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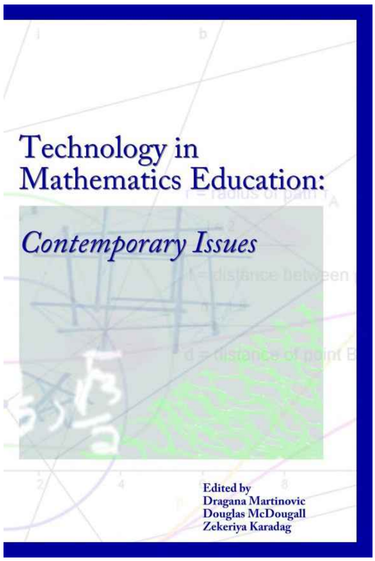

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Technology in Mathematics Education: Contemporary Issues

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Technology in Mathematics Education: Contemporary Issues

Edited by

Dragana Martinovic,
Douglas McDougall, &
Zekeriya Karadag

Informing Science Press

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Technology in Mathematics Education: Contemporary Issues

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Preface

This book consists of chapters that present diverse technologies suitable for the teaching and learning of mathematics, as well as their implementation at different levels of schooling. The invited authors submitted their contributions, which were then refereed by an international group of reviewers. The eight chapters of the *Technology in Mathematics Education: Contemporary Issues* introduce new technological tools such as Interactive White Boards in elementary schools (Chapter 1, by Bruce et al.), iPod Touch and Netbook laptops in Grade 9 mathematics classes (Chapter 2, by Jarvis and Franks), and collaborative pen-based Tablet PC in post-secondary, college classes (Chapter 3, by Carruthers). In Chapter 4, Bu et al. describe a comprehensive framework for conceptualizing the pedagogical uses of mathematics software, GeoGebra, in teacher education and in the professional development of in-service teachers. Mathematics content knowledge and processes are described in the context of GeoGebra applications in teaching and learning of Geometry (Chapter 5, by Surynková) and through the use of Geometer's Sketchpad and GeoGebra by secondary school teachers (Chapter 6, by Sherman). Benefits of dynamic mathematics software use are seen in developing proofs by university students (Chapter 7, by Kondratieva) and understanding probability concepts (Chapter 8, by Radakovic and McDougall).

Computer Algebra Systems, such as Maple, Math Lab, Mathematica, Derive, and TI-Inspire calculators, help users to focus on concepts rather than on procedures for performing calculations. Dynamic Geometry Systems, such as Cabri and The Geometer's Sketchpad, were developed to help students explore mathematics (in particular geometry) and construct their own knowledge. However, integration of dynamic technology in algebra and statistics education encouraged us to suggest a term that would be inclusive of other mathematics disciplines, besides geometry. Martinovic and Karadag (2010) coined a new term: *Dynamic and Interactive Mathematics Learning Environments (DIMLE).* We believe that DIMLE covers all of the dynamic software packages employed in mathematics education, regardless if their primary focus was learning geometry, algebra, statistics, or other.

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As the editors, we wish to have readers inspired through the work published in this book and thank all of the contributors for helping to bring this book to life.

Reference

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Chapter 1

Understanding Interactivity in an IWB-mediated Classroom

Cathy Bruce, Tara Flynn, Rich McPherson, & Farhad Mordechai Sabeti

Introduction

Researchers worked with a total of 20 teachers in a research program that spanned three years, to examine how elementary teachers integrated the interactive whiteboard (IWB) into their practice and how this IWB-integrated practice evolved over time in the mathematics classroom. In the first two years of the program, 18 teachers worked in small teams using a lesson study approach to develop, test and refine lessons that incorporated the IWB in difficult-to-teach areas of mathematics. In the third year of study, the researchers worked with two teachers to intensively observe the types of use and the nature of interactivity between the teachers, the students, and the IWB. Using a design research methodology – in which products of research are continually refined and reexamined at marked phases of research activity to gain insights into the complex nature of the classroom environment – researchers zigzagged back and forth between theory and data (Bruce, 2007), as they developed a working framework to describe how IWBs are used in mathematics classrooms and their role as a mediator of learning.

Literature Review

History of IWB Use and Research

Interactive whiteboards (IWBs) are increasingly widespread in schools. In the UK, a nation-wide education initiative worked to install IWBs in every school in the early 2000s; similar initiatives are occurring internationally, including in Canada, the United States (see Schuck & Kearney,

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2007), and South Africa with comparable technologies (see Slay, Sieborger, & Hodgkinson-Williams, 2008). In Canada, the implementation of IWB technology in classrooms has been somewhat slower, as it was initially coupled with teacher and school interest in the technology in a more grass-roots movement. Most recently, Canadian educational institutions have been allocating significant resources to the acquisition and spread of IWBs in classrooms.

Because IWBs are a relatively new classroom tool, the research on their effectiveness is limited and is mostly reports on teacher use, rather than student use or learning by using IWBs. There are good reasons to suspect that IWBs, like other learning technologies, will enhance the learning environment; Tamin, Bernard, Borokhovski, Abrami, and Schmid (2011) conducted a meta-analysis of over 1055 studies to determine that "the average student in a classroom where technology is used will perform 12 percentile points higher than the average student in the traditional setting that does not use technology to enhance the learning process" (p. 17). At the same time, there is little quantitative evidence that use of the IWB, in particular, leads to increases in student achievement. Researchers consider the possibility that motivational aspects of the IWB and the pupils' obvious enjoyment of lessons may have misled the teachers into thinking that more learning was taking place than was actually the case. As a result of limited student achievement data, some researchers question the justification of using this costly technology when other kinds of projection technology would presumably facilitate the same level of learning (Higgins, Wall, & Smith, 2005). The high level of resources targeted for acquisition of IWBs lends urgency to the question of whether and how IWBs contribute to student learning.

One possible challenge is that the IWB can reinforce traditional teacher-directed pedagogy, with the IWB operating as a presentation tool (Moss et al., 2007; Smith, Hardman, & Higgins, 2006), which does not necessarily address the learning needs of most students. On the other hand, studies also illustrate that, when thoughtful and ongoing professional learning is combined with IWB implementation, teachers can learn to maximize the use of the tool to enhance student learning through multi-modal representations and inquiry approaches (Bruce, 2011; Bruce, Ladky, Ross, Mackenzie, & Flynn, 2008; Bruce, McPherson, & Sabeti, 2011; Bruce, McPherson, Sabeti, & Flynn, 2011).

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The earliest research on IWB use was summarized by Higgins, Beauchamp, and Miller (2007) as primarily being qualitative in nature with an emphasis on teacher testimony, as well as action research-based approaches (see Glover & Miller, 2003, for example). According to Higgins, Beauchamp, and Miller (2007), the most widely claimed advantage of the IWB across multiple studies is that IWBs motivate pupils because learning is more enjoyable and interesting, resulting in improved attention and behaviour. Benefits of IWB use identified through these earlier studies included the ease of use for whole class teaching due to dynamic visual demonstrations (Kennewell & Beauchamp, 2003), improved engagement of students, and the ability to use of a range of multimedia resources (Ekhami, 2002). Much of this research focused on student engagement in particular (Glover, Miller, Averis, & Door, 2007; Hodge & Anderson, 2007; Wood & Ashfield, 2008). A key problem identified by authors of these studies was that teachers were not given sufficient supports to integrate IWB technology with their existing pedagogy (Slay, Sieborger, & Hodgkinson-Williams, 2008). The IWB was therefore often used as a static device with limited evolution of teaching practice (Holmes, 2009). High costs, the time required for teachers to learn to use the IWB, and continued lack of professional learning have all been identified as ongoing challenges to effective IWB use (Beauchamp, 2004; Glover & Miller, 2009).

The Case of Mathematics and IWB Use

Research on interactive whiteboard use in mathematics in particular suggests that IWBs have the potential to accelerate learning through dynamic and multiple representations (Goodwin, 2008) that are unambiguous and clear to students (Holmes, 2009) and that also encourage mathematics communication (Bruce, 2008). The unique features of the IWB may be particularly well suited to exploration in mathematics above other content areas, because of the affordance of visual representations and the ability to manipulate these representations. Smith, Higgins, Wall, and Miller (2005), suggest the particular suitability of the IWB to learning in mathematics when they note that the kinesthetic interactivity available to IWB users may especially enhance learning if this interaction is directly relevant to the subject matter. They cite Greiffenhagen (2002) who reminds us, for example, of the frequent need for drawing lines, shapes, and figures in mathematics, and that the manipulation of these contribute to the understanding of certain mathematical properties.

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Pratt and Davison's (2003) study lends further depth to the discussion about the fit between content and technology in the particular context of mathematics education. Pratt and Davison described the "visual and kinesthetic affordances" of the IWB. Visual affordances relate to "the size, clarity and colourful impact of the computer graphics, writ large on the whiteboard" (p. 31). Kinesthetic affordance relates to "the potential impact of dynamically manipulating the screen in such a way that the teacher's (or child's) agency in the process is far more impressive than merely following a small mouse arrow" (p. 31). Pratt and Davison worked with 14 teachers who were identified as enthusiastic users of the technology and noted that in interviews the teachers were not necessarily "tuning into learning" but were largely focusing on their feelings of being able to control the attention of the class with the IWB and their excitement about features of the new technology. In other words, teachers were taking advantage of the visual potential of the IWB in the whole class setting, but not necessarily the kinesthetic potential of the touch screen. Additional affordances of the IWB noted by others include "being a catalyst for classroom discussion, movement from informal to formal language use, and the development of mathematical ideas" (Serow & Callingham, 2011, p. 162).

Teacher Practice Using IWBs

In quantitative studies comparing concept attainment in classrooms with IWBs and classrooms without, no major changes in pedagogy were noted (Kennewell, 2007). In this paper researchers actually found more traditional, whole group teaching, less group work, less take up of responses, and a faster lesson pace with superficial student contributions. In their early review of the literature, Smith et al. (2005) found that the use of the IWB did not automatically transform teacher practice from a traditional pedagogy to a more constructivist, student-centred practice: "far from transforming classroom practice, the new technology appears to have been uncritically absorbed into teachers' pre-IWB practice" (p. 96). Moss et al. (2007) point out that "good teaching remains good teaching with or without the technology; the technology may enhance pedagogy but only if teachers and pupils engage with it and understand its potential in such a way that the technology itself is no longer viewed as the ends but as another pedagogic means" (p. 94). Glover, Miller, Averis, and Door (2007) also reiterate the importance of focusing on good pedagogy, stating that teachers need training in order to understand the relationship between ap-

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proaches to interactive learning and student attainment of concepts and procedures. Another study led by Miller and Glover (2006) found that 'best practice' of teaching with the IWB involved linking the IWB activities to student work at their desks where students can interact with mathematics ideas and one another but also return to the IWB for explanations and illustrations related to mathematical questions.

The potential of the IWB to support student learning is highly influenced by teacher competence with IWB technology - competence seen as necessary for teaching students who live in a technologically advanced world (Holmes, 2009). In recognition of this challenge, the research focus has turned to:

> the process of teacher development associated with both the introduction of the IWB and the development of its use. This focused on technical as well as pedagogical change, and included the position of pupils in this process of development and their own use of the technology. (Higgins, Beauchamp, & Miller, 2007, p. 216)

More recent studies indicate that a greater emphasis on teacher collaboration and stronger support systems within schools support teachers in implementing novel teaching practices using IWBs, where the sharing of IWB lessons and experiences allow teachers to feel comfortable and supported as their own pedagogy evolves (Lewin, Scrimshaw, Somekh, & Haldane, 2009; Slay, Sieborger, & Hodgkinson-Williams, 2008; Warwick & Kershner, 2008).

Further, Slay, Sieborger, and Hodgkinson-Williams (2008) suggest that professional learning supports need to be in place for educators to use the technology effectively, with a particular focus on pedagogy that maximizes student learning through the visual and interactive features of the IWB. Teachers who have opportunities to learn to integrate pedagogy with IWB use may, for example, encourage more diverse responses from students in mathematics when they have students use the IWB to support the generation, illustration, and comparison of alternative solutions and solution strategies during lessons. Most recently, the author and her collaborators focused their attention on how crafting interactive lessons that enable students to investigate and manipulate IWB tools to illustrate their thinking, is of greater relevance to student understanding (Bruce, McPherson, & Sabeti, 2011).

