the teacher training students), the Number Producer was given as one of several problems to illustrate mathematical thinking as part of the history of mathematics. One idea was to make the students give some thought to how new mathematics develops. Since this was the first assignment of the course, there were no immediate complaints, and all the students went away to make an attempt at the problem.

However, it turned out the students spent a lot of time on it and found it very hard. Most of them got frustrated with it and thought it was too difficult for them. They tried seeking help from others. Most of them managed to hand in partial solutions, while a few didn't hand in anything at all. As one of the students said, "This is a problem where we had to think for ourselves and couldn't look up a formula." Many students had searched for help in textbooks without luck. To these students the problem was challenging, in the sense we described in Section 4.1.3—the problem solver is not aware of procedural or algorithmic tools that are critical for solving the problem, and therefore will have to build or otherwise invent a subset of mathematical actions to solve the problem.

As for the solutions, some students just wrote the answer, whereas others elaborated so that one could follow their process. From this, it was clear that they asked themselves good questions in order to figure out how the machine worked, and hence learned something new. And so they built what are for them new mathematical ideas and went beyond what they previously knew.

In the discussion that followed, several points were made. Some felt that this sort of problem could be destructive in the sense that some students lose confidence when they cannot produce an answer at all. Further, they spent a lot of time on the problem, and some felt it was a waste of time when they could not find any answers. However, it turned out that several students had started thinking about why this challenge was given, and one said, "I didn't interpret it as traditional maths, but thinking back I realize that maybe it was." Also, they learned what it was like to not always be able to solve a problem completely; they were obviously used to handing in almost perfect solutions to assignments.

Didactical Analysis

The background of the participants and the setting in which the challenge is given has implications for learning. For example, in the case of the Number Producer, the teacher candidates were not used to and hence didn't expect to be challenged the way they were, whereas the interviewed students were certainly expecting a challenge. Another difference was that the teacher candidates had more time to think about the problem, but they did not have the interviewer with whom to discuss their insights. This makes the learning processes and the outcomes very different, and hence influences the effect of the challenge on learning.

Another point to be made about the effect on learning from this example is motivation. The person presenting the challenge and the people receiving it must have some sort of agreement beforehand: Why do this? The interviewed students receiving the Number Producer were very clear on their motivation (in a non-typical classroom situation). The teacher training students (who were in a typical classroom situation), it turned out, were not.

Variation to the curriculum requires interesting and useful challenges in order to have a positive effect on learning. For example, one of the teacher candidates faced with the Number Producer said, "It wasn't an interesting exercise, but one can't expect all exercises to be interesting to everyone."

Still, the Number Producer is an example of a challenge that can be an addition to the curriculum. For one thing, it doesn't require a lot of background theory. All the ingredients are explained in the problem statement. The first few questions helped the students find out how the machine works, whereas the final few questions were themselves new challenges. These new challenges are easier to accept if you have done the first questions, then you want to apply the things you have learned. Learning something new and being challenged on it immediately enhances learning. "Now that you have understood how the machine works, can you find a number which produces itself?" It is in human nature to learn, and we all need to be challenged on what we learn, otherwise we lose interest.

4.4.2 Example: Pattern sequence

The Context

The turmoil following the 2001 crisis in Argentina led to many students dropping out of school. The severity of the situation is indicated by the following statistics: 35 per cent of youth between the ages of 15 and 24 neither study nor work; 13 per cent of teenagers abandon school; the unemployment rate among those under 29 years old is 13 per cent, among whom 54 per cent live in poor households.

In 2004, the government of Buenos Aires started the Back-to-School program for secondary students, in line with the Zero Dropout Plan (for more information, see www.buenosaires.gov.ar/areas/educacion/desercioncero), which targets adolescent school leavers living on the margins of society. Its aim is to provide a curriculum equivalent to that of compulsory secondary education that will lead them to gain the necessary official certificates and grades to get a dignified job. Since 2002, in the city of Buenos Aires, education has been compulsory until the students are 16 (Law no. 898 on compulsory secondary education).

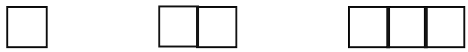
The Zero Dropout Plan targets an important sector of the young population. The participants in the Back-to-School program must be at least 19 years old and have interrupted their education for at least a year, but be interested in and committed towards completing their secondary education and show commitment towards it. A significant number of these students have experienced failure in both their primary and secondary schooling, and many of them combine their education with family and work responsibility.

Some students dropped out of school many years ago, while others have poor primary education. Many of them do not even know the multiplication and division algorithms nor can they use basic procedures for subtraction. Some have a criminal record or suffer from drug addiction. Attendance is poor and a high rate of absenteeism interferes with continuity. Particularly in first year, many have difficulty in adapting to the school environment.

Pattern and Sequence Problems

The initial problems given to the students required them to describe the general step or the result of a regular process, such as the addition of the first n natural numbers or the calculation of the number of elements of a certain geometric configuration. The geometrical context helped students recognize the equivalence of different descriptions of the pattern.

The teacher presented the following sequence of figures built with matches and explained how they should be further assembled.



(a) Determine the number of matches needed to form the sixth figure in the sequence.
(b) How many matches would be needed to build the 100th figure in the sequence?
(c) Find a formula for the number of matches in the nth figure.
(d) Is it possible for one of the figures to be composed of 1549 matches? 1500 matches?

Mathematical Analysis

Algebra can be understood as a tool to model and handle problems of a certain type. The process that students go through to obtain a formula for the number of elements of a collection is reflected in the form of the expression found. At the same time, this process helps students to appreciate the meaning of a "letter" used as a variable and get a feel for the correct use of algebraic expressions. Moreover, different approaches to the same problem may illuminate a discussion on the equivalence of different expressions and how algebraic expressions can be transformed.

From this perspective, for the students in the project, a challenging activity is the production and validation of formulas using natural numbers. We intend that students look for patterns, find formulas to describe them and produce arguments to validate them. The teacher is not expected to "teach" the formulas nor the students to "apply" them; rather the students have a chance to speculate, create, test and validate them. The problems are designed to admit multiple approaches and formulas for the same process.

