GEOMETRICAL PROOFS, BASIC GEOMETRIC CONFIGURATIONS AND DYNAMIC GEOMETRY SOFTWARE

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Abstract

Knowing geometrical facts and understanding their proofs are of equal importance for learning geometry. One way to target both goals is to study basic geometric configurations (BGCs). BGC is a geometrical figure that depicts a statement along with auxiliary elements pertinent to its proof. Benefits of using BGCs in teaching geometry can be enhanced by employment of applets produced with a dynamic geometry software. We discuss innovative practices that apply this idea to teaching proofs at the university level. A sample classroom scenario illustrating our approach is presented. Results of a survey of 13 students who had taken the course highlight a potential of this practice and suggest directions for further research.

Key words/ themes: promoting conceptual understanding of mathematics through explorations, visual proof, computer technologies for learning/doing mathematics.

Motivation and overview of our innovative practices

This paper concerns innovative practices and development activities in teaching Euclidean Geometry at the undergraduate university level. As a consequence of the current grade school mathematics curriculum, students enter this course with a limited amount of geometrical knowledge often restricted to a list of formulas and facts for certain geometrical figures such as the triangle and circle. Several students have some familiarity with formal deduction (e.g. two column proofs), but they often lack understanding of the meaning of this process. Thus the university course aims to give students a richer background and experiences in geometry as well as allowing students to understand the essence of proving practice. Learning to proof is vital for learning mathematics (Rav 1999; Balacheff 2010). First, proof as a means of validation, reinforces a precise and highly logical way of thinking based on axioms, definitions, and statements, which link and describe the properties of mathematical objects. Second, proofs include mathematical methods, concepts, and strategies also applicable in problem solving situations (Hanna & Barbeau 2010). Despite their central role in mathematics proofs often receive insufficient appreciation and epistemological understanding from students, who rely on empirical evidence rather than on formal deductions of mathematical theorems (Coe & Ruthven 1994). "Pupils fail to appreciate the critical distinction between empirical and deductive arguments and in general show a preference for the use of empirical argument over deductive reasoning." "Proof is not 'used' as a part of problem-solving and is widely regarded as an irrelevant, 'added-on' activity" (Hoyles & Jones 1998, p. 121).

This state of affairs identifies the needs for "mathematical activities that could facilitate the learning of mathematical proof" (Balacheff 2010, p. 133) and "problem situations calling for an interaction between visual methods and geometrical methods" (Laborde 1998, p.114). One possible approach "is centered around the idea that inventing hypotheses and testing their consequences is more productive ... than forming elaborate chain of deductions" (Jahnke 2007, p.79). The process of making conjectures and inventing hypotheses requires mathematical intuition, which develops through students' experiences not only in formal logical manipulations but also in experimental explorations of objects and ideas (De Villiers 1990). At the same time, "current curricular trends, promulgating proving processes based on experimentation and conjectures, will lead to an effective learning of proof, with proof attaining its full meaning in the learners' understanding only if these processes are set within a genuine process of building 'small theories'... From these axioms/hypotheses would be elaborated hypothetico-deductive networks, which would then be confronted with the initial experimentations and conjectures" (Tanguay & Grenier 2010, p.41). A productive way of incorporating experimentation and proving needs to be found so that "*proofs do not replace measurements but make them more intelligent*" (Janhke 2007, p.83). This is possible due to several roles (besides validation statements) that proofs may play in mathematical thinking (Hanna 2000; De Villiers 1990). First, at the informal deduction stage, proof as explanation of empirical observations is most appropriate. Next, students "should build a small network of theorems based on empirical evidence" and become accustomed to "*hypothetico-deductive method* which is fundamental for scientific thinking" (Jahnke 2007, p.83). At this stage, the proof functions as a "*systematization* (the organization of various results into a deductive system)" and "*construction* of an empirical theory", preparing students for rigorous proofs aiming at establishing truth by deduction or "*incorporation* of well-known facts into a new framework" (Hanna 2000, p. 8).