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Evolution of Teacher Use of the IWB

Several studies have attempted to document the evolution of teacher practice as teachers gain facility with the IWB. Beauchamp (2004) saw a progression, beginning with the use of the board as a substitute for a chalkboard. In Beauchamp's continuum, teachers became increasingly sophisticated in their use of the IWB, moving from an apprentice user (an early stage where the teacher is able to capitalize on her/his comfort with computers when learning to use of the IWB), to an initiate user (involving the teacher's awareness of the potential of the IWB to enhance existing pedagogy and change practice), to an advanced user (who has the confidence and freedom to 'play' with features of the board more creatively). The pinnacle of Beauchamp's continuum would see teachers as synergistic users of the IWB, involving students in highly interactive learning contexts where "both teacher and pupils are able to construct meaning and dictate the direction, momentum and scale of the next step in the lesson" by physical and cognitive interaction with the board (p. 343).

Miller and Glover (2007) developed a similar continuum in which teachers moved from the supported didactic stage (where the IWB is used to enhance traditional teacher-directed lessons), to an interactive phase (where the IWB is further integrated in the teaching with deliberate efforts to involve pupils, nevertheless, the full potential of the board is not realized), to the enhanced interactive stage in which the IWB is a fully integrated feature of the learning environment. Teachers in this continuum move towards higher frequency of use and the maximization of the interactive capacity of the board.

What Do We Mean by Interactivity?

If we are interested in learning about how the interactive whiteboard is used, then it would be fruitful to reflect on the very notion of interactivity in this context. To interact is to "act upon one another" – but in the case of the IWB, who is doing the acting, and who is being acted upon? There are layers of interactivity at play, from the concrete (actually touching the board) to the abstract (interacting with ideas represented there). These layers can be difficult to unpack; for example, Moss et al.'s (2007) examination of interactivity with the IWB showed that teachers' understandings of interaction appeared to deal directly with the concrete act of manipulating the board and did not include interaction with concepts or ideas. And yet, the metacognitive unpack-

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ing of these layers has implications for practice and exploiting the most powerful features of the IWB. For example, Moss et al. note, "Where we observed best practice, departments or individual teachers were aware of this dimension and had consciously set aside time to reflect on the most appropriate use of the technology in this context" (p. 7).

We recognize that researchers also have multiple interpretations of interactivity. Tanner, Jones, Kennewell, and Beauchamp (2005) define interactivity as the degree to which the learner has opportunities to actively interact with the teaching/learning situation – the more active, the more meaningful and the deeper the interactivity: "We conceive interactivity as demanding a degree of *active* participation by learners who contribute to the development of collective understanding" (p. 722 [italics added]). We emphasize the word 'active' which is at the root of interactivity. Kennewell (2007) describes the role of the IWB in the classroom "in the full interactive/dialogic teaching approach"; interestingly, Kennewell's descriptions implicitly position the IWB as an actor in the teaching situation. From Kennewell's vantage point, the IWB plays the role of consultant (providing information), an organizer (providing a tight structure of activities with unpredictable outcomes), a facilitator (providing a looser structure for activities that involve student choice), and finally, as a repository for student ideas that can later be revisited.

The IWB may indeed be an actor in the learning situation, as per Kennewell, but any role the IWB plays is of course a result of teacher decision making. This intersection between teacher decisions and the IWB is at the crux of our earlier question: who is doing the acting, and who is being acted upon in the case of the IWB? Removing the students from the equation for a moment, we might also ask, in the case of the IWB, who or what is the mediator of the learning – the teacher or the interactive whiteboard itself? For Hennessy, Deaney, Ruthven, and Winterbottom (2007), the mediating role is played by the teacher. This is a question that is not easily answered, but that draws us back to the nature of interactivity with the IWB itself.

Drawing on the work of Smith, Higgins, Wall, and Miller (2005), our understanding of interactivity distinguishes between 'technical interactivity' (physical interaction with the IWB) and pedagogic interactivity (interaction between students and others in the classroom designed to bring about learning). An over emphasis on technical interactivity (and under emphasis on pedagogical considerations particular to the features

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of the IWB) has the potential dangers of (a) increasing the pace of lessons with lecture lean, (b) prioritizing trivial activities that have attractive graphics, (c) giving superficial attention to content and concepts being addressed, and (d) provoking brief student response format as opposed to rich discussion (Moss et al., 2007; Smith et al., 2006). On the other hand, data analysis in our study suggests that pedagogic interactivity not only involves interaction between students and others but can very clearly involve the IWB as a mediator for communication, illustrating and generating mathematics understanding amongst participants in the learning moment. Our data concurs with findings from Tanner et al. (2005) and Kennewell, Tanner, Jones, and Beauchamp (2008), who explored the degree of control students had over the lesson trajectory in whole-class teaching: the more the students had agency and contributed to the development of the lesson and related understanding, the more profound the interactivity.

Method

Our research program focuses on IWB use in mathematics classrooms for teachers and their students (6-14 years old). The results shared in this chapter involve research spanning three years with 20 teachers and 500 students in Canada. In the first two years, teacher teams learned about IWB use through a lesson study professional learning program that was organized through the district school board. Four teams of teachers (4-6 teachers per team, Grades 1-10) worked to generate problem-based lessons using manipulatives and IWBs as mediating tools to tackle difficult-to-teach mathematics concepts. In the third year, two teachers were followed over the course of the academic year to document the use of the IWB in their mathematics classes. We explored the question that rippled through many education communities – what distinguishes the IWB as anything other than a "glorified" chalkboard or overhead projector? This is an important pedagogical consideration that drives our research. We are interested in understanding how interaction with the IWB maximizes student learning.

Data sources included teacher interviews, student interviews, classroom observations, and video capturing of teacher meetings, mathematics lessons, and students working in small groups. Interview data were transcribed, field notes and observations were generated electronically, and video data were rendered. Data analysis was organized in three phases.

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Phase 1

For analysis in phase 1, researchers used open and axial coding strategies in a grounded theory approach, where the codes were generated by viewing and reviewing all data sources (see Charmaz, 2003). Simultaneously, we generated and tested a theoretical framework that described IWB use by students and teachers. We employed a design research methodology where the framework was informed and continually refined based on findings in a series of iterations (see Collins, Joseph, & Bielaczyc, 2004). Design research focuses on the development of 'products' of educational design, based on cycles of testing and refinement. The design research methodology allowed us to go back and forth between data and theory to develop the framework for interactivity with the IWB as a product of classroom-embedded research; through continuous testing and revision we gained increasing understanding of the role of the IWB in the complex environment of the classroom.

In phase 1, we initially focused on teacher use of IWB, especially use that exploited the truly interactive features of the board. In 2004, Beauchamp published a five-step continuum that illustrated how teachers transition from a beginning stage (where the IWB was a blackboard substitute) to an expert 'synergistic' user (where both the teacher and students adeptly used the IWB to construct meaning and had agency over the lesson trajectory) with experience. We were also interested in Glover, Miller, Averis, and Door's (2007) three-stage continuum (from supported didactic, to interactive, to enhanced interactive). Our research initially involved an envisioning of a similar continuum, and we conjectured that teachers would move through increasingly sophisticated levels of interactivity with the IWB over time. But we quickly moved away from seeing the teacher's IWB use as a continuum, and rather began to notice fluidity between the types of IWB use and the nature of the interactivity depending on the specific learning situation at hand. Teachers, and students, used the IWB in a range of ways depending on their immediate goal. As a result, we became increasingly focused on student use of the IWB and the ways in which students interacted, in whole class or small group settings, with the IWB.

Phase 2

Our initial coding and theoretical framework development in phase 1 led us to identify two broad types of IWB technological interactivity; in phase 2 analysis, we categorized some instances as 'productive', in

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which the IWB operated as a tool to produce and represent math ideas that were unique in response to the learning situation. In contrast, reproductive instances were situations where the math demonstrations could have been 'reproduced' with another technology (including the chalkboard or chart paper and markers) or were pre-generated and static in nature (such as a scanned workbook page where students could fill in answers using an IWB pen). We also found instances when technological glitches or user proficiency actually interrupted the flow of the lesson, and we called these problematic instances. Our phase 2 coding involved a simple count of IWB use instances by teachers and students in 372 minutes and 50 seconds of video clips. In the 778 instances of IWB use that were coded, we found 71 problematic instances, 15 reproductive instances and 692 productive instances of IWB use (see Bruce, McPherson, & Sabeti, 2011; Bruce, McPherson, Sabeti, & Flynn, 2011).

Phase 3

In phase 3 of analysis, researchers used pattern matching (Mark, Henry, & Julnes, 2000) to generate and test a typology and framework that built on phase 2 analysis to not only incorporate technological interactivity but to also consider pedagogical interactivity (Smith et al., 2005).

Findings

Technological Interactivity

We identified two broad categories in the data describing technological interactivity with the IWB: productive and reproductive. Productive uses of the board involved the fluid use of the uniquely interactive features of the IWB with the goal of generating and/or representing new learning. Examples of productive use included teacher or student use of a dynamic IWB feature such as a mathematics content tool, internet link, Flash-based tool, or generative work based on an interactive program. The following vignette (Bruce, McPherson, & Sabeti, 2011) describes an instance of productive use of the board:

> Using a YouTube video of skateboarders, the teacher screen-captured a skater in mid-flight and imported the image into Notebook software. She then used a virtual protractor on the skateboard picture. She then used the straight line tool to emphasize the base of the angle, using the protractor to measure

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> the angle between the skateboard and ground. This demonstration engaged students completely – most students were focused on the angle measurement. In the debrief with the teacher and students, both identified this as an 'aha moment'. (Field note, November 17, 2010)

Eighty-nine percent of all IWB use instances coded were productive. It is important to note that the teacher participants in this research were highly motivated teachers with the goal of maximizing IWB use, which partially accounts for the high percentage of productive instances. In this way, conditions for this research were optimal, rather than natural.

Reproductive instances involved demonstrations using the IWB, which could easily have been substituted by another technology such as an overhead projector or use of the blackboard and chalk (for example, students are presented with a series of ten computations on the IWB; they record their answers at their desks and one-by-one come to the board to fill in the correct answers). A third category, problematic instances, also emerged and these involved technology glitches such as poor connections between the IWB and computer, and distractions such as looking for a virtual tool without success.

The analysis pointed to two particularly important outcomes of technological interactivity involving productive use of the IWB:

(i) **The IWB provided** ***visual support for communication and shared student reasoning.*** The data revealed that the IWB facilitated a sense of shared experience: students had the opportunity to view multiple solutions and solution strategies on the large screen (for collective viewing and debate). This supports the 'visual affordance' findings of Pratt and Davison (2003). Importantly, the IWB operated as a mediating tool to help students co-construct mathematics understanding, and that resulted in a collaborative learning environment.

(ii) **The IWB provided** ***opportunities to increase agency, including student risk-taking in pairs and small groups.*** Interestingly, when students worked at the IWB, they took greater risks in their mathematics thinking and were more persistent in solving problems than the observed non-IWB pairs and groups. This was surprising to researchers because of the public nature of the work happening at the board; we conjecture that this may be due to the ease of manipulating representations on the board,

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> where work is not fixed or necessarily permanent. The efficiencies of the IWB also enabled students to explore multiple solutions to problems within the same timeframe that their non-IWB peers were exploring one solution or solution strategy, resulting in increased independence, ownership and agency (see Bruce, McPherson, & Sabeti, 2011; Bruce, McPherson, Sabeti, & Flynn, 2011).

Our findings led us to increase our focus on student use and interactivity with the IWB and to consider the IWB as a tool for producing shared understanding in a constructivist setting. This leads us to concur with Hennessy et al. (2007) who write that "the strength of the IWB lies in its support for shared cognition, especially articulation, collective evaluation and reworking of pupils' own ideas, and co-construction of new knowledge" (p. 298).