Cognitive Analysis

The work of the students illustrates how equivalent formulas are found for the number of matches required for n squares. For example, students who saw each new square as resulting from the addition of three matches produced the formula $n.3 + 1$. Another formula $2.n + n + 1$ came from students who counted the horizontal matches in pairs, added in the vertical matches completing the squares and finally the initial vertical match. Other students gave the formula $4 + 3(n - 1)$, noting the four matches for the first square and the three additional matches for each new square. Finally, some students gave $4.n - (n - 1)$, counting four matches for each square and then subtracting the number of vertical matches that were double-counted.

Harmonizing the equivalent expressions provided a basis for introducing the notions of common factor and distributivity. Thus, in showing the equivalence of $4+3(n-1)$ and $3n+1$ many students used the concept of multiplication as repeated addition. They considered $3(n - 1)$ as $(n - 1) + (n - 1) + (n - 1)$ and recorded the sum vertically, as for natural numbers:

$$\begin{array}{r} n-1 \\ n-1 \\ \underline{n-1} \\ 3n-3 \end{array}$$

Then they added by associating on the one hand the ns and on the other hand the -1s leading to establish the equation $3(n - 1) = 3n - 3$. We observed that the students implicitly made use of the commutative and associative properties in connection with addition although they had not learned the symbolic formulation. From this, the common factor and the distributive property, which the students had not yet worked with, could be formalized. Eventually, students would write such equations as $4 + 3(n - 1) = 4 + 3n - 3 = (4 - 3) + 3n = 1 + 3n = 3n + 1$.

Proving the equivalence of two formulas is a gateway to algebraic manipulation. When a formula involving a variable arises in some context, students can check special cases numerically.

We conclude with an examination of the work of some students who answered the questions as to whether the sequence contained a diagram requiring 1549 or 1500 matches.

While some calculations were tentative, the following one led to a correct answer:

$$\begin{array}{r} 301 = 100 \\ \times 5 \\ \hline 1501 = 500 \end{array}$$

By this, the students wished to express that if 301 matches are needed for the 100th figure, then for the 500th figure, they would need $301 \times 5 - 4$, the subtraction for the number of matches that repeat when concatenating five series of 301. They multiplied 16 by 3 for the matches needed for an additional 16 squares, and these added to the 1501 yielded 1549 matches, the number required for the 516th figure in the sequence.

Didactical Analysis
The problem was developed in one of the reinsertion schools in Villa Lugano, a neighborhood of the city of Buenos Aires. The teacher we collaborated with had a strong commitment to the project, positive expectations of her students and a sound mathematical education. We collaborated in designing problems that were to challenge her students mathematically.

We expect that the performance of the students on this problem would help provide a benchmark for suitable mathematical challenges. We plan to formulate what is a challenge from a theoretical perspective as well as from the perspective of the teacher and students. Together with them, we will study from a socio-cultural perspective how mathematically challenging activities can motivate students to participate in the mathematics classroom and how a particular way of handling interactions among the participants can contribute to a classroom culture that facilitates participation as a step towards learning.

4.4.3 Examples: Probability

The following two examples demonstrate how challenging mathematical problems can be used to engage students in a post-secondary, introductory probability course. In such a course, students have difficulties in seeing connections between basic probability models and word problems of a varying verbal content, that are based on these models. Furthermore, a typical dilemma for students in this course is to combine, in a proper way, intuitive and strictly mathematical approaches to problem solving. In order to stimulate students' creativity, the following course project was offered to students at Community College of Philadelphia. The students have the option of selecting a challenging problem from external sources or attempting one suggested by their instructor. In the examples below, students selected the first problem, and the instructor offered the second. Both problems were solved and presented to the class by students.

Foot-and-Mouth Disease Problem

- One person per hundred people has the infectious Foot-and-Mouth disease.
- The probability of a person with this disease testing positive is 0.9, and the probability of a person who does not have this disease testing positive is 0.2.
- What is the probability that a person who tests positive has the disease?

Three Cards Problem

- Suppose you have three cards: a black card that is black on both sides, a white card that is white on both sides, and a mixed card that is black on one side and white on the other.
- You put all the cards in a hat, pull one out at random, and place it on a table. The side facing up is black.
- What is the probability that the other side is also black?

Mathematical Analysis

The level of difficulty of both problems is higher than that of standard problems in this course. The solution of the Foot-and-Mouth disease problem involves such notions as conditional probability, complete probability and Bayes' formula.

This is one solution presented by a student:

We define events as "*Yes*"**:** a person has the disease; "*No*"**:** a person has no disease; "*Pos*"**:** a person is tested positively; "*Neg*": a person is tested negatively.

We can see $P(Yes) = 0.01$ and $P(No) = 0.99$.

Then

$$P(Pos/Yes) = 0.9 \quad P(Pos/No) = 0.2.$$

What is $P(Yes/Pos)$?

Now the "branch probability"

$$P(Yes/Pos) \times P(Pos) = P(Pos/Yes) \times P(Yes) = P(Pos \cap Yes).$$

Using Bayes' formula:

$$P(Yes/Pos) = \frac{P(Pos/Yes) \times P(Yes)}{P(Pos)},$$

that is,

$$P(Yes/Pos) = \frac{P(Pos/Yes) \times P(Yes)}{P(Pos/Yes) \times P(Yes) + P(Pos/No) \times P(No)},$$

that is,

$$P(Yes/Pos) = \frac{0.9 \times 0.1}{0.9 \times 0.01 + 0.2 \times 0.99} = \frac{9}{207} = \frac{1}{23} = 0.043 \approx 4\%.$$

An alternative solution was also presented based on a tree diagram and evaluating branch probabilities.

The Three Cards problem is a well-known example of a counterintuitive problem. This problem is discussed broadly in the literature (Nickerson 2004) and on the Internet (en.wikipedia.org/wiki/Three_cards_problem). There are various solutions of this problem based on notions of reduced sample space, conditional probability, and multiplication of probabilities.

This problem contains two types of challenge: mathematical and psychological. While the mathematical challenge is to find a solution, the psychological one is to be confident of one's solution even if it may disagree with one's intuition. An effective approach to solving the problem is to consider six faces, three black and three white, with probability 1/6 for each face.

One of the solutions, based on Bayes' theorem, is as follows: if event E is to draw a card black on both sides, and event F is to see a black face, then

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{P(F/E) \times P(E)}{P(F)} = \frac{1 \times 1/3}{1/2} = \frac{2}{3}.$$

Another possible solution is based on the formula for conditional probability and the reduced sample space for faces.