My approach to teaching Euclidean Geometry emphasizes the use of *basic geometric configurations* (BGCs) - fundamental geometric facts expressed in drawing (Kondratieva 2011). Such drawings contain auxiliary elements and labels (equal angles, equal segments, perpendicular and parallel lines) that allow remembering the statements along with the ideas of their proofs. BGCs are the stepping stones to proving or solving geometric problems. But figures support visual thinking only if a learner grasps the mathematical structure they represent (Arnheim 1969). This calls for the following teacher's actions : (1) Asking students to explain what relations they observe in a figure and what they think about the role of the auxiliary lines drawn on the original figure. (2) Constantly showing connections to already learned geometrical facts and focusing students' attention on the key ideas used in a particular solution. (3) Demonstrating several proofs or solutions of the same problem in order to show connections between geometry, trigonometry and algebra. (4) Directing students' attention to the implications, converse and equivalence of statements. (5) Helping students summarize their findings in the form of a mathematical statement. (6) Surprising students with an unexpected conclusion or asking them to correct errors in a flawed reasoning (Kondratieva 2009, 2011).

Taking into account that "computers can offer a new context for designing innovative activities to address the main problem of linkage between empirical experiments and deductive reasoning" (Osta 1998, p 111), I introduce my students to dynamic drawings (applets) produced in GeoGebra (GG). Several types of tasks helping to connect visual evidence and geometrical facts can be considered: (i) moving from a verbal description on a geometrical figure to a drawing; (ii) explaining the behavior of drawings in geometrical terms (interpreting or predicting); and (iii) reproducing a drawing or transforming a drawing (Laborde 1998, p. 115). The students employ my GG applets and create their own drawings that help them to understand and interpret BGCs. Many BGCs allow "dynamical visual proofs, which are based on 'drawing in movement' that can be properly performed in a dynamical environment" (Gravina 2008).

The novelty of this approach consists of combining the methodology of the BGC approach with the advantages offered by the geometry software, in order to balance empirical and deductive practices. First, students read and analyze sample proofs and identify BGCs and key ideas pertinent to the proofs. At the same time students construct interactive applets in GG with the requirement to make the constraints described in the statement indestructible by dragging. This forces them to use geometrical properties of the object they draw. Students are asked to show auxiliary lines and measurements pertinent to the idea of the proofs. Students are encouraged to invent alternative proofs to the statements they analyze and interpret with help of GG. Students are given examples of all these activities in class. They discuss BGCs with their teacher using both static and dynamic drawings. As the semester evolves, the students are provided with fewer hints for problems and are asked to continue building GG applets and experiment with them in order to find their own solutions. In this way students gradually adopt the Euclidean (synthetic) geometry tradition of proofs and learn to recognize and apply BGCs. The students learn to observe and explain individual empirical facts, then build, and check their "small theories" based on many dynamic and static drawings.

The six point circle theorem in the classroom

In this section we illustrate our approach with a concrete example of a study of the *six point circle theorem*. This theorem states that: *"For a triangle ABC with midpoints on the sides L, M, and N and the feet of the altitudes D, E, and F, all lie on a circle."*

Below we give a sample dialog between a teacher and a small class of undergraduate students equipped with dynamic geometry software (DGS).

Teacher: Good morning, class! Do you know how many circles one can draw given 3 arbitrary points?

Anna: I think only one. Yes, you can always draw a circle through 3 points, but only one.

Teacher: Everybody agree?

Nick: I agree. But the points must not be collinear; otherwise you can't draw a circle.

Anna: Oh, I forgot this exception. But other than that...

Teacher: Can you explain why?

John: One can construct the center of such a circle and thus find the circle. We did it last time. It's the intersection of perpendicular bisectors of the sides of the triangle with vertices at given 3 points. This is a basic geometric configuration we constructed at home in Geogebra (see Figure 1).

Matthew: I also noticed that Geogebra has a build-in function "Circle through Three Points" that allows drawing a circle by 3 points.

Teacher: Very good. I hope everyone made the applet and recalls this fact now. Today we are going to experiment with other construction. Please draw a triangle ABC. Find the midpoints of the sides and call them L, M, N. Draw a circle through L, M and N. Draw an altitude CF with foot F on AB. Drag vertices of the triangle. What do you notice?