Pedagogical interactivity - Developing a framework for teaching and learning mathematics with the IWB

We developed the skeleton of a theoretical framework for IWB use, which has been continually revised through consultation with teacher participants and through testing of the framework against field notes and video data collected over the three years. Initially, the research team considered the framework to be a continuum of growth similar to the work of Beauchamp (2004) and Miller and Glover (2007), in which teachers would increasingly maximize the interactive features of the board and turn its control over to students by degrees. In its first iteration, our framework consisted of five essential stages of a continuum where we would observe teachers progressing through the stages over time. Stages of use identified in the continuum were:

> Stage 1: Non-dynamic demonstration (the IWB acts as a static screen for visual support with limited interactivity);
>
> Stage 2: Dynamic demonstration (the IWB acts as a computer screen with interactivity demonstrated by the teacher);
>
> Stage 3: Student practice (students repeat what the teacher essentially has demonstrated);
>
> Stage 4: Student investigation (students investigate mathematical ideas and problems with the use of the IWB); and,

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Stage 5: Facilitating discourse (teachers and students use the IWB to facilitate and support math communication).

We then amplified the framework by including examples of lower and higher risk instances of IWB use for each stage. Lower-risk examples were relatively teacher-directed and led to more predictable student actions and outcomes. Higher-risk examples were relatively student-directed and led to less predictable student actions and outcomes possibly requiring more immediate teacher decision-making and scaffolding.

However, we quickly discovered that the theoretical framework was not a continuum; teachers were not static in their use of the IWB but, in fact, they moved through these various types of use within a single lesson. Immediate context, the needs of students, and requirements of the learning moment were very important in teacher decisions around the use of the IWB. This revision was exciting because it moved the framework away from a deficit model and towards an asset model. Rather than one end of the continuum representing "better" or more sophisticated teaching than the other, the framework became a description of types of use for various purposes: a descriptive model which better reflected IWB use on a practical, lesson-by-lesson – even moment-by-moment – basis.

Our dissatisfaction with the accuracy of the first several iterations of the framework led to additional revisions at different stages throughout the research. In this way, the framework itself became a dynamic tool for continually testing our understanding of what was happening during the instructional moments with the IWB in a design research approach (Collins, 2004).

We return to Smith et al.'s (2005) distinction between 'technological interactivity' and 'pedagogical interactivity' to better categorize the types of interaction with the IWB observed in the study. According to Smith and colleagues, technological interactivity refers to physical interaction with the device (the IWB), and pedagogical interactivity refers to interaction between students and others in the classroom. We expanded the definition of pedagogical interactivity to include the triangle of interaction between students, teachers, and the interactive whiteboard. Once we amplified the definition of pedagogical interactivity, we were then able to categorize our data into four different types of pedagogical interactivity.

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i. Teacher demonstration: involves the presentation of ideas (as in a slideshow, to provide a non-dynamic example) or the demonstration of a task (dynamic example), where the teacher is the sole user of the board in a whole class setting.
ii. Student practice: involves student manipulation of the board in whole class or small group settings to perform a task constructed by the teacher. A dynamic example of this might be students using an online tool or virtual manipulative; a non-dynamic example might involve the display of student computation solutions;
iii. Student investigation: students use the IWB as a tool to actually explore and construct or develop their understanding of a concept, either as a whole class or a small group. The outcome and process are not known, though tools may be provided (and the nature of these tools largely decides whether it is a dynamic or non-dynamic use).
iv. Consolidation of ideas: usually a whole class strategy, the teacher or students use the features of the board to consolidate their thinking and make generalizations. In this case, the board may be a repository for student solutions, which can then be compared and discussed.

To better illustrate what pedagogical and technological interactivity look like in an IWB classroom we developed a matrix and populated each cell with precise examples from video data (drawn from all three phases of the research) (see Figure 1).

The four categories of pedagogical interactivity identified in the data are teacher demonstration, student practice, student investigation, and consolidation.

Technological interactivity: physical interaction with the device		
	Predominantly Reproductive IWB is a presentation tool; This IWB activity can also be achieved using different technologies including chalkboards, chart paper and overheads	**Predominantly Productive:** IWB is an interactive tool that illustrates and generates ideas because of IWB use; More difficult to achieve with other technologies

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Pedagogical interactivity: interaction between students and others in the classroom designed to bring about learning with the IWB as part mediator		
Teacher Demonstration	The Grade 3 teacher presents a slideshow that includes key information from the lesson on repeating numeric patterns and presents a problem for students to solve. The didactic lesson approach supports visual representations.	The Grade 4 teacher shows saved patterns that students generated on the IWB during the previous day. Students categorize the patterns into three sets (growing, shrinking, or repeating patterns), justifying their thinking, while the teacher highlights and moves each pattern into one of the three categories to generate a table of classified student samples.
Student Practice	Students in a Grade 2 class work in pairs at the IWB to solve 5 addition questions that have been loaded into IWB compatible software (such as Notebook) from the mathematics text. The students use the IWB pens to record their solutions. Some students use a calculator available in the gallery of items for use with the software.	Students in Grade 1 practice skip counting by 5's in chorus while one student simultaneously clicks on the large interactive 100's chart on the IWB to turn every 5th number over so that there is a large visual representation of the vocalized skip counting (5, 10, 15, 20, 25…)
Student Investigation	Grade 10 students are asked to work on the IWB in pairs to analyse a photograph displayed on the IWB. They are asked to use the parallel lines theorem to find relationships between various angles. Students use the IWB pens to highlight each example they find.	Students in a Grade 7 class investigate three figures that look like triangles. Some students are working with paper figures at their desks and tools such as rulers, scissors and protractors to determine which of the three representations are actually triangles. A group of three students work at the IWB with the same three representations as well as virtual tools to assess the three figures, deciding which of the three are triangles. The group at the IWB record their investigation using the screen capture tool and data recorded with IWB pens and draw tools to illustrate their thinking. This group saves their file, ready to present their thinking to their peers on the IWB.
Consolidation	After four groups have used the IWB to illustrate their thinking, the teacher has captured and placed the four different student representations of the same linear growing pattern onto one screen on the IWB. The teacher then asks the students to describe how the four representations are similar and different. After this analysis, the students then discuss the effectiveness of each representation in terms of when each might be most	Two groups of students have classified a series of geometric figures on the IWB into a Venn diagram with intersecting sets. Other students have done the same task at their desks. The whole class is now discussing their sorting strategies. The first group using the IWB presents their sort to the class using the spotlight feature in the IWB software (where one area is in view and the other areas are hidden from view). This enables the students to focus on each set in the Venn diagram

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Pedagogical interactivity: interaction between students and others in the classroom designed to bring about learning with the IWB as part mediator		
	useful.	independently, then the presenters show the intersecting areas. A second group then puts the image on dual screen so that both groups' sorts can be viewed simultaneously. The students discuss which sort makes the most sense and generate a third sort by cloning and dragging the objects from both previous examples into a third Venn diagram that combines aspects of both previous diagrams.

Figure 1. Framework for technological and pedagogical interactivity with the IWB for mathematics teaching and learning

Teacher demonstration fits most closely with traditional classroom practices, with the teacher presenting information and modeling mathematics thinking to students (presentation tool). In the IWB-mediated environment, this involved slide show presentations, and the use of web-based instructional video clips. A more student-centred teacher demonstration included teacher capturing and displaying of student work from a previous lesson/activity and featuring/highlighting/summarizing student ideas that were generated on the IWB (interactive tool).

Student practice in both familiar and unfamiliar contexts has been found to be an essential feature of mathematics learning (Boaler, 2006). We observed many opportunities for students to practice doing mathematics with the IWB. In some cases, this involved performing operations (similar to writing answers on a chalkboard). In other cases, students used learning objects such as virtual tools or manipulatives (e.g., a hundreds chart or a virtual geoboard with moveable 'elastics') to represent their understanding in a context established by the teacher with known or anticipated outcomes.

Another distinct type of pedagogical interaction with the IWB involved *student investigation* of mathematical ideas or problems. We consistently observed students investigating mathematics with the IWB in small and whole group settings. Notably, when students worked in pairs and in small groups at the IWB, their exploration of math ideas was powerful. This was as simple as accessing images and using tools to explore the mathematics in those images. A more 'productive' use of the board for student investigation involved students generating new understandings using virtual tools to make and test conjectures, and exploring multiple solutions. Context (such as a problem or a purpose for the investiga-

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tion) was most often provided by the teacher, but the task was usually open-ended and the range of outcomes largely unknown. Student choice and selection of tools were an important observed characteristic in this type of pedagogical interactivity.

Although participants in our research found the *consolidation* to be one of the most challenging features of their mathematics lessons, focused attention in this area led to teacher and student use of the IWB to make consolidation lively, focused, and highly interactive. For example, screen capture features allowed both teachers and students to compare and contrast solutions and solution strategies brought forward by students. The ease of access to powerful tools enabled students to adopt the role of 'teacher' and facilitated their sharing of mathematical ideas, solutions, and strategies. In this way, the IWB was the focal point for communication that facilitated debate, justification, and furthering the norms of a positive community of learners.

In summary, focused attention on the nature of interactivity with the IWB, combined with professional learning opportunities that incorporate technical and pedagogical considerations for IWB use in the classroom, offers practitioners tremendous opportunity for maximizing the interactive effect of this technology. The visual and kinesthetic affordances of the IWB have proven to support teaching and learning in mathematics in particular. Interestingly, in analysis of mathematics lessons, we have learned that high quality teaching does not in fact involve one type of technical/pedagogical interactivity but in fact, involves purposeful teacher decisions that lead to multiple uses depending on the context and the needs of the learning situation.

Implications for Research and Teaching Practice

We see three specific ways that the *pedagogical interactivity* framework might now be used:

a) an orienting tool for practitioners new to the IWB;
b) as a professional learning tool for more in-depth exploration; and
c) as an analytical tool for research observations of IWB use and interactivity;

a) Previous iterations of the framework have been posted online for ease of use by educators and researchers. This allows educators to see potential types of interactive whiteboard use and interactivity with ex-

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amples of researched effective methods for using the IWB. Informal feedback at this stage has revealed an appetite for this kind of organizing tool to give teachers a window into how they might incorporate novel and/or enriching uses of the IWB to support student learning. We hope to formally launch this more refined iteration of the framework as a knowledge mobilization strategy (see www.tmerc.ca for the framework and accompanying video samples);

b) As a professional learning tool for inquiring practitioners, the framework could be used to more deeply explore uses of the IWB in a collaborative inquiry approach. For example, teachers and/or researchers interested in maximizing student use could track the amount of time students spend on each type of use in the framework to ensure variety and to maximize the potential of the IWB for student learning (such as ensuring sufficient time for consolidation).

c) Researchers investigating IWB use might find this framework useful as an analysis tool. For example, we are interested in returning to our full data set and conducting code counts of instances of IWB use for each cell in the framework. This could be combined with student interviews directly linked with each type of use to assess which types of use seem to be most effective for which groups of students.

Academic Contributions

The academic contributions of this research are threefold. First, the cycle of developing, testing, and refining the IWB ***Pedagogical interactivity*** framework confirms the value of using a design research methodology in technology-related educational contexts. Technology is a fast-moving field, which complements the dynamic nature of product development in design research. We recommend design research as a highly effective methodology in this field.

Second, we hope that our articulation of the framework development process has illustrated the importance of zigzagging (Bruce, 2007) between theory and evidence. Rather than presenting research findings as a fixed body of knowledge or a simple storyline, we have tried to reflect the organic nature of theory development grounded in the complex environment of the classroom.

Third, this framework of interactivity can be further tested in other research contexts by colleagues in the Information and Communication Technology (ICT) research community. We are sometimes tempted to

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innovate in research along with the technology, but the danger is in losing track of solid previous work. Healthy research practice not only involves knowledge creation but also interacts with and builds on previous knowledge. In producing this theoretical framework, which is grounded in classroom contexts and previous research of colleagues in the ICT community, we hope to create a springboard to construct even further knowledge about the affordances of IWB use and interactivity in classrooms.

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Chapter 2

Interactive Notes and Self-Differentiated Instruction: Findings from a Case Study Involving the Use of Mobile Technologies in Grade 9 Mathematics Education

Daniel Jarvis and Douglas Franks

Abstract

This chapter reports on the findings of a qualitative research study that was conducted for an Ontario district school board in the fall of 2009. The study involved analyzing the perceived effects on teaching and learning of two different handheld/mobile technologies (iPod Touch mobile devices, Netbook laptops) that were implemented in a Linear Relations unit in two Grade 9 Academic Mathematics classrooms. Findings of interest include the development of 'interactive notes,' the effects of the technologies on teacher/student communication, and the positive existence of, what we have referred to as, Self-Differentiated Instruction.