A quite popular approach employs the idea of labeling faces of three cards. If B1 and B2 are black faces on the black card, and B3 is a black face on the mixed card, then one can see that the probability of a black face being B1 or B2 is 2/3.

The Three Cards problem was offered in two introductory probability courses and a calculus-based probability and statistics course, 50 per cent of whose students were undergraduate majors in mathematics. In all three classes, 25 students in each, approximately 60 per cent of students gave the same wrong, "intuitive" answer (it was ½), and three students in each class presented the correct answer and solution.

Cognitive Analysis

In the introductory probability course, many students are future elementary school teachers. However, some of them may be placed in a category of "remedial and struggling" mathematics students with a strong math anxiety. The goal of this course project is to combine methods of challenging and collaborative learning to help students develop logical thinking and creativity, both of which are critical for future teachers.

In the course project, there was a general opinion among students that a word problem with an appealing content, even if it is difficult, is more stimulating than an easier but boring, non-contextualized problem. For instance, the Foot-and-Mouth disease problem was uptodate in its content since this disease was discussed widely in the press at that time. In addition, the result, showing that the probability that a person who tested positively has the disease is as low as 4 per cent, stimulated an active discussion about reliable interpretation of test results.

During the work on the project, students appreciated having independence and a stress-free atmosphere. All students, including individuals with a weak background, found that they benefit from solving and presenting challenging problems. Capable students, not previously identified in class, can be recognized as informal group leaders in this process.

Didactical Analysis
In the project, students were randomly divided into study groups and were asked to find challenging, curriculum-related problems, satisfying certain criteria, and present their solutions to the class. Each group was also encouraged to solve and present a second problem from the set of challenging problems offered by the instructor.

The project started with class discussion about its purpose and possible outcomes. Students had four weeks to prepare the assignment. At the end, students completed a questionnaire and evaluated the course, choice of problems, quality of presentations and effectiveness of the project.

Importantly, most of the groups selected interesting, amusing problems no matter how difficult they were, and prepared their presentations carefully. Students clearly described ideas and concepts related to each problem, accurately explained methods and formulas applied and operated validly with mathematical notions many of which they found difficult in routine class studies. The audience met presentations with a great interest; every presentation generated a number of questions and was accompanied by animated discussion. In evaluating the project overall, students found this activity stimulating and helpful for their success in the class. We believe that carefully selected challenging problems can be incorporated into the introductory probability curriculum and may be used dynamically throughout the entire course.

4.4.4 Examples: weekly problems

The problems discussed below are different from the ones discussed so far, as they were given in a class taught in English to students with a different first language. For the students in Papua New Guinea's University of Technology in Lae, English is a second or third language, so learning mathematics may require multiple translations. They may find it hard to interpret a problem, but can normally solve it once it is explained to them. The use of mathematical logic, the conversion of word problems to mathematical ones and their solution is very challenging for them (Sukthankar 1999).

In 1992, Sukthankar and her colleagues started a feature in the University of Technology weekly publication, *Reporter*, called "Fun With Mathematics," which contained mathematical quizzes. The problems were designed to create interest in mathematics and to encourage maximum participation by the appropriate provision of clues. They tried to add problems which would not only challenge students to improve their mathematical skills but also teach them how to translate word problems symbolically with proper mathematical interpretation and correct use of technical English words.

They found that a little help from the lecturers made a big difference in the number of participants who used the clues to research a particular topic and arrive at a solution. There were prizes awarded every week to the winning students. The student response to these quizzes was excellent, and consequently, the Department of Mathematics and Computer Science decided to extend this feature of mathematical quizzes to the weekly publications of other universities and tertiary institutions in Papua New Guinea. This idea led to the establishment of the Annual Mathematics Competition for all tertiary institutions in the country.

Below are five sample problems chosen from the weekly quizzes:

Problem 1: If three dice are thrown, what is the probability that the sum of numbers on the top faces is not more than 15?

Problem 2: Tim and John celebrate their birthdays today. In three years, Tim will be four times as old as John was when Tim was two years older than John is today. If Tim is a teenager, what is his age?

Problem 3: In a test given to a large group of people, the scores were normally distributed with mean 70 and standard deviation 10. What is the least whole number score that a person could get and yet score in about the top 15 per cent?

Problem 4: The numbers p, q, r, s and t are consecutive positive integers, arranged in increasing order. If $p + q + r + s + t$ is a perfect cube and $q + r + s$ is a perfect square, then what is the smallest possible value of r?

Problem 5: Sheep cost \$40 each, cows \$65 each and hens \$2 each. If a farmer bought a total of 100 of these animals for a total cost of \$3279, then how many sheep, cows and hens did he buy?

Mathematical Analysis

Methods for solving all these problems are different. The solutions involve knowledge of counting elements in the sample spaces, solutions of simultaneous equations, normal distributions, properties of prime numbers and mathematical logic.

In Problem 1, the first thing that the students needed to note was that it was much easier to count the number of sums greater than 15 than the number of sums less than or equal to 15. Then they had to ensure that no arrangement was missed or repeated while calculating the number of ways. This is a good problem in which to learn how to calculate the sample space systematically. The students always had problems understanding phrases like “more than,” “less than,” “not more than,” “not less than,” “at least,” “at most,” and so on. The solution of Problem 1 was a double challenge since it required correct interpretation and then a mathematical solution.

Students found it hard to translate the apparently confusing wording of Problem 2 into a mathematical equation in two variables. Let t and j denote the ages of Tim and John. When Tim was two years older than John is today, John’s age was less than John’s present age j by the difference between Tim’s present age and his age when he was two years older than John, namely $t - (j + 2)$. Thus, John’s earlier age was $j - (t - j - 2) = 2j - t + 2$. From the given conditions we get $t + 3 = 4\ (2j - t + 2)$, which gives us $5t = 8j + 5$. Hence $5(t - 1) = 8j$ and since Tim is a teenager, this implies that $t = 17$.

Problem 3 is a typical example of a straightforward probability problem, involving conversion of a normal distribution to a standard normal distribution and finding the probabilities using the standard normal distribution probabilities table. Although it is not so much mathematically challenging, it was a challenge for the students to interpret the problem correctly.