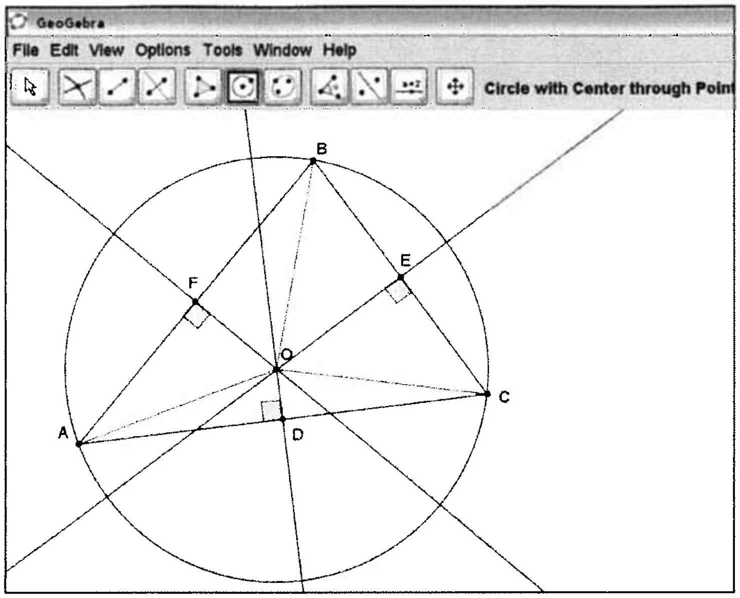

Figure 1: Basic geometric configuration showing that circum-center O lies at the intersection of perpendicular bisectors of the sides of a triangle.

Students follow teacher's instruction and use various build-in functions such as "Midpoint or Center", "Perpendicular Line", "Circle through Three Points" to create their applets. Students are required to obey all given constrains and then experiment with their drawings and report their observations.

Kelly: I think point F always lies on the circle defined by the midpoints L, M, and N.

Teacher: It looks like that, indeed. Now we have to either explain this fact or find a situation where this is no longer true.

The students play with their applets for some time. Often, students quickly see the fact and get more and more convinced by viewing results of dragging points. At the same time they may have difficulties to find an explanation of the observed geometrical behavior. When Teacher notices that no reasonable explanation emerges from this activity, she decides to give students a hint in the form of her own applet, prepared for this lesson. This applet embraces a BGC because it represents a statement and an idea of its proof. The goal of the teacher is to guide the students letting them to see this idea and articulate it in their own words.

Teacher: Please take a look at my drawing (Figure 2). What do you observe?

Mathew: I see a quadrilateral with vertices at the points of interest. Should we try to show that this quadrilateral is always cyclic?

Teacher: Perhaps. What can you say specifically about this quadrilateral?

Nick: It is a trapezoid. The line MN, connecting midpoints of the two sides of the triangle, is parallel to the third side of this triangle.

Teacher: Yes, good. What do we need to prove now about this trapezoid if we had to show that it is cyclic?

Kelly: To be cyclic it needs to be isosceles. But is it?

Teacher: Let's look at the figure and find out.

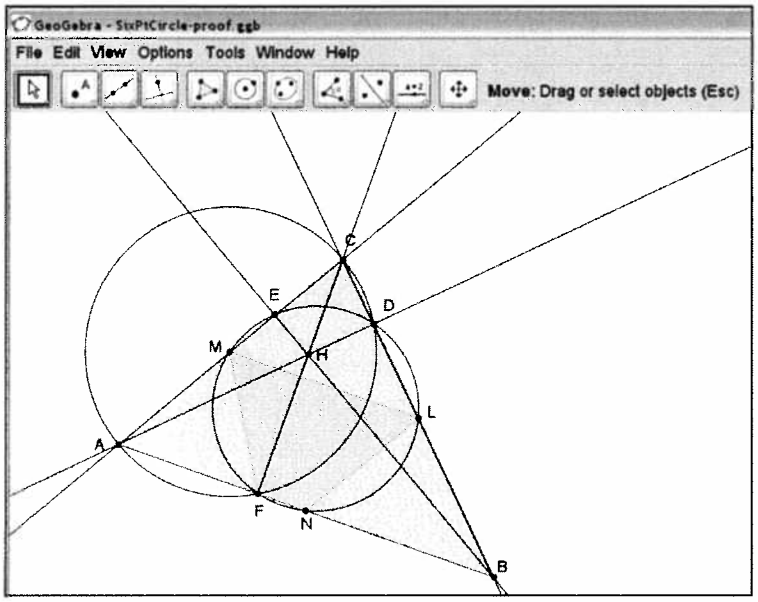

Figure 2: The six point circle theorem: case of acute triangle.

John: Since segment LN is a midline in triangle ABC, its length is a half of length of the side AC. Now what about MF? On your hint-figure it looks like MF is a radius of the circle with diameter AC. If this is the case, then we are done.

Nick: Right. Point F is the foot of the altitude CF, so CFA is 90 degrees angle and thus the hypotenuse CA is a diameter of the circum-circle. This is a BGC we recently learned. (See Figure 3.)