Technology in Education

The use of electronic technology in schools has been occurring since at least the 1970s, with the introduction of personal desktop computers, and basic handheld calculators. The last 30-year period has seen a great proliferation of hardware and software, with education often having a strong focus for the use of these products. While computer labs are still commonplace in schools, mobile learning has gained momentum over the past decade. Robson (2003) noted that mobile learning "started with the introduction of Casio graphing calculators in 1986 and is poised to become stronger and more significant with the availability of handheld networked devices . . . that have the capabilities previously associated with desktop computing" (para. 5). In a summary of research

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on technology in education written in the middle of the first decade of the 21st Century, the US-based North Central Regional Education Laboratory (NCREL) stated that a shift in students' learning "from" computers to "with" computers was occurring, largely as a result of the ever-growing availability of "new information and communication devices" (n. d., para. 20) to students both at home and in the school.

With the recent introduction of sophisticated handheld graphing calculators such as the TI-Nspire, "smart" devices such as the iPod Touch, NetBook computers, and Tablet PCs, these early millennial prognostications are rapidly coming to pass. Kamenetz (2010) observed, in a commentary on technology and education, that children in many countries now "belong to a generation that has never known a world without ubiquitous handheld and networked technology" (para. 2). Definitely, almost all school-aged Canadian students are among those who now live in a "digital world" (Franklin & Peng, 2008, p. 70). Prensky (2001) coined the term "digital natives" (p. 1) to describe K-16 students at the time who were among the first generations to be growing up in a digital world. This expression has its critics, however. For example, Franklin and Peng (2008) suggest that, as of late in the first decade of the 21st Century, the term may no longer apply to school-aged children as it in effect understates the situation: "the digital world is ubiquitous to their very being" (p. 69), so rapidly is technology changing and infusing almost every part of their lives. Jenson, Taylor, and Fisher (2010) take a different view of the notion of "digital native," claiming that it represents an overstatement: It is "clear . . . that *not all students are 'digitally native.'* This term denotes a privileged position in terms of gender, socio-economic status, and geography" (p. 11). Suffice it to say, however, all of these critics take the position that contemporary education must involve the use of technology and the development of digital competency.

Research on the use of technology in classrooms is still quite limited, likely because it is rapidly changing. Educational institutions such as schools need both time and resources to effectively integrate these technologies into classrooms for use by teachers and students in specific subject areas. Since calculators, including graphing calculators, are the handheld technology of longest and most common use in teaching and learning mathematics, and have consequently been researched most

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extensively, we first offer a brief review of the classroom use of this mobile technology.

Calculators/Graphing Calculators

The mathematics classroom has been the focal point for extensive calculator use for a number of decades now. Waits and Demana (2000), pioneers in the use of technology in the mathematics classroom, reflected in the National Council of Teachers of Mathematics' (NCTM) turn of the century yearbook on why this should be so, especially in contrast to the use of computers in classrooms. Graphing calculators, they noted, were portable, relatively inexpensive, and reasonably powerful: "Every student could own his or her own . . . personal computer with built-in mathematics software" (p. 53). This stood in marked contrast to much less available, and more expensive, desktop computers.

Reflecting on over 20 years of working with teachers and technology, Waits and Demana (2000) offered four important observations:

- "Change can occur if we put the potential for change in the hands of everyone" (p. 53).
- For change to occur on a large scale, "it takes practiced teachers to change the practice of teachers" (p. 53).

  This point is based on the authors' observation that teaching practice is very difficult to change from the outside, because teaching is a complex profession heavily constrained by issues that are often local in nature. A top-down approach to professional development is often ineffective—it is best achieved by turning it over to "practicing teachers who had succeeded in embedding the appropriate use of calculators [technology] into their own practices" (p. 53).
- "Calculators cause changes in the mathematics that we teach" (p. 54).
- "Calculators cause changes in the way we teach and in the way students learn" (p. 56).

Ellington (2003) conducted a meta-analysis of the effects of calculators on achievement and attitude levels in elementary and secondary mathematics classes. She analyzed the results of 54 published studies that met her criteria for examination. Her review included studies on three

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types of calculators—basic, scientific and graphing—and her comments on the latter are particularly revealing. She found that calculators were particularly beneficial when they had a "pedagogical role" (p. 456), for example, when they were integrated into mathematics study and not simply used for checking work or for drill and practice. It also made a significant difference to learning and positive attitudes when calculators were permitted during testing. The author found that, when calculators were integrated into the testing process "the results based on graphing calculator use were significantly better than the results of basic or scientific calculators in two areas: conceptual skills and problem-solving skills" (p. 457). (Ellington did not provide specific examples of mathematical concepts; instead "conceptual skills" is her inference from the studies analyzed that an understanding of relevant mathematical concepts would be required to solve the problems the students were given.) She also determined that graphing calculators had a "more significant [positive] influence on student attitudes" (p. 457) toward mathematics than did the other types of calculators. Ellington concluded that teachers—especially those in Grades 6 to 12—"should design lessons that integrate calculator-based explorations of mathematical problems and mathematical concepts with regular instruction" (p. 457).

Waits and Demana (2000) and Ellington (2003) wrote specifically about the handheld technology of calculators, and especially graphing calculators, but we wonder if the points they make might not also apply well to some much more recent handheld technology. Might smartphones and smaller computers such as Netbooks, for example, best serve mathematics learning when their use is thoroughly (i.e., primarily, given the context of the lesson/exploration) pedagogical? Can their capabilities be fully accessed by teachers and students in the mathematics classroom—and beyond?

Examples of Recent Use of Mobile Technology in Mathematics

Franklin and Peng (2008) conducted a four-week study of the use of the iPod Touch in Grade 8 mathematics classrooms, centred on the question: "Are mobile devices such as the iPod Touch a useful tool in the mathematics classroom for providing video content to support the learning of mathematics both formally and informally?" (p. 72). Working in teams, students developed videos on algebraic equations, with

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particular attention to concepts such as slope, absolute value, and elimination (a method of solving equations). The authors experienced a number of challenges, from technology infrastructure issues to individual staff resistance and bureaucratic delays. They concluded, nevertheless, that, while mathematics achievement levels were not tested, "the use of iPod Touch to build math videos was viable" (p. 79), and that participants and observers alike "were impressed by the ability of students to present difficult concepts in a visual format and then discuss them with friends" (p. 79). Franklin and Peng reported that the teachers involved believed that, through this experience, the students had gained in their understanding of the mathematics concepts involved.

In North Carolina, Project K-Nect has been underway since 2007 (Davis, 2010). The project offers the following description on its Internet homepage:

> Project K-Nect is designed to create a supplemental resource for secondary at-risk students to focus on increasing their math skills through a common and popular technology—mobile smartphones. Ninth graders in several public schools in the State of North Carolina received smartphones to access supplemental math content aligned with their teachers' lesson plans and course objectives. Students communicate and collaborate with each other and access tutors outside of the school day to help them master math skills and knowledge. (Project K-Nect Home, para. 1)

The smartphones and service are supplied free of charge to the students and their schools by a wireless technology provider. The project is said to have academic goals (improving mathematics skills and knowledge), social goals (empowering at-risk students), and technological goals (assessing the efficacy and viability of mobile devices as "digital assets" in learning) (Project K-Nect Summary, para. 2).

In brief, teachers are able to distribute to students via the smartphone mathematics problems tied to their mathematics lessons. If a student has difficulty solving the problem, he or she can access additional resources, including peer support, through the use of their mobile device. School math blogs and student- or teacher-created videos demonstrating mathematics topics such as algebra concepts can be readily accessed. Students are also able to videotape themselves solving a problem and can, thus, show others where they are having difficulties or

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alternative solutions to the problem. Some students have created and posted mathematics problem-based movies to raise students' level of interest and engagement (Davis, 2010; Project K-Nect Summary, paras. 4, 5, 6). A feature of the project is that problems are randomly selected from a database; thus all students are not solving the same problem. Text and voice capabilities also have been disabled, and teachers are able to fully monitor a student's use of the phone. Students do have instant messaging capability in order to communicate with others in the Project, and they have access to a school mathematics blog on which they can post questions, videos, audio, and text (Davis, 2010; Project K-Nect FAQ, para. 1).

Although reporting appears to be largely anecdotal to date, as described in separate Spring 2010 interviews with a project coordinator and a North Carolina mathematics teacher on the CBC Radio One program, *Spark*, results are described as positive in terms of motivation and learning. Students' ability to understand and explain the mathematics has reportedly increased, as has their attitude toward mathematics. Limited reported data on student achievement also have indicated that Grade 9 students using the smartphones in algebra studies outperform those with the same teacher who do not use this mobile technology (Davis, 2010).

Learning with Technology

Waits and Demana (2000) and Ellington (2003) made two important claims for the successful adoption of technology: it needs to be effectively available to all, and it needs to be appropriately integrated into instruction. Robson (2003) and NCREL (n.d.) made the same points in their discussions of the adoption of new technology (handheld devices, mobile learning) by educational systems. Robson stated that:

> 1. The technology must be pedagogically effective and viewed as an improvement; and,
>
> 2. The technology must be available and accessible. (para. 7)

Robson (2003) identified that some researchers believe that mobile learning ("m-learning") (e.g., Tatar et al., 2003) was typically based on a traditional instructional model of "content retrieval and delivery" (para. 12). A more effective model for the use of powerful handheld technology, they argued, was one centered on hands-on "projects and collabo-

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rative groups" in which the media were designed to support inquiry. Rather than using the devices for writing free-form text, students should be engaging in such activities with this technology as creating graphs and animations and asking thoughtful questions. The optimum form of communication was "face-to-face discussion supported by shared attention to data, drawings, graphs, and text" (para. 12).

NCREL (n. d.) authors noted the gradual shift from students learning "from" technology to learning "with" technology (para. 17). Learning with technology implies a focus on integrated, inquiry-based learning, opportunities for enhancing students' problem solving and reasoning skills, and removing the need for learning to be confined to schools and to specific devices. "Technology access is increasingly centered on the learner experience" (paras. 20-21).

It may be that with the advent of powerful mobile devices, technology will increasingly play a central, even "revolutionary" role in education. As Robson (2003) observed,

> There is every reason to believe [mobile learning] will gain new impetus and wider applicability with the introduction of ubiquitously connected handheld devices that have the power of a personal computer. There may be no other choice if the educational system is to adapt to the learning styles and meet the demands of future generations of digital natives. (para. 28)

Our research project sought to analyze how the handheld/mobile devices were used by these "digital natives" within the mathematics classrooms.

Research Context

In the fall of 2009, meetings occurred with the Director of an Ontario District School Board to discuss possible research foci for some work around technology in local area schools. The authors agreed to visit both an elementary school and a secondary school within the participating District School Board to observe students, teachers, and the various uses of technology in mathematical instruction. Subsequent meetings at the Board office narrowed the research focus to a modest qualitative case study at the secondary school level in which three teachers would implement lessons involving three different digital technologies (iPod Touch, Netbooks, TI-NSpire graphing calculators) within a specific mathematics unit (Linear Relations) and course (Gr. 9 Academic Math-

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ematics). A mathematics coordinator worked closely with the researchers throughout the project, in terms of planning, logistics, and communication with teachers and administrators. A graduate research assistant transcribed the interviews and also took part in the literature review research. Jarvis and Franks originally met with four mathematics teachers at the secondary school to discuss the potential of a research study and to collect ideas and information from these individuals. Upon further reflection and discussion with the coordinator and the board, it was decided to invite three of these teachers who would be able to adjust their course schedule and who were willing to implement certain technologies within 10 lessons of a Linear Relations unit of study in the Grade 9 Academic curriculum. All three accepted this invitation to participate in the study.