The solution for Problem 4 is based on properties of prime numbers. If we denote the five consecutive numbers as $n-2$, $n-1$, n, $n+1$ and $n+2$, then from the given conditions, we have to find the least n such that $5n$ is a perfect cube and $3n$ is a perfect square. The smallest n that satisfies both of these conditions has to be divisible by 5^2 and also, since 3 divides n, must be divisible by 3^3.

Hence $n = 5^2 \times 3^3$. Most students started by writing the consecutive numbers as $n, n+1, \ldots, n+4$. They soon realized that it was getting too complicated to derive from the given information $5n+10$ as a perfect cube and $3n+6$ as a perfect square, the smallest possible $n+2$. Then of course, they chose the sequence $n-2, n-1, n, n+1, n+2$ and arrived at the solution.

Problem 5 deals with properties of integers and logical elimination process which are used in simple number theoretical problems very often. A sheep costs \$40, a cow costs \$65 and a hen costs \$2. Let s, c and h be, respectively, the number of sheep, cows and hens. We note in passing that c must be odd and that the units digit of h must be 2 or 7. These facts can be used to help narrow down the search, or as a check on the answer. We have two equations

$$40s + 65c + 2h = 3279$$

and

$$s + c + h = 100.$$

Subtracting twice the second from the first gives $38s + 63c = 3079$. Since $19(2s + 3c) + 6c = 19(162) + 1$, we see that $6c - 1$ must be divisible by 19. Hence, c must have a remainder 16 when divided by 19. Since c is odd and $63c < 3079$, we must have $c = 35$. This quickly leads to $s = 23$ and $h = 42$. The same problem can also be solved in many ways using congruence and divisibility properties.

Cognitive Analysis

The students knew how to calculate the number of ways to get a particular sum with a two dice problem, but to extend to three dice was challenging for most of them. To find the number of ways to get a sum equal to 8 on two dice A and B, some students took the following systematic approach of listing combinations like (6,2), (5,3), (4,4), (3,5) and (2,6). This method uses an approach of starting with a 6 on the first die and then decreasing the numbers to 5, 4, 3 and 2 and getting the appropriate numbers on the second die to make the sum 8. By this method, no combination is missed and all possible combinations are counted.

The same idea is extended for the three dice problem. By the same method, it was easy to calculate the 10 combinations that give the sum 15 on three dice: (6,6,3), (6,5,4), (6,4,5), ... Not all students could think of this combinatorial method. Instead, some tried to pick up the combinations giving the sum 15 at random and had no reliable way of checking whether they had considered all the possibilities. Once shown how to arrive at a definite answer by a systematic counting approach, they appreciated the combinatorial method.

The phrase "not more than 15" was confusing for some students. We normally do not notice this problem with students who have English as their mother tongue. They were not sure whether the expression, "the sum on the top faces of three dice not more than 15," meant $3 \leq \text{sum} \leq 14$ or 15!

In Problem 2, one needs to interpret carefully the verbal expression into an equation. Students needed help to get the equation $5(t-1) = 8j$. Some could easily derive the answer $t = 17$ from the equation since Tim is a teenager and 8 has to divide $t-1$.

For solving problems on normal distribution, it should be noted that a problem of the following type was easier for students to solve:

"If X is normally distributed with mean $\mu = 16$ and standard deviation $\sigma = 4$, find the probability P $(X < 10)$."

However, it would take them longer to solve if it was worded as follows:

"The weekly salaries of 5000 employees of a large corporation are assumed to be normally distributed with mean \$640 and standard deviation \$56. How many employees earn less than \$570 per week?"

The students had been exposed to solving quizzes involving elementary number theory, geometry and mathematical logic. Therefore more than half of them could solve Problem 4 correctly. For the rest, it was challenging but within their reach!

Students solved problem 5 in different ways. It is a good exercise to find the properties of numbers using mathematical logic. By observing carefully the costs of each sheep, cow and hen, the number of animals and the total cost, the method reduces the number of choices for integers to be the number of sheep, cows and hens bought totaling to 100 with total cost \$3279.

Didactical Analysis

An important outcome of these efforts was that students realized that solving math problems could be fun. They were involved in small study groups and had personal consultations with lecturers. They felt that they were enjoying mathematics as a subject and it is not as intimidating as they had earlier thought.

We also used this opportunity to concentrate a bit more on our female students and find out the reasons for their lack of active involvement in classroom mathematics learning. Mathematics is still regarded as a male subject, especially in Papua New Guinea. Boys always dominated classroom discussions and were expected to do better in education than girls. Girls have almost never taken part in any mathematical discussion and for most of the time were silent listeners. They seemed to lack the ability to initiate any mathematical activity (Sukthankar and Wilkins 1998).

During an academic semester in 1997, ten first-year female students from the University of Technology who identified themselves as having low self-concept in their ability to learn mathematics were studied (Sukthankar and Wilkins 1998). During the first half of the semester, they were interviewed and their performance was closely monitored as well as their manner of study and classroom participation.

Then in the second half, they were especially encouraged to participate in the weekly mathematics quizzes; their lecturers also gave them additional help. We learnt from the interviews that the use of computer algebra systems for their course work and understanding of mathematical concepts was beneficial. They were also given extra help to prepare for the term tests. After the tests, their strength and weaknesses were discussed and they were appropriately tutored. They were also given special problem solving sessions and were encouraged to enroll for the Annual Mathematics Competition. We found that over the period of almost six months, there were some positive changes in their attitude towards mathematics. They participated in the Annual Mathematics Competition. This time we found that almost all of them were very enthusiastic to compete, there was an urge to do better and their final results were very considerably beyond their expectations. During the interviews, we found that the main cause of their inability to do mathematics was deeply rooted in the social and cultural factors of their society. Overall, they felt incompetent and had a low self-esteem, and could not see the relevance of studying higher mathematics once they could do basic mathematics. A change in attitude improved their performance and as a result they felt more confident to take up higher studies.

4.5 Example: the challenge of a contradiction and schema adjustment

As for the examples presented in the previous section, the presentation of the following example results from considered reflection on practice.

4.5.1 Inconsistency, contradiction and cognitive development

In addition to developing schemas, it is important to ensure a certain flexibility and richness in a learner's overall schema system. A poor or rigid schema system may force a problem solver to use a very specific representation, and as a consequence, to choose a non-optimal or inadequate solution method or approach. One example of this is the so-called *Einstellung* effect or mechanization of thought, when a solver, based on her repetitive practices, forms a certain stereotype and tends to use the same method again and again without noticing a novel element that critically changed the situation.