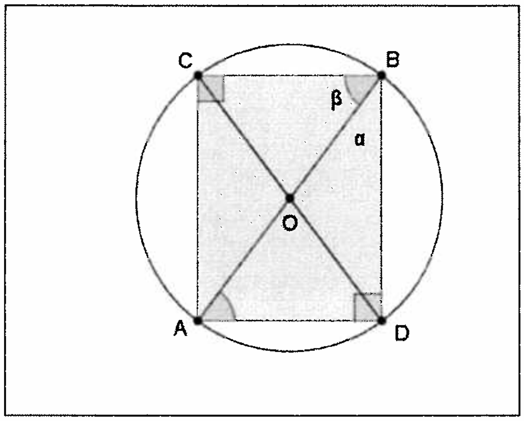

Figure 3: Basic geometric configuration showing that hypotenuse of a right triangle is a diameter of the circum-circle.

Teacher: Very good. Let us summarize. What is the key idea of the proof? What are BGCs required in completion of this proof? Please write down your proof and make sure that your statements do not contradict the observations made with the applet. What difference do you observe in the proof for the cases of acute and obtuse triangles?

Students drag points on the applet to make the triangle obtuse and observe that all elements of the proof remain the same (Figure 4).

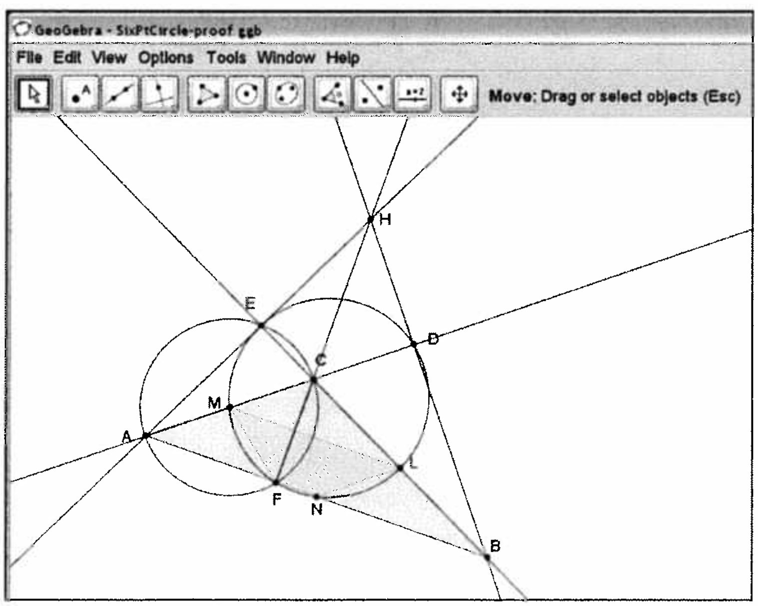

Figure 4: The six point circle theorem: case of obtuse triangle.

Teacher: As a matter of fact, we just proved so called the *six point circle theorem*. Why does it refer to six points?

Anna: We proved that three side midpoints and foot of an altitude lie on a circle. But we can repeat this for other altitudes in the triangle and will get the foot of each of them on this circle defined by the midpoints. I can show you this on my applet.

Teacher: Very good. Next time we will study the *nine point theorem*, which is an extension of today's discussion, so please save the applet you made today for the next class.

Conclusions

The conversation given in the previous section exemplifies our innovative practices in teaching geometry. Its main component is a guided discussion of properties and details presented in a prefabricated dynamic figure. All negotiations and experimentations with the applet aim at helping students to form a visual mental image of a geometrical statement along with an argument supporting this statement.

A survey of 13 students taking the Euclidean Geometry course in the Fall of 2010, reveals the following results (in parentheses the number of positive responses is shown) : "I am learning a lot of new things about geometry in this course" (13); "I am developing my understanding of and ability to proof to a higher degree" (12); "The use of GeoGebra helps me to make a connection of symbolic and visual representation" (12) and "to interpret and understand theorems in geometry" (11); "GG is aesthetically pleasant" (11), "insightful" (12), "helping to activate my knowledge"(11); "GG encourages me to make and test conjectures"(11), "facilitates exactness of my mathematical thinking"(11); "GG allows me to try a larger range of possibilities compare to pen and paper approach"(13).

The survey results confirm a potential usefulness of the proposed approach and concur with the observation that experimentation and conjecturing in dynamic geometry environment must be done in conjunction with other activities (e.g. analysis of BGCs and deductive proofs) that help students to build their networks of theoretical knowledge. Further research is required in order to highlight details and nuances of synchronization of the heuristic and logical components of students' work within the innovative practice described in this paper.

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