Research Methodology

Our qualitative case study involved a written open-response student survey; a series of 11 individual interviews and focus group meetings with teachers, students, and a mathematics coordinator; researcher journal observations; and a collection of instructional artefacts (e.g., handouts, instructional websites, instructional mathematics videos) pertaining to the various technologies. Both the university and the school district gave ethical approval to the project. An open-response written survey was designed and conducted to solicit initial feedback and ideas from students themselves, in order to assist the researchers in preparing relevant and meaningful questions for the subsequent individual interviews with teachers/coordinator and focus group meetings with students. Teachers were then asked to implement a series of lessons involving technology within the Linear Relations unit of study in the Grade 9 Academic curriculum. One teacher would use handheld Apple iPod Touch[1] technology, another on mobile Netbooks[2] (i.e., small lap-

[1] The iPod Touch is a portable media player, personal digital assistant, and Wi-Fi mobile platform designed and marketed by Apple Inc. The iPod Touch adds the multi-touch graphical user interface to the iPod line. It is the first iPod with wireless access to the iTunes Store, and also has access to Apple's App Store, enabling content to be purchased and downloaded directly on the device. . . . The iPod Touch and the iPhone, a smartphone by Apple Inc., share the same hardware platform and run the same iOS operating system. The iPod Touch lacks some of the iPhone's features and associated

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tops) technology, and a third on graphing calculators. The teachers were asked to try to incorporate the above, respective technologies in 8-10 lessons, if possible, throughout the unit. The third teacher was ultimately not able to fully participate in the study, and so our results will focus on the iPod and Netbook technologies. Interviews were scheduled with the teachers individually, the math coordinator, and then seven small focus groups of students as per their availability during school hours. The researchers, upon request, were also given copies of sample instructional handouts/lessons, were shown the various instructional websites maintained by the teachers, and were given access to, and brief demonstrations of, the instructional technologies being used in the study.

Data Analysis

The analysis of the data gathered from the school, students, and teacher participants was guided by the purpose of the study, the results of the written surveys, researcher observations, and the transcripts from the interviews. Using qualitative data analysis software, *Atlas*.ti, transcript data were reviewed by the researchers and organized according to 26 emergent themes. The researchers utilized the various forms of data (i.e., class surveys, 11 individual and focus group interviews, and classroom artefacts like handouts and websites) to draw conclusions about how the handheld/mobile devices were used by students within the mathematics classrooms, describe opportunities/challenges related to the use of various technologies, and present recommendations regarding the pedagogical use of handheld/mobile technologies.

Technology Summaries

We discuss separately the two forms of instructional technologies that were implemented within the participating classrooms. For each technology, we will present a synthesis of the perceptions of how it was implemented, its strengths or opportunities relating to pedagogical

apps, such as access to cellular networks (and speaker on older models). As a result, the iPod Touch is slimmer and lighter than the iPhone.

[2] Netbooks are a category of small, lightweight, and inexpensive laptop computers suited for general computing and accessing Web-based applications typically with long battery life.

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goals and classroom practice, and challenges presented by the technology in focus.

Apple iPod Touch Technology for Mathematics Learning

Apple iPod Touch handheld mobile devices were used six times in one of the three classes for the mathematics learning within the Linear Relations unit of the Grade 9 Academic curriculum. The teacher maintained an instructional website on which he posted digital versions of each day's mathematics note. The lesson was often done on an Interactive Whiteboard, a hard-copy distributed to students on paper, and then the digital version made available via the Internet on the instructor's school website. Videos of the various math skills/problems in the unit, which featured the voiceover of the math coordinator along with screen capture documentation of point-and-click manipulation of related software, were also made available on the instructor's website. These videos of skills and various problems being solved could be manipulated by students either as a hint-based tutorial system, or as a checking and/or reviewing mechanism. The teacher also experimented with what he referred to as an "interactive note" in which the lesson in Word copy included direct hyperlinks that would take the student directly to the YouTube-based video, allowing students easy and quick access to both forms of visual representation (print and audio/visual) of the mathematical content and new learning. The teacher describes an average classroom experience:

> **Teacher:** Basically, I would teach a lesson and use a SMARTBoard or whatever as the lesson, and then when it came time for the worksheet, what I would do is distribute the worksheet, distribute the iPods, and have kids work through the assignment. While they were working through they could go online, they could use the iPod, go online—all of this was online, the videos were imbedded, so I did need the Internet. So, they would go online, they would go to my website, open some links, and that's going to take them to some of the videos where you had some walkthroughs for the solutions.

Students would work in pairs or triads and although they mostly completed the worksheets/assignments individually, they were encouraged to talk to each other or to ask the teacher if they needed assistance. As one student noted, "It's not really like working in pairs or groups. It is

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mainly independent but then what happens is our desks are set up in threes kind of. It is like one, two, three in a row and if you are having problems and [teacher] has a line, 'You can just ask the person next to you'." Since much of the iPod review involves an audio feed, students are often working individually, by default. One teacher noted that, "[M]ost of them have their own headphones, and we encourage that just for hygiene reasons." Another teacher indicated that this isolation may not always be a good thing:

> **Teacher:** One thing that you do not want to do is have them plugged in, and tuned into this, and they don't feel like a part of the class. You could use these in a group situation: they are conducive to centers, if you will. If you had maybe one iPod and a jack where you could plug in a couple of different headphones and they're all watching the same thing. I have not done that like that, but you could certainly use them for that. I have thought that would be neat.

Students in the iPod classroom describe how they remember using the device in class:

> **Student:** Well, everybody had access to it on certain days—if there was a video tutorial for it.

> **Student:** Yeah, we would use it as we were doing the work, if we came to a question that we did not understand then we would pull out the iPod and go to the video and it would explain it to us.

> **Student:** Especially if the teacher was busy too because then he would not have to be going all over the classroom all the time, there is something for you already there.

Another focus group provided a similar recounting of iPod use in class, noting the video utility:

> **Student:** I guess we just use it every day, and then [the teacher] gave us the lesson and then we would go on the website and we could watch videos on how to do it.

> **Student:** Yeah, he would give you a worksheet and then you get the iPods.

The Apple iPod Touch handheld devices were used half-dozen times during the unit of study, usually as a method for retrieving video clips

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from the instructor website which were pre-recorded by the math co-ordinator to walkthrough a mathematics question. The teacher also experimented with what he referred to as an "interactive note" in which the actual worksheet created as a Word file contained hyperlinks for questions which, when clicked, would take the student directly to the YouTube video as it popped up on the iPod Touch screen. Students mostly worked independently, with headphones, primarily using the iPod Touch videos as a tool for checking answers or as a series of walkthrough hints consulted when stuck.

Perceived strengths of the iPod Touch technology for learning mathematics

The perceived strengths of the Apple iPod Touch handheld devices were their size, ease of use, reliability, and the fact that they represented a popular and "cool" machine that a number of students already owned personally and were therefore familiar with in class.

The size of the Apple iPod Touch made the handheld device very easy to use, as well as taking up much less space on a student's desk. As two students so indicate:

> **Student:** These were just easier because they were smaller. Especially like an iPod—just put it in the corner of your desk and you have lots of room still to work. Not like a laptop where it takes up this much of your desk, and you have to lean your paper back over the edge of your desk.
>
> **Student:** [iPod Touch] basically does everything a laptop can do, it is just smaller and more convenient, so I would really prefer that much more.

One of the teachers noted how the ease of use of the iPod Touch is also linked to students' previous experiences with technology outside of the classroom:

> [T]he students likely prefer the iPods, not only for the wow and cool factor, but ease of use. They are more familiar with how to operate an iPod than they are with how to operate a standard computer which we might think is something out of this world really. We might think of that as something really obscure because we're thinking, 'Well, this is the computer technology generation right?' Well, really they are the handheld

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> technology generation and this is where they are moving, this is what they're used to, this is what they are comfortable with. Most of them can probably T9 better than they can home row. T9 would be with the cell phone, texting using the 9 digits right—so, most of them can probably T9 faster than they can home row. [i.e., use 9-digit number keys versus regular 'home row' keyboard]

In review, iPod Touch handheld devices were admired for their ease of use, quick boot-up speed, reliability while functioning, relatively small size particularly for use on traditional school desks, and their contemporary "wow/coolness" factor. Overall, students spoke very highly of the device.

Perceived challenges with the iPod Touch for learning mathematics

Four perceived challenges relating to the iPod Touch were the input limitations (i.e., no keyboard), the short battery life, the distraction factor, and the fact that as an Apple product they did not permit students to view/use any flash-based software/animations (e.g., CLIPS resource).

The Apple iPod Touch handheld devices require a lot of energy and hence the batteries have to be recharged frequently, creating an obvious challenge for both students and the classroom teacher.

> **Teacher:** Charging these things right now, it is a bit of a pain, but fortunately I have a co-op student and they charged them for me. But you have got to plug them all in individually.... One at a time. Well, each computer has about four USB ports so you would plug them all in. So out of four different computers you could have about twenty going at the same time.... There is a charging cart which would speed things up, facilitate charging them as well as if you wanted to download some videos.

While the distractions of the Internet were not unique to only the iPod technology, this did present some potential challenges to students' ability to focus.

> **Teacher:** I would definitely say that the iPod Touch is less controllable versus the computer being more controllable.... [O]ur server can tell where these [students] are at all times. So,

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> our servers can tell the addresses that they are on and the files that they're accessing. It is a little bit harder for students to on a whim go and close—some don't know the Alt F4 trick so they still have to navigate all the way up to the x in the top corner to get out. So, if we are just doing a browse around it is not usually a big deal, whereas with these, if you have Facebook open you hit the button and boom you are done. So the distraction is there with the iPods.

Students in the iPod Touch class generally seemed to feel that the Internet distraction element was minimal throughout the month, indicating that a fast-paced lesson, teacher availability, and the desire to use the iPod and video review during class time to better understand and complete the daily assignment were all reasons that may have contributed to a general lack of off-task behaviour in this regard.

> **Student:** I was more focused on finishing the worksheets, so you did not have it for homework later, instead of doing other stuff.

> **Student:** I have seen other people, but when I am on it, I am on it for that reason. I am not the strongest math student, so I'm down to work, and I don't care about anything else.

> **Student:** Our generation is so used to multitasking, where you can do pretty much both things. Most of the time you have like a tab function, when you press something it opens all the windows that you have open in Safari or whatever, and then you can just scroll back to that. Most of the time you are just using the math website. I have seen some people on Facebook, but it is usually after they are done and they are just waiting.

One can perhaps conclude that, when the lesson is kept moving by first the presentation of the lesson on the IWB/board, followed by the immediate use of the handheld device for the worksheet assignment, and regularly circulation by the teacher to help students, the distraction element seems to be diminished.

Although the video review content was not affected specifically, one final challenge with the iPod Touch devices was their inability to play Flash-based content. The math coordinator noted some of the difficulties surrounding the lack of flash-based capability on the Apple iPod Touch devices:

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> **Coordinator:** There is still the drawback that it does not support Flash and a lot of the good math resources require Flash and that is where, going back to your original question, that is an issue with the iPod—that you can not access anything that is interactive Flash.... CLIPS and Gizmos and any kind of software like that.... [T]here are also a lot of the OERB [Ontario Educational Resource Bank] materials that are Flash-based as well, that are automatically eliminated. So, when you think about that—the iPad, nobody is really all that excited about it yet in the math world, but there are going to be things that come up in five years.... If you have iPads and then you have a couple of desktops [computers] around the room, and if there was something that required interactive action, they could just go over and play around on that.

In review, the Apple iPod Touch devices lack a physical keyboard input for text-focused activities, take longer and are somewhat more cumbersome to recharge, have the potential to lead to Internet distractions, and are incapable of playing Flash-based programs. Notwithstanding, there still appears to be a number of perceived advantages of this technology, particularly in relation to the video reviewing/playback.

Netbook Laptop Technology for Mathematics Learning

Netbook laptop technology was used once or twice per week throughout the month-long Linear Relations unit of the Grade 9 Academic curriculum. The teacher maintained an instructional website on which he posted assignments, homework, and links to files that he had created using various software programs. The lesson was often done on an Interactive Whiteboard, followed by the assignment of an activity, which required the use of one or more prepared files accessed via the laptops. Using a combination of software titles (e.g., The Geometer's Sketchpad (GSP), PowerPoint, Camtasia, and SmartNotebook), the teacher created several of his own narrated mathematics videos to review certain skills/concepts, and these were meant to be used by students either as a hint-based tutorial system, or to check work and/or for review. He referred to these as "interactive notes," the difference here being that he himself had produced the video component. He describes the classroom experiences:

> **Teacher:** Well, I used the Netbooks primarily for their access to many different programs that the iPods weren't allowed to

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> access. I used Flash to go over a unit review; I used Geometer's Sketchpad to go over a few ideas interactively with the students. We had a Flash tutorial that we went through regarding slope, and relating slope to modelling linear relations. Also, prior to the this study starting, we did a fair amount of work with the Netbooks and the Microsoft productivity package to put together a substantial lab report leading into modelling linear relations. I would say that I used the Netbooks [about] 9 times throughout a unit of 16 or 17 teaching days. So it was fairly heavy saturation having it in the classroom.