For instance, when asked to find the area of a right triangle with hypotenuse equal to 12 units and altitude drawn to the hypotenuse equal to 7 units, a solver uses the usual area formula 12(7)/2 without noticing that a right triangle with such measurements simply does not exist! (Applebaum and Leikin 2007). Familiarity with inconsistent questions can cause a solver to focus in the future on making sense of the given information before proceeding towards a response or conclusion. Checking data for consistency should be completed prior to selecting a formula or solution method.

A flexible schema can lead to efficiency. For example, it is inefficient to solve the quadratic equation, $x^2 - 123456790x + 123456789 = 0$, by calculating the discriminant and using the standard formula for roots. If one observes that the constant term is just one less than the middle coefficient, one can use the Viete theorem or the factor theorem to obtain the answer immediately without calculation.

A novice problem solver can easily overlook a trap offered by a problem. In contrast, an expert's schema system includes, besides methods and procedures, possible error and verification techniques that make use of multiple representations and often prevent the solver from using flawed reasoning and making false statements.

In the rest of this section, we illustrate how inconsistent and contradictory propositions can be used for further development of a learner's cognitive system.

Based on her practices, a learner forms a set of domain-specific expectations about the nature of problems and statements. She develops ways to judge and form an opinion about what is likely to be true and what is not. Often, a statement that surprises a learner or challenges her expectations will stimulate the whole process of understanding the subject. It may also help to break the learner's stereotypes and uncover clarity in the realm of explicit rules and formal theories.

Say, for instance, one is able to illustrate that $1 = 2$ by certain mathematical manipulations and reasoning. The problem then becomes one of locating the error (logical, algebraic or arithmetic) that leads one to the impossible outcome. Notice that the main psychological feature of the situation that distinguishes it from other forms of intellectual inquiry is the presence of the appealing voice of the problem, the voice that essentially passes the ownership of the question directly to the learner. The very fact of the impossibility of the conclusion forces the learner to search for an inconsistency in the reasoning apparently accepted as truthful just a moment ago. The following problems illustrate the situation.

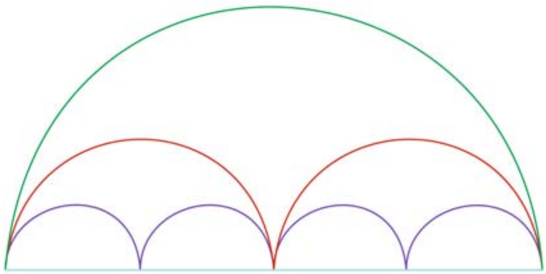
4.5.2 *What do you do if you have to prove that 1 = 2? and other paradoxes*

This section gives four paradoxical problems of different nature. They are followed by a short comment about mathematical reasons and instructional implications.

Problem 1: Consider the following algebraic derivation:

1. Let $a = b$.
2. Then $a^2 = ab$.
3. Then $a^2 - b^2 = ab - b^2$ or, equivalently, $(a + b)(a - b) = b(a - b)$.
4. Then $a + b = b$.
5. Since $a = b$ due to step 1, we have $2b = b$.
6. Thus, $2 = 1$.

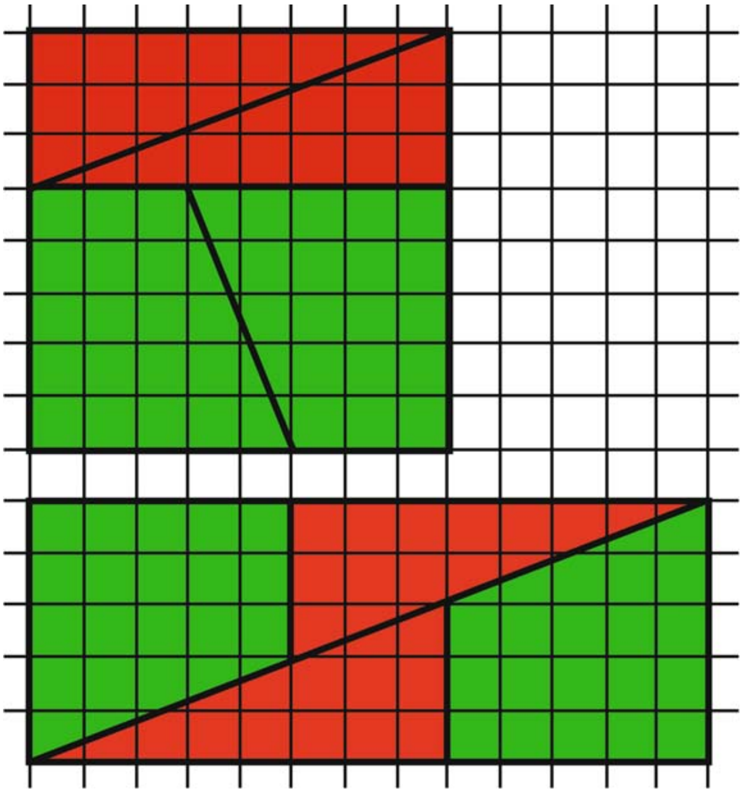
Problem 2: Draw a semicircle of diameter 2. Then draw two semicircles, one on each of the halves of the diameter. Then draw four semicircles, one on each quarter of the new diameters, and so on.



Note that the length of the very first semicircle is π, and so is the sum of the lengths of the next two semicircles, as well as the sum of the next four. One can reason then in fact it will remain true for any positive power, n, of 2. On the other hand, when the power n is getting larger and larger, the curve consisting of 2^n semicircles gradually approaches the segment of length 2. This apparently proves that $\pi = 2$.

Problem 3: Three traveling salesmen have car trouble and are forced to spend the night at a small town inn. They go in and the innkeeper tells them, “The cost of the room is \$30”. Each man pays ten dollars and they go up to the room. The husband of the innkeeper says to her, “Did you charge them the full amount? Why not give them five bucks back since their car is broken and they hadn’t planned to stay here.” She then brings the men five one-dollar bills and each man takes one while the other two dollars rest on the table. Originally each man paid ten dollars: $10 \times 3 = 30$; now each man has paid nine dollars $9 \times 3 = 27$ and there are two dollars sitting on the counter: $27 + 2 = 29$. The last dollar has disappeared. (Note that this problem was also used as an example in Chapter 1.)

Problem 4: Areas paradox: this figure "proves" that $64 = 65$.



The tasks presented in this section may be viewed as illustrations of challenging conceptual tasks in the sense that Kadijevich (1999) describes.

4.5.3 Brief comments on the paradoxes in Problems 2 to 4

Problem 2 is deeper and trickier than Problem 1. It appears in the framework of real analysis, and leads to the old philosophical questions. Does a segment consist of a collection of points? Is a point just a circle with radius zero? How do we justify the limit whenever such an operation appears in our reasoning? What is convergence and why do we talk about different types of convergence?

Now, Problem 3 perfectly illustrates the joke about the existence of three kinds of people: those who can count and those who can't. The resolution turns on properly allocating the amounts. The \$27 consists of the \$25 kept by the innkeeper and the $2 not returned to the men. It does not include the $3 given back to the men.

Finally, Problem 4 is an interesting visual illusion. If one looks at the side lengths of the triangles and rectangles involved in the figure, one notices that they are 3, 5, 8, 13, some of the Fibonacci numbers $F_{k+1} = F_k + F_{k-1}, k > 1$. It is remarkable that the illusion is based on the property of Fibonacci numbers $\left|F_k^2 - F_{k+1}F_{k-1} = 1,\right|$ which implies smallness of the difference of the slopes, $\left|\frac{F_k}{F_{k-1}} - \frac{F_{k+1}}{F_k}\right|$, especially if one pick large values of k.

4.5.4 Analysis of Problem 1

Mathematical Analysis

The algebraic expressions $AX = BX$ and $A = B$ are equivalent only if X is not 0. In our example, $X = a - b$ is zero since $a = b$. Thus reduction from $AX = BX$ to $A = B$ is not possible. Since a forbidden step was made (passing from line 3 to 4) a contradictory conclusion occurs. Note that the reduction from line 3 to 4 still makes sense if $a = b = 0$. But then the reduction from line 5 to 6 is not possible for the same reason.

Cognitive Analysis

If $A = B$ then $AX = BX$ for all X. Students often tend to mistakenly treat an implication (if-then statement) as an if-and-only-if statement. Thus the reduction from $AX = BX$ to $A = B$ could be taken mistakenly as an equivalent to the initial one.

The reduction $AX = BX$ to $A = B$ works in the majority of cases (all but $X = 0$). Students tend to ignore this special case and proceed formally. If an algebraic example is relatively long and the student is relatively new to the activity, she would tend to follow the main route and ignore the rare case. Her working memory would be occupied by other tasks such as factoring and the assumption that this case could be temporarily put aside. At the moment of reduction the joy of finding similar factors on both sides of the equation dominates the fact that this factor is equal to zero.

Technically, the students know about the rule that division by zero is not allowed. However, it is often a dead rule, one on a list of other rules. Students may accept it formally and easily overlook it in practice. When the paradox is demonstrated and the contradiction in line 6 reveals itself, the student tries to find why the contradiction occurs. She knows that 2 is not equal to 1, and that forces her to resolve the contradiction, to find where something went wrong. The fact that there are only six lines supports her hope for success. A paradox presents a kind of self-appealing (self-contained) challenge.

Compared to an algebraic exercise that just requires simplifying an expression, this one, which leads to a contradiction, provides a motivation to check the derivation and locate the mistake. When a student finds that violation of a certain rule leads to a contradiction, the student gets to understand the reason behind why the rule is worth remembering and obeying. This example illustrates how a paradox serves as a disequilibrator of learner's schemas, and how understanding of an algebraic rule develops from rethinking (restructuring) a schema.

Didactical Analysis

The mathematical challenge of Problem 1 may be given to students familiar with algebraic derivations. Students with experience and success in similar algebraic problems are expected to be able to resolve the contradiction. The fact of the contradiction is obvious. To ensure that the whole problem belongs to the ZPD (Zone of Proximal Development (Vygotsky 1978, also see Section 6.2.2.3)) of a learner, the teacher provides sufficient training in algebraic reductions and makes sure that students can do and check their algebra. Some students will tend to substitute numbers in place of letters to check the derivations. This is a possible approach as long as the student does algebraic substitution consistently.

In the experimental setting (Kondratieva 2007), the paradoxes from Problems 1 and 4 were given to first- and second-year university students to be resolved in class during a ten-minute period. The students were not tested nor taught algebra or geometry immediately prior to the task since they had all passed a placement test, and therefore, it was implicitly assumed that they had already been trained in the subjects. The experiment showed that:

1. Everyone was intrigued and motivated by the contradictions.
2. Not everyone was able to find the reasons for the contradictions. Some students were able to locate the wrong line in Problem 1, but no clear explanation was given. Even fewer students were successful with Problem 4.
3. Good students found that the problems were not difficult but were nevertheless interesting. They said that they learned to stay alert while doing formal derivations or trusting a pictorial proof.
4. Some students composed their own examples of paradoxes using similar ideas. Such a task was not assigned, and the fact that they did so voluntarily illustrates an important human tendency to mimic-and-create during the process of acquisition of new knowledge.

4.5.5 Concluding remarks

While there are different levels of difficulties in the apparent contradictions we have considered, they all have in common the intrinsic call for a resolution, when, rephrasing Aristotle's metaphorical idea, the mind experiences itself in the act of making a mistake. And then it makes sense.

The role and place of paradoxes in the process of cognitive development can be identified within Piaget's theory of equilibration, which refers to the Kantian epistemological proposition that the knower constructs her knowledge of the world. Paradoxes disequilibrate a learner's schemas, and that is the starting point of the process of accommodation of a portion of new information. Then the learner will go through stages ranging from "beyond belief" to acquisition of knowledge with justification.

If we want students to learn how to verify and validate their solutions and to critically read others' work, we need to familiarize them with situations involving contradictions and paradoxes. They then need to know how to handle such situations and how to analyze and arrive at possible resolutions and explanations. That is why an exposition of paradoxes is so valuable.

We conjecture that the phenomenon of paradoxes drives the whole of human intellectual development because the challenge of a contradiction is the mainspring of learning on both individual and historical levels. Therefore, these types of challenges cannot be ignored but instead need to be carefully analyzed and promoted as instructional tools inside and beyond the classroom.