Although the Netbook laptops were equipped with keyboards for text entry, they were nonetheless critiqued for issues relating to input, mostly focusing on the small tracking pad that was used with the laptops for software control in certain activities. Students describe challenges:

> **Student:** Sometimes I could not really place the things—I had to redo it to get it right.
>
> **Student:** Yeah, I am used to the mouse, I am not used to the laptops.
>
> **Student:** The touchpads were so tiny compared to a normal laptop so sometimes my fingers would go, and I would be like, 'Why isn't it working?'

This teacher made the laptops available to students throughout a month of lessons; he designed a series of his own tutorial-type demonstration videos, and students used the Netbooks to complete GSP activities and to view and review the teacher's GSP tutorial files.

Perceived strengths of the Netbook laptop technology for learning mathematics

The keyboard input, which was uniquely available on the Netbook laptops, was seen as an advantage for typing up word-based problems or solutions in mathematics class.

> **Student:** I think those are better because they have everything that this has [iPod] plus more word documents and stuff like that, so that you can type stuff up.
>
> **Student:** Yeah, because we do homework on the laptops and you can't really do it on that [iPod].

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Other strengths were that the battery charge lasted longer than on the iPods, and that all Flash-based programs/demonstrations from the Internet could be viewed with the Netbooks.

Perceived challenges with the Netbook laptop technology for learning mathematics

The main perceived challenges with the Netbooks were the difficulty with login and network access; the problems experienced with the small tracking pad; the inability to save work locally to disk after the general username/password was implemented to free up space taken up by personalized settings/backgrounds/etc. and speed up software processing; and the issue with potential Internet distractions.

Not only was the Internet access problematic until it was eventually sorted out with router signal strength, but the Netbooks were described as overall being slow, clunky, and unreliable.

> **Student:** Sometimes your username would not work so that was difficult. When you log on you have to put in [assigned username], sometimes that wouldn't work.

> **Student:** Sometimes it would take long to load the lesson, so it wasted some of our time. Sometimes the computer would take five minutes to load something.... When we were doing the videos it would still load slowly.

> **Student:** You cannot even save your work because you have to go under the [password] thing and then you cannot go under our usernames, so we cannot save it on our hard drive.

Similar to the concerns shared around Internet distractions on the iPod Touch, the Netbook laptops were also described as having this same particular challenge for students during class sessions. In the statements that follow, the teacher describes how he feels that students are on-task and not being distracted by the Internet as they work on his activities.

> **Teacher:** I was surprised. Other than the first few minutes when you give them out, they might be checking their email or they are checking their Facebook, or whatever it is they are doing. Once it was mentioned, 'This is it, let's get to work here,' I mean they have all signed a contract saying that they're not going to do that.... They are going to use it for only academic purposes, so maybe it is off-task and they are working on their

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> English. But for the most part, once we got going they would stay focused on their task.

The teacher using the Netbooks also developed an online homework system for distribution and submission. While he maintained that it was an effective means of organization and communication, some students felt frustrated with this new system describing factors such as software incompatibility, lack of printer ink or paper, viruses, and their own forgetfulness in printing off the work to complete.

The Netbook laptops were used as a method for students to access software programs, to view and review prepared video tutorials, and to type out answers to online homework. While there appears to have been complicated issues surrounding login, slow processing speed, and the clear dislike of some students for online homework communication, the Netbooks were also appreciated for their ability to access Internet, including Flash-based online content, and the way in which a keyboard allowed for text input.

Other Significant Issues

While coding the survey and interview data, a number of significant themes around particular issues began to emerge, five of which are highlighted below.

Technology experiences in elementary school

When we asked students to recollect and share the types of technology-related experiences that they remembered from elementary school, we were not surprised at the wide range of experiences that were noted. Two main factors enter into this reality: we suggest the accessibility of different technologies at the various elementary "feeder" schools and the varying degrees of teacher enthusiasm and background training in relation to the various technologies for mathematics instruction, whether available or not. What follows is a sampling of student comments, demonstrating the wide range of prior exposure:

> **Student:** The most I remember using is just an ordinary calculator.... Not even a scientific calculator. Just one of those basic ones.

> **Student:** I didn't use a graphing calculator in Grade 8, but in Grade 9 we did.

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> **Student:** We used the SMARTBoards, and they were talking about getting the iPods.
>
> **Student:** We had computers, but we never used them for math.
>
> **Student:** We used calculators and the SMARTBoard.
>
> **Student:** We didn't really use anything, most of it was done on chalkboard. Every now and then, when we weren't using the same room—we were in a portable—so sometimes we would go in the meeting room and there was a SMARTBoard, or sometimes a projector.
>
> **Student:** We didn't use any. We just used the chalkboard.
>
> **Student:** We didn't use any. We used a projector where you write all of those little papers.
>
> **Student:** We used to get those quizzes with the Senteos [clickers], or whatever they're called, in Grade 8, but that's pretty much it.
>
> **Student:** We used laptops every day in three of my classes.... math, geography, and language.

Obviously, there exists a large gap in previous technology use in both a general sense (overall curriculum) and in a more specific sense in terms of mathematics instruction. From the above-listed interview excerpts, we note an exposure to and prior knowledge of technology for mathematics learning ranging from a complete absence of technology to individual laptops for students with highly regular IWB usage in classrooms. We see this discrepancy as further supporting the rationale for technology being used in all secondary school mathematics courses, thereby addressing this experiential gap with which students enter high school and also ensuring the competencies which they will require as they exit secondary education en route to post-secondary learning, or to the workforce.

The perceived effects of technology on teacher-student communication

A second theme of interest that emerged from the data was the perceived effects of technology on teacher-student communication. Somewhat ironically, technology was viewed as both a means to im-

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prove communication between the teacher and student (i.e., freeing up the teacher to have more time to spend with individual students while other students used the video tutorials to review/check instead of waiting to have questions answered by the teacher), and yet in other ways, a potential detriment to this pedagogical relationship in the classroom (i.e., teachers perhaps on occasion deferring to the electronic files/devices when asked for help by students, instead of answering them directly).

> **Teacher:** The best thing would be to help one-on-one for each of the questions, but that is not possible. But definitely I would say that [the device] gives you that advantage.

> **Student:** You can ask a lot more specific questions and get the answers you actually asked for.

There were times when students indicated that one teacher in particular would sometimes defer to the devices/machines instead of wanting to immediately answer the question.

> **Student:** Sometimes if you had a question he might just be like, oh, 'You have the video, just watch the video.' Sometimes he might not help you if you have a different question.

> **Student:** Well, you have to watch the video first. You're not going to just ask a question before you watch the video.

> **Student:** But if you already watched the video and you don't understand, there is a human on there describing how to do it.

> **Student:** [I]t was a lot like that at the beginning of the year too. You would ask a question but sometimes he wouldn't always explain it the way you wanted it explained. I guess with the computers he thought we could just find it easier, like look on the Internet and find it in what we were doing. That wasn't always the case.

> **Teacher:** If you're comfortable using this to answer a question you do not have to put your hand up, but if you want to you certainly should have that option.

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No doubt there was a context for such student-teacher dialogues, perhaps with factors that were unknown to us (e.g., student tone, prior behaviour), yet we share this example primarily because it addresses what we feel to be an important issue in the educational technology debate: that humans can and should not be completely replaced with technology in the classroom, but rather need to work with the available technologies in supporting students with more comprehensive and heretofore impossible strategies in the classroom.

The perceived effects of technology on student engagement/learning

A third theme that emerged from the data was that of the perceived *effects of technology on student engagement and learning*. By and large, the overarching opinion of the various technologies involved in the study was that they improved student engagement in learning mathematics.

> **Teacher:** It is difficult sometimes to draw that on a chalkboard and show kids what you mean versus showing them a video. And if there's some audio to go along with that, that talks them through. Actually seeing what is happening and the ability to see a lot of things very quickly and compare different situations. So, you are looking at different graphs, 'Well, what happens when this is this slope, and what happens when this is the y intercept, well what if that changes?' So, you can show five different graphs in a very short amount of time and get that point across effectively.

> **Student:** Well, if someone is a visual learner. For me, if the teacher just stands up at the SMARTBoard, I don't pay attention because it is not something that attracts me. But if we were using that [mobile device], and it is something that is hands-on and we know how to use it, I think that we would use it a lot.

> **Student:** It would definitely be better for visual learners because it would be all hands-on work and it's not just sitting there having to listen to the teacher day after day.

> **Student:** It made class more fun.

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> **Student:** It was not just the same old boring lesson. You are like, 'Oh, good we get to use the computers today.'
>
> **Student:** I found it was more fun when we had the computers for sure. I would rather go to class and work with a computer than sit there with a pen and paper.
>
> **Student:** Some people will not pay attention to the teacher but when they are holding the iPod they are interested.
>
> **Student:** When you're just sitting there in class, you blank out because you are just writing stuff down and looking at a blackboard but when you are more interactive and more into it, you are paying attention instead of just sitting there writing notes.
>
> **Student:** I liked it better because if we were actually doing it with pen-and-paper we would have to get the ruler out and then we would have to draw the line in the different colours there, but it is more fun to do it on the computer when it does not take so much time.
>
> **Student:** I liked it. I found it easier.
>
> **Student:** Yeah, I mean you bring an iPod into the classroom and you have got sort of student engagement right there. You have got kids asking, 'Are we going to use the iPods?' and, 'Sir, when are we going to use the iPods again?'... They are excited about doing the math on the iPods, so they are a little more motivated.

In terms of the quality of the learning, or to what degree the learning is enhanced, or deepened, by the use of mobile and handheld devices in mathematics teaching and learning, this led to a variety of responses, each referring to different factors for learning which they felt were critical.

> **Teacher:** ... As a starter at the beginning of class, or to get them interested in the topic. Using these to kind of explore.
>
> **Student:** Yes it opened up new ways to figure out problems and usually when you answer one problem, the answer gives you about 30 more questions. More or less you learn faster, I

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think.... Basically you get things done faster, and usually you pay more attention.

Student: You can have more examples. You can build the framework in your mind for how to do the work and most of the time it works really well and people can learn at their own speeds.

Student: People have strengths and weaknesses. So you can focus on your weaknesses.

Student: Yeah, and I find sometimes that if he explains it one way, and then you watch the video, it helps more because it's explained in a different way.

Student: I think the videos helped.

Student: [I]t just helps hearing somebody else explain it. [Teacher] is a really, really good teacher, but it just helps to hear another person explain it with different words and stuff. When I am at home and I do not know how to do it, it helps me learn and reach my goals better and stuff in the math program.

Student: I know that it helped me because I am a visual learner.

Student: It just helped explain things better. If you did not understand, it was a lot easier to pull up examples instead of just writing them on a SMARTBoard or trying to talk us through it. It is easier to just see it in a video sometimes.

Student: It helped me learn math because if I was on the iPod going through and I had solutions and things, there is a calculator in it too so I could just go back to the calculator and try it again if I messed up.... You could learn more because you are learning it faster.

Some students were not as convinced that the various technologies helped to increase or deepen their learning of mathematics, for a number of reasons including a general affinity for math that did not require technology, a perceived sense of "fun over serious work" which did not increase learning, and an affinity for traditional pen-and-paper over the newer technologies.

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> **Student:** I found it was fun... but it did not stick in my mind because I thought it was fun, I did not think it was serious.
>
> **Student:** It was more fun to use the laptops. I probably would have understood it better if we didn't use them—I am more of an auditory learner, not visual.
>
> **Student:** It may be visual but I had a hard time with it because I am used to pen-and-paper, I am not used to technology. That screwed me up and that is why I got a bad result on my test.
>
> **Student:** It probably would not have affected me very much because I like math in the first place. [teacher] is a pretty good teacher and he explains it well. I guess if I didn't understand it, I could just ask him right away and he could say it in a different way that makes sense.