4.6 Conclusion

The preceding examples of challenging mathematics problems together with the descriptions and analyses of students' responses to them illustrate that students benefit socially and cognitively from engagement with challenging problems. The qualitative analyses suggest that the gains are evident in the short term and are intellectually important over time. Students build adequate and sophisticated strategies to solve challenges.

From a cognitive perspective, through meaningful engagement over time with problems within a strand of mathematics, students build effective and important problem-solving schemas. They develop insights into the mathematical structure of related problems and this knowledge becomes schematized. Moreover, students need to develop flexible schemas since rigid ones may inadvertently cause a problem solver to choose a non-optimal or inadequate solution method or approach. Resolving inconsistent and contradictory propositions or paradoxes can support the development of flexible schemas.

Most research on schema construction has been done using traditional psychological paradigms, investigating how and (more often) to what extent individuals can construct and apply schemas in a short period of time. The research of Weber et al. (2006), which forms the basis of the examples in Section 4.3, differs from this paradigm significantly, looking at how students developed schemas over time, all the while solving challenging problems. They believe that this change in perspective radically altered the nature of their findings. If their research participants were given straightforward problems, they would not have had the need to develop the useful representations for these problems that were critical for their schema construction.

If they were only given a short period of time to explore these problems, the schemas also would likely not have been constructed. In fact, students initially did not see the deep connections between the various problems on which they worked. Hence, looking at the processes that individuals use to form and use schemas in relatively short periods of time is looking at only a subset of the processes used in this regard. The work of Weber et al. (2006) demonstrates that studying the way that students solve challenging strands of problems over longer periods of time provides a more comprehensive and useful look at how students can construct and use problem-solving schemas.

As this chapter has illustrated, challenging mathematics problems are suitable for a range of audiences and didactical situations. They are apt as interview questions for entrance into university mathematics programs to obtain windows into how students think mathematically; as investigations for teacher candidates to further develop their own mathematical understanding and to acquire insight into how learners learn mathematics; as supplements to or material integrated throughout a course; as a means to reinsert marginalized students into mathematics, providing them with a context with which to entertain their minds; and, by placing mathematical challenges in a university's daily or weekly newspaper, as vehicles to popularize and create interest in mathematics among students studying the subject in a language other than their own.

Challenging mathematics problems can be instruments to stimulate creativity, to encourage collaboration and to study learners' untutored, emergent ideas. We have also shown that they are appropriate for secondary and post-secondary students as well as for high-achieving and low-achieving learners. From a didactical perspective, it is important that the problems require little specific background and generally can be attempted successfully by students of varying mathematical backgrounds.

Economic and social capital need not be markers of who can participate in mathematics. In Fioriti and Gorgorió (2006), from which the example in Section 4.4.2 is excerpted, the authors indicate how it is possible to engage socially excluded youngsters with challenging mathematics problems so that they are reinserted into school settings and thereby widen their possible social and academic participation in their society. Clearly, there are a host of socio-economic realities that need to be addressed to truly democratize academic and social participation. However, engaging students of diverse backgrounds in challenging mathematics problems contributes to this larger goal.

Making mathematics less exclusionary and more inclusive depends on shifting from traditional pedagogies and procedural views of mathematics learning (Boaler and Greeno 2000). It requires reversing a common belief among teachers that higher-order thinking is not appropriate in the instruction of low-achieving students (Zohar et al. 2001). If challenging mathematics problems were used in settings such as formal classrooms and other informal arenas, learners might begin to recognize mathematics as accessible and attractive (cf. Zohar and Dori 2003). They would have opportunities to build mathematical ideas and reasoning over time, develop flexible schemas and inventive problem-solving approaches, and become socialized into thinking mathematically.

As Resnick (1988) suggests: "If we want students to treat mathematics as an ill-structured discipline [that is, one that invites more than one rigidly defined interpretation of a task]—making sense of it, arguing about it, and creating it, rather than merely doing it according to prescribed rules—we will have to socialize them as much as to instruct them. This means that we cannot expect any brief or encapsulated program on problem solving to do the job. Instead, we must seek the kind of long-term engagement in mathematical thinking that the concept of socialization implies". (p. 58)

If mathematics educators and teachers adopt a long-term perspective on the development of problem-solving schemas, then a paramount goal of mathematics education—to further learners' effective problem solving—would be more achievable.

References

Anderson, J.R., Reder, L.M., Simon, H.A.: Situated learning and education. Educational Researcher **25**, pp. 5–11 (1996)

Applebaum, M., Leikin, R.: Looking back at the beginning: Critical thinking in solving unrealistic problems. The Montana Mathematics Enthusiast **4**, 2, pp. 258–265 (2007)

Boaler, J., Greeno, J.G.: Identity, agency, and knowing in mathematics worlds. In: Boaler, J. (ed.) Multiple perspectives on mathematics teaching and learning, pp. 171–200. Ablex, Westport (2000)

Bourdieu, P.: The forms of capital. In: Richardson J. (ed.) Handbook of theory and research for the sociology of education, pp. 241–258. Greenwood, New York (1986)

Cai, J., Lester, Jr., F.K.: Solution representations and pedagogical representations in Chinese and U.S. classrooms. Journal of Mathematical Behavior **24**, 3/4, pp. 221–237 (2005)

Chi, M.T.H., Feltovich, P.J., Glaser, R.: Categorization and representation of physics problems by experts and novices. Cognitive Science **5**, 2, pp. 121–152 (1981)

D'Ambrosio, U.: Etnomatemática: Elo entre as tradições e a modernidade [Ethnomathematics: Link between tradition and modernity]. Autêntica, Belo Horizonte, Brazil (2001)

Davis, R.B., Maher, C.A.: How students think: The role of representations. In: English, L. (ed.), Mathematical reasoning: Analogies, metaphors, and images, pp. 93–115. Lawrence Erlbaum Associates, Hillsdale, NJ (1997)

De Corte, E., Greer, B., Verschaffel, L.: Mathematics teaching and learning. In: Berlin, D.C., Calfee, R.C. (eds.) Handbook of educational psychology, pp. 491–549. Macmillan, New York (1996)