Both teachers and the coordinator expressed some degree of qualified approval of the various technologies used in the mathematics classrooms. It is interesting to note the variation in student responses, which of course also depend on the teacher's preparation for, and use of, the technology, the number of times a given technology was used in class, and the way in which the debriefing of the technology explorations/demonstrations were handled in class. Overall, the iPod Touch students seem to have reported most favourably in their views regarding positive effects of technology on learning.

The development of two forms of "Interactive Notes"

A fourth theme that emerged from the teacher and student interviews was that of the *creation and implementation of "interactive notes."* Both teachers (in iPod and Netbook classes) used this term, yet both had created a digital resource that was unique in some respects, which drew upon their own individual technology strengths and which also was related to their specific goals.

The teacher implementing the iPod Touch technology was in the habit of providing digital Word documents mathematics lessons, particularly to accommodate students with Individual Education Plans (IEPs). The teacher extended these documents to include live hyperlinks which would then take his students directly out to YouTube videos that the coordinator had prepare beforehand, based on unit problems. In one particular instance, the teacher further experimented with a review

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strategy whereby he handed out a "lesson sheet" which had blank portions (not just missing words, but missing worked examples) that needed to be filled in by watching the video tutorial. It was this document that he actually referred to as an "interactive note," i.e., for which the students needed to interact with the video tutorial. We submit that both of these ideas have potential merit: the physical document (handout) that requires digital interpretation (watching the review videos and transcribing into written form for the note), and the digital document (hyperlinked Word file) that requires physical interpretation (activating links; playing/pausing/rewinding video clips). In the discussion excerpts that follow, the teacher describes what he observed in his class as students began to use the video tutorials/clips. He also shares his own beliefs regarding how and why he considers the use of the iPod Touch technology in the classroom to be a highly effective approach to mathematics learning and review, for a number of reasons.

> **Teacher:** [Students] had a copy of the static note.... Some of them would write [the lesson] on their own. Most of the time they will have an outline just to speed things up a bit.... A lot of their IEPs say that they need copies of notes—well all of that is available there and online as well. They have got that certainly. In terms of using the iPods in class, primarily for videos.... Now in addition to using it as a walkthrough for some of the questions and the worksheet, I gave one lesson where basically I did a quick intro, gave them a supplementary worksheet—not so much a worksheet, but a lesson sheet, and they had the lesson on the iPod. They listened to the iPod.... So there would be questions throughout the video and they sort of got a guided sheet that they're working through at the same pace as the video is going, and they might have the questions written down and the lesson is on the iPod and they're watching the iPod, they're pausing it, they're filling out their own note. So, it was not a worksheet, it was sort of an *interactive note* where they were going at their own pace, so I did use that as well.... [I]t gives the kids a chance to pause it, which a lot of them, if they do need me to pause, they may not say that in a lesson.... They are all going to get to the objective of that note, but some of them take different paths. Some will pause it, rewind it, check things over again.... Anything new takes a bit of getting used to. A lot of them would try the question and then check it if they were not sure. A lot of them, if they knew that they had the

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> right answer would just go on to the next question and only use the iPod when they felt that they were challenged.... This gives them the solution, not just the [final] answer. Because if they do not get that answer in the back of the book, they do not know where they went wrong and they cannot figure it out. This way they can actually see where they went wrong.

The second teacher using the Netbooks was much more familiar with a variety of software tools and, due to the fact that his class was experiencing YouTube access issues, he creatively designed his own video tutorials and then connected these videos via embedded hyperlinks to documents that he had written and posted on his website for access with the Netbooks. So, compared to the iPod class where digital notes were linked out to YouTube videos, in this case they linked across to other applications/files on the hard disk within the local laptop. The teacher explains the process of creating the video files and then provides a very insightful glimpse into student questioning that he believed actually changed as a result of the kind of activities that he was asking his students to navigate and complete.

> **Teacher:** I mean hyperlinks back and forth between the note. So they would watch a certain session and then they would go back and forth, and that is what I consider an online note, I call that an interactive note.... Camtasia, PowerPoint, a number of different programs that I use... It is all self-contained.... The reason why I did self-contained was exactly to get around that YouTube access issue. I was having those problems, so I came up with that solution.... I would have to say of the whole unit, probably the best day would have been when we reviewed the interactive equations note with looking at how to make line equations from two points, using point-slope. They had homework that went along with that and they had the note, and they would use the note and the homework which was later built into the whole unit review note.... I was not answering less questions but I was answering more higher level questions. So those lower level, 'What process do I follow?' questions—students were taking a little bit of ownership over and using the file to help themselves out with. But then the, 'Hey, what would I do if I had a more complex situation?' or, 'What would I do if I had this?' Those were the kinds of questions that I

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> started to get because those were the ones that were not covered in the online note.

When interviewing the mathematics coordinator on this particular topic, he commented on how he noticed students begin to use the videos differently over time, and also described the large final review that he prepared for the teacher's class using YouTube videos, and which could be equally accessed from an iPod Touch or from a child's home via the Internet.

> **Coordinator:** I guess I have to go back to what I overheard one of the students say. It was a very interesting point, we have not trained the students how to use this, and one said, 'Well, you could just watch the answer.' Well yeah, you can, but that is not going to help you on a test. I know that [iPod teacher] used it as review, he came up with the idea, 'How about if we gave them a quiz with the iPod so that it is similar questions, it is like an open-book test, so then it becomes good review, and later on you put the iPod away so that this is a way that we can start.' It is like the training wheels, and then we're going to pull them off for the test and the exam.... The nice thing is that they can access it with the iPod in class, and they can go home and access it as long as they have access to high speed Internet.... The videos went with it, so there are worksheets, and now what they can do, is they can follow along and check their answers as they go.

In all of the above instances, the "interactive notes" idea, whether it involved links to web-based YouTube videos or to instructional video files stored on a computer's local hard disk, or even on a local server, would ultimately be judged on how, why, and when these videos would be used by students in the classroom sessions. And so it is to this important set of questions that we now turn our attention.

The perceived effects of technology on students' ability to review lessons

The fifth, and perhaps the most important, theme that emerged from the data analysis was that of the *students' ability to review lessons* via the iPod Touch and Netbook technology. Overall, this appeared to be viewed as an example of student empowerment in which students were placed in a position of at least partial control of their own learning. It

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was also described as a logistical advantage insofar as many of the videos were available online 24/7 for class-based and external course review.

> **Teacher:** Well most of the bottom groups, they are the ones using this technology more. A lot of it is reviewing their answers, checking their solutions to make sure that they're on track. A lot of the gifted students, they have that confidence, 'Yeah, this makes sense, let's go to the next question,' they do not use it as much, they do not need to use it as much. Now that sort of brings up the question, well what do they do? They are going to whip through it, and the other end of the class is going to be a bit slower, so you just have different worksheets.... But it is kind of nice though when you are working on something to go around, they're working on their worksheet, sure they are working at their own pace and they have got this kind of teacher talking to them in the iPod, but I can go around and do one-on-one stuff now... and that's a huge advantage. So, I see so-and-so here, they are not getting this stuff right here. Well I do not have to stop, and then everybody else is waiting for me, they are still working and I am working with that student on that one problem that they seem to have, and then work that out, and then I can find somebody else and just do a lot of one-on-one. I find that is very effective.... I guess it would just be quite often they come up against challenges when they are solving a problem and that is where a lot of kids quit. If they cannot get to that final answer and there are some steps along the way that they struggle with, a lot of kids will give up. This gives them the chance to see the solution and not just the answer and it helps them through maybe those struggling points throughout the solution and I think that would give them a little more confidence.

The use of video tutorials was also described as a form of self-directed, or we can perhaps even add the notion of *self-differentiated learning* in terms of how, when, and why the videos were used, depending on students' own self-perceived learner needs and related goals for the mathematical activity.

> **Teacher:** There is one of me and there are 30 of these things [iPod Touch]. Maybe that is what they are talking about in

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> terms of it helps them to learn math—it helps them to get through each of the questions if they are struggling with it, instead of skipping the question and moving on to the next one.... The nice thing about that is that it would give them a chance to stop. When I am teaching a lesson, I try and go at the pace that is comfortable for everybody, but it is hard to reach everybody, and some kids need to see stuff over and over again. So having these videos is a good chance for them to pause it, stop it, rewind it, replay it, relearn it, and see things over again. They also, I mean they would have the ability to go at home and most of the kids have their own iPod Touches so they would be able to access this from home as well.

The use of the video tutorials seemed to provide students with a form of *empowerment*, both in class and at home, as clearly expressed by several students in the following interview excerpts.

> **Student:** Well, I really like it because if the teacher is really busy then it is right there, you do not have to wait. I am a really shy person so that helps me a lot. So, I do not have to ask the teacher and he is really clear when he is describing how to do it on there. So it helps me a lot. I do not think that we are allowed to take them home, so if people do not have access to the Internet that could be a problem, but apart from that I think it is really great.

> **Student:** I think it gives you more opportunities just because ... if you cannot find the answer, you can go on his website and it will explain how to do it.... I play volleyball all the time so I am gone. That weekend when I get home, I will go and print off the lesson or print off the worksheet, so it is kind of giving you more opportunities.

> **Student:** If you were sick one day, he would put the notes on there so you could copy down. He will put up, when we have tests, just to check and see what you missed.

> **Student:** It is good because he shows you how to get to everything. If you are at home and you cannot figure out a question or something, you can go and watch the video.

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> **Student:** He put everything on a video and then you could click on that and watch it instead of just watching one long video of it. So, it focused on a part if you needed help with it.

It was interesting to listen to accounts by students and teachers about how the videos were actually used. For some, it was a hint system, others a checking system, and still others a reviewing mechanism.

> **Student:** If I had a problem with a question, I would watch only half of a video, so I would not see the answer, and then I would do the rest and then I would watch to see if I got it right.

> **Student:** Watching and doing…. we would try the question first, and then if we did not get it, then we would check.

> **Student:** Sometimes I would do the question and then just to see if I did it right—not necessarily to help me.... more of a checking.

> **Student:** It just depends on the person. If you do not understand it, you are going to use it, but if you understand it, you will just go through it and check your answers after.

> **Student:** Well, if the teacher is busy, if they have to do something on the computer or anything, you do not have to ask, and it is right away, it is right there, it is really simple to just go on it.

> **Student:** Personally, for me, I can do it without an iPod, but using the iPod is easier. If I draw a blank or something, you can watch a video, like a how-to video and just refresh yourself and go through it. That is what I did.

> **Student:** I would just watch the video. I would do my question and then watch the video to see if I got it right. If I got stuck I would watch it.

> **Student:** Yes, if you were in the middle of a question and you forgot what you were doing, you would just rewind the video and watch it again.

Several students noted how valuable the video tutorials were as reviewing tools for the end of the unit of study, or again later on for reviewing the unit material for the final examination.

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> **Student:** It would be easier to do that on the iPod because then you do not have to write or anything, you can just watch it and take notes, or whatever. You could just still watch it and remember.
>
> **Student:** It helped me so much. We just got our review and there's stuff that we did like eight months ago, I do not remember how to do it. I did my review last night and every single question from a long time ago I had my iPod out and I was copying down from it.

Interviewing the mathematics coordinator often provided us with a broader perspective on what he saw happening with the teachers and the students using the technology in the various classes. In the following excerpts, he discusses the significance of video permanency.

> **Coordinator:** I think that the nice thing about creating content for a course like this, or for what we've been doing here is that it's permanent. Once the digital content has been made, it's always available. It isn't like a particular type of software or creating a TI83 lesson, or whatever, that's going to be obsolete in a few years. A lot of what's being created here is transferrable into future videos. I mean even if it's just clips embedded into something else, later on we're developing a library of resources that we can tap into for years to come. So, that makes it much easier to justify putting in the effort.

The teacher from the Netbook class also supported the idea of sharing instructional videos that were created by teachers with the whole board, or maybe even beyond, via Internet. In essence, he was arguing that instead of limiting server space for teacher files to a minimum size on the local school server, the board should consider housing these at the board level so that all teachers could benefit.