Fioriti, G., Gorgorió, N.: Mathematical challenge to widen excluded students' foreground. Submitted paper, ICMI Study 16 (2006)

Francisco, J.M., Maher, C.A.: Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. Journal of Mathematical Behavior **24**, 3–4, pp. 361–372 (2005)

Hayes, J.R.: The complete problem solver. 2nd edn. Lawrence Erlbaum, Hillsdale, NJ (1989)

Hinsley, D., Hayes, J.R., Simon, H.A.: From words to equations: Meaning and representation in algebra word problems. In: Carpenter, P., Just, M. (eds.) Cognitive processes in comprehension. Lawrence Erlbaum, Hillsdale, NJ (1977)

Kadijevich, D.: Conceptual tasks in mathematics education. The Teaching of Mathematics **2**, 1, pp. 59–64 (1999)

Kiczek, R.D., Maher, C.A., Speiser, R.: Tracing the origins and extensions of Michael's representation. In: Cuoco, A.A., Curcio, F.R. (eds.) The role of representation in school mathematics, pp. 201–214. National Council of Teachers of Mathematics, Reston (2001)

Kondratieva, M.: Understanding mathematics through resolution of paradoxes. Mediterranean Journal for Research in Mathematics Education **6**, pp. 127–138 (2007)

Lakatos, I.: Proofs and refutations: The logic of mathematical discovery. Cambridge University Press, New York (1976)

Lobato, J., Siebert, D.: Quantitative reasoning in a reconceived view of transfer. Journal of Mathematical Behavior **21**, 1, pp. 87–116 (2002)

Maher, C.A.: How students structure their investigations and learn mathematics: Insights from a long-term study. The Journal of Mathematical Behavior **24**, 1, pp. 1–14 (2005)

Maher, C.A., Martino, A.M.: The development of the idea of mathematical proof: A 5-year case study. Journal for Research in Mathematics Education **27**, 2, pp. 194–214 (1996)

Marshall, S.P.: Schemas in problem solving. Cambridge University Press, New York (1996)

Mayer, R.E.: Thinking, problem solving, cognition. 2nd edn. Freeman, New York (1992)

Mayer, R.E., Wittrock, M.C.: Problem-solving transfer. In: Berlin, D.C., Calfee, R.C. (eds.) Handbook of educational psychology, pp. 47–62. Macmillan, New York (1996)

Nickerson, R.: Some instructive problems: Three cards. In: Nickerson, R. Cognition and Chance: The psychology of probabilistic reasoning, pp. 157–160. Lawrence Erlbaum, Mahwah, NJ (2004)

Novick, L.R., Holyoak, K.J.: Mathematical problem solving by analogy. Journal of Experimental Psychology: Learning, Memory, and Cognition **17**, 3, pp. 398–415 (1991)

Nunokawa, K.: Mathematical problem solving and learning mathematics: What we expect students to obtain. Journal of Mathematical Behavior **24**, 3–4, pp. 32–340 (2005)

Owen, E., Sweller, J.: What do students learn when they solve mathematical problems? Journal of Educational Psychology **77**, 3, pp. 272–284 (1985)

Powell, A.B.: "So let's prove it!": Emergent and elaborated mathematical ideas and reasoning in the discourse and inscriptions of learners engaged in a combinatorial task. Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick (2003)

Powell, A.B.: Socially emergent cognition: Particular outcome of student-to-student discursive interaction during mathematical problem solving. Horizontes **24**, 1, pp. 33–42 (2006)

Powell, A.B., Frankenstein, M. (eds.) Ethnomathematics: Challenging eurocentrism in mathematics education. State University of New York, Albany, New York (1997)

Reed, S.K.: Word problems: Research and curricula reform. Lawrence Erlbaum, Mahwah, NJ (1999)

Resnick, L.: Treating mathematics as an ill-structured discipline. In: Charles, R.J., Silver, E. A. (eds.) The teaching and assessing of mathematical problem solving, pp. 32–60. National Council of Teachers of Mathematics, Reston (1988)

Schoenfeld, A.H.: Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In: Grouws, D.A. (ed.) Handbook of research on mathematics teaching and learning, pp. 334–370. Macmillan, New York (1992)

Schoenfeld, A.H., Hermann, D.J.: Problem perception and knowledge structure in expert and novice mathematical problem solvers. Journal of Experimental Psychology: Learning, Memory, and Cognition **8**, 5, pp. 484–494 (1982)

Smullyan, R.M.: Lady or the tiger? and other logic puzzles, including a mathematical novel that features Gödel's great discovery. Knopf, New York (1982)

Sukthankar, N.: Cultural factors in mathematics education contributing to the shortage of women professionals in Papua New Guinea. Educational Review **51**, 2, pp. 173–181 (1999)

Sukthankar, N., Wilkins, N.: Why are we dumb in mathematics? A qualitative case study. University of Technology Science Reviews **2** (1998)

Uptegrove, E.B.: To symbols from meaning: Students' long-term investigations in counting. Unpublished doctoral dissertation, Rutgers, The State University of New Jersey, New Brunswick (2004)

Vygotsky, L.S.: Mind in society: The development of higher psychological processes (Cole M., John-Steiner V., Scribner S., Souberman, E. trans.). Harvard University Press, Cambridge, MA (1978)

Weber, K.: Student difficulty in constructing proofs: The need for strategic knowledge. Educational Studies in Mathematics **48**, 1, pp. 101–119 (2001)

Weber, K.: Problem-solving processes, proving, and learning. Journal of Mathematical Behavior **24**, 3/4, pp. 351–360 (2005)

Weber, K., Maher, C., Powell, A.B.: Strands of challenging mathematical problems and the construction of mathematical problem-solving schema. Submitted paper, ICMI Study 16 (2006)

Zevenbergen, R.: "Cracking the code" of mathematics classrooms: School success as a function of linguistic, social, and cultural background. In: Boaler J. (ed.) Multiple perspectives on mathematics teaching and learning, pp. 201–223. Albex, Westport, CT (2000)

Zohar, A., Dori, Y.: Higher order thinking skills and low-achieving students: Are they mutually exclusive? Journal of the Learning Sciences **12**, 2, pp. 145–181 (2003)

Zohar, A., Vaaknin, E., Degani, A.: Teacher's beliefs about low achieving students and higher order thinking. Teaching and Teacher Education **17**, pp. 469–485 (2001)