> **Teacher:** Yeah, I do not believe that they should be on the [secondary school] server if they want to be shared through the whole board. I mean why not have them out there, why not have them public domain so that other teachers can access and use this stuff? Isn't that the goal right? Developing lots of public domain content that can be used?

The use of video clips for reviewing mathematics skills/concepts led to a number of very interesting observations, chiefly among these the em-

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powerment of students to take control of their own learning both in class and external to class via the Internet. Videos were used by students of varying mathematical aptitude (IEP special needs through gifted/highest achieving students) in a number of different ways: digital walkthrough lesson, a play-pause-rewind hint system, a catch-up tool when absent, and as a method of reviewing the lesson, or unit, or in preparation for the final examination.

In summary, we have discussed five emergent themes from the survey and interview data. Elementary school experiences with technology in mathematics, or in school curriculum more broadly, were shown to vary drastically among students. Perceived effects of technology on student-teacher communication, on student engagement, and on student learning by and large were shown to portray a positive impression of the technologies being used in the classrooms. Finally, the ability of students to take control of their own educational experience through the use of *interactive notes* and videos, and via *self-differentiated learning* was highlighted.

Notable Obstacles

Perhaps one of our most meaningful contributions as researchers, in retrospect, was the opportunity to work with the mathematics coordinator and the board computer technicians in trying to troubleshoot and to present ideas for improving the technological conditions which were necessary for the study, and hence for future use of handheld/mobile technologies in mathematics teaching, to actually occur. In brief, there were two main obstacles that needed to be addressed: (i) the lack of strong-signal, consistent wireless Internet connectivity (with three related sub-issues); and, (ii) the accessibility of YouTube videos.

Wireless connectivity

Wireless connectivity requires an adequate number of nodes or routers placed at strategic locations near/around the classrooms. However, if the goal of school-wide access for students is to be realized, a more large-scale approach would apparently be to install an Enterprise system that allows each student to logon to the Internet and have their account stay open and follow them throughout the various school locations and throughout the course of an entire day. One other related issue is that the Apple iPod Touch devices maintain a record of previ-

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ous connections, so, in the case where there were two available signals in the iPod classroom, the devices needed to be adjusted to force a certain signal feed. This would also be the case if a school-wide Enterprise system was installed. The teacher and the coordinator had comments about this issue:

> **Teacher:** It was a really difficult problem up to the point where we switched to [being] closer to the login server.

> **Coordinator:** The short-term solution might be the mid-term solution as well, because it has worked and that was adding a second node right in [iPod Teacher]'s classroom, and there was one down in a room down below, so it was probably about 30 feet or 40 feet away, and so half the class, or a number in the class, were accessing the one in the room, and the other half of the class were accessing the other one.... The problem is if you get a whole school working on an iPad and all of a sudden this becomes popular, a school decides to go that way, they are going to need to have the Enterprise solution that they have in hotels. You login once and it follows you through the building, you do not have to go to a different node and you can move around the building. That would be nice too, but it is more expensive, and I think that is the long-term solution.

Another issue related to wireless connectivity is whether or not the handheld/mobile tools of choice are eventually to be allowed to travel with the students, like calculators, both in and out of school. In allowing students to sign out devices, as is now being done in many jurisdictions, the ability to access the Internet, YouTube instructional videos, etc. obviously becomes more tangible. However, along with this distinct 24 hr/day access advantage comes a number of difficulties including overall cost, damaged/lost/stolen devices, and issues relating to the censorship of the Internet outside of school.

> **Teacher:** If they are at school they all have access.... Now, if they do not have the computer then they are not going to have the Internet, unless they are stealing their neighbour's. If it was right on the iPod again, and if they could take those home, then that would solve those issues.

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Finally, a third issue that pertained, although more tangentially, to wireless connectivity was the way in which the Netbooks were programmed to house student data by virtue of individual passwords and preferences. This situation, coupled with limited RAM installed on the devices, meant that both boot-up (starting the devices) and system performance was overall very sluggish, making Internet/video access all the more difficult for the user. Therefore, it was decided in consultation with the board technicians that an alternate route would be pursued, namely, wiping off individual passcodes/preferences in favour of a shared network password and requiring USB sticks for external backing up of files. Not an ideal solution, as reported by the Netbook Teacher and by students during the interviews, but one that did ultimately allow students to enjoy faster machines and Internet access.

In summary, the three issues relating to wireless connectivity that emerged from the data were signal strength in the math classrooms, the idea of extending connectivity to the entire school with an Enterprise type solution and/or to the home via an assigned device system, and the changeover from individual settings on Netbooks to a shared username/passcode system that increased performance.

YouTube filters and video access

A second major technology infrastructure issue was that of accessing YouTube video through the board and school system filters. Open Internet/YouTube access within a school context, which in this case was necessary for the handheld/mobile device research, often involves a debate regarding censorship.

> **Coordinator:** Yeah, if you go and type it in, I am pretty sure it is blocked, but if you go on a website where it has a YouTube video sometimes it just refuses to load. But probably [Netbook teacher]'s math is not locked.... Netbooks have a different Internet key on them so it has more websites open that the math teachers use, and other people in the school.

> **Teacher:** Yeah there [originally] was a Barracuda filter that you could not watch YouTube videos... I think that was opened at the beginning of the year.

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YouTube access remained a problem for the Netbook teacher throughout the unit, leading to the above-noted creative strategy of cross-linking his own prepared videos within the system.

> **Teacher:** The reason why I did self-contained [videos] was exactly to get around that YouTube access. I was having those problems, so I came up with that solution.... I definitely would have hyperlinked things to YouTube.... The second [issue] would have to be simultaneous access to certain resources, specifically YouTube. I could get one person watching the video and the rest would be blocked.... There was a filter, and that was sorted out and now it's simultaneous access to the same video.... And not on the iPods. On the iPods you can access any video at any time. But through the way that the Netbooks connect to the network, I am guessing there is some kind of watch over how much traffic is going through a certain port.

These difficulties were corroborated with student commentary, as shown in the following excerpts:

> **Student:** Yeah, there would be a link to YouTube, but sometimes it wouldn't work.

> **Student:** We could have used it—I guess it was how fast the Internet was—if it was working or not, if the connections were on.

> **Student:** Sometimes you couldn't, sometimes it would block the videos.... At first they would block all the videos, and they had to fix that.

On the issue of censorship and administrative/parental concerns, the Coordinator explained:

> **Coordinator:** So, they fixed those and that is great, but the history there is that there is too much paranoia amongst parents and upper administration and teachers themselves about either the potentials or the negative aspects of YouTube, as opposed to the potential educational aspects. There is content that I would prefer my fifteen-year-old child not see on YouTube, however... it may be inappropriate but it's not immoral. I think there is a big difference between the two. However the educational potential there, I think, outweighs that.... So, if they go to the videos that I posted, automatically if there is one on poly-

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> nomials—they would see another series of [related] videos. It was just that I started with YouTube and I know that TeacherTube was a possibility—maybe down the road, it is just that not many teachers have come onboard.

In summary, wireless connectivity was sorted out to the satisfaction of both teachers using handheld/mobile technologies that required access to the Internet. This was achieved by installing an increased number of strategically-placed signal nodes, and represented, at least to the coordinator and board technicians, a workable short-term solution. YouTube filtering, which had existed previously in the school, was managed to allow access to this Internet feature in both classrooms, yet a peculiarity of the Netbook programming/set-up did not ultimately allow multiple students to access YouTube-based videos simultaneously, and this issue unfortunately persisted throughout the research unit. That being said, the Netbook teacher did therefore experiment with his own form of "interactive note" which did not require Internet connectivity, perhaps leading to some valuable insights regarding multi-platform delivery of visual, interactive content within a technology-rich curriculum.

As we have already seen above, a more long-range plan for wireless connectivity may very well involve more than just increased nodes being installed throughout every school. Some discussion was had regarding a system such as Enterprise (common in hotels) which, although more expensive, would allow students and staff to lock into an Internet connection with a single user profile that would thereby allow them to access a strong, consistent wireless connection throughout the school grounds, a reality that would open up whole new doors for handheld/mobile technology. With the advent of digital textbooks via products such as Apple iPads, Kindles, Torch, etc., the notion and related costs of complete wireless connectivity may become more and more palatable for district school boards in Ontario. In terms of YouTube filters, the researchers are suggesting that the board continue to remove the filter, at least for certain classes, as the next stage of research will also likely require this situation.

Having conducted other research involving professional development (PD) models (Jarvis, 2009; Jarvis & Franks, 2011), we recommended that future board-based initiatives involve the following key components: (i) school- or classroom-embedded experiences with instructional technology; (ii) choice of technology and/or particular self-assessed

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area/need for development; and, (iii) PD initiatives that extend over time, with some form of accountability component built into them (e.g., pairs/triads/small groups of teachers working together on some aspect of their teaching with technology, and sharing ideas/resources/failures/ progress). As one teacher noted in the study, providing choice to both teachers and to students would be ideal.

> **Teacher:** You have got to almost offer the choice to the staff as well, because there's no point in putting someone in a traditional math class that wants to use the SMARTBoard and the technology, and vice versa. So, if we marry the teachers with the—and this is the ideal world—where you would marry the teachers with the students that want to learn that in particular way.

A second related point is that both teachers and the coordinator indicated that they possessed very different levels of computer skills. Again, this speaks to the need of allowing choice and voice in professional development initiatives, so that teachers can select and/or request personally meaningful topics that will immediately influence their teaching practices in classrooms.

> **Coordinator:** So, both were using the SMARTBoard, I would say daily, at the least.... The difference there is [Netbook teacher] is very gung-ho and has the technology to do the podcasting and the video production, and [iPod teacher] does not have access to that. So, the [Netbook teacher] was actually trying to create some of this material that he was going to use within his [class].

Professional development aimed at changing teacher beliefs and practice is a complex phenomenon (Ball, 1996; Fullan, 2005; Jarvis, 2009), and one that is only beginning to be better understood in mathematics teaching. When the use of instructional technology is added to the list of other reform-based components such as problem-based learning, manipulatives, cooperative groupwork, etc., it is no wonder that today's teachers feel somewhat overwhelmed. As district school boards contemplate the expenditure of large sums of money on promising instructional technologies, they are wise to consider the professional in each classroom who will either implement these products with growing excellence, or who may sabotage, perhaps even unknowingly, these ex-

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pensive initiatives due to a lack of confidence and/or prior knowledge/experience, which are both necessary components.

Overall, we saw a positive response to instructional technologies among students and staff members, particularly in the case of the iPod Touch implementation, but also in the case of the Netbook implementation insofar as the system glitches did not lead to frustrating difficulties with video access.

Recommendations

Based on our case study data analysis, we offered the following key recommendations relating to handheld/mobile technology effectiveness in teaching mathematics:

Multiple Tools Available in Mathematics Classrooms

One of the seven Mathematical Processes to be found within the Ontario Mathematics Curriculum: Grades 9-10 (Ontario Ministry of Education, 2005) is *Selecting Tools and Computational Strategies*. The document specifically notes, "Students need to develop the ability to select the appropriate electronic tools, manipulatives, and computational strategies to perform particular mathematical tasks, to investigate mathematical ideas, and to solve problems" (p. 14). This ability to "select appropriate tools" is encouraged from the very youngest ages in the Kindergarten right through Grade 12. This is best facilitated, of course, within a context that indeed offers a variety of tools from which to select.

We recommended that a variety of electronic tools be present in every secondary school mathematics classroom. In this way, rich problems can be approached in a variety of ways depending on the nature of the task set before students, or even on problems that they may set for themselves. The technology-rich classroom might feature a full class set of iPods or iPads, but it might also feature only half-dozen of these handheld machines, along with several desktop computers loaded with modeling, geometry, and statistical software, a cart of graphing calculators and CBR detectors, an Interactive Whiteboard, and a bin full of algebra tiles and compasses. This concept of multiple tools for the purpose of deliberate selection in problem solving is highlighted in the following statements shared by the coordinator, the mathematics teachers, and the students.

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> **Coordinator:** Then the question is, if you have iPads and then you have a couple of desktops around the room, that if there was something that required interactive action, they could just go over and play around on that, or if they could go up and play on the SMARTBoard and use some of this interactive material, and go up as a group, as five of them, and play with it.

> **Teacher:** For sure. You talk about differentiated instruction—you do not want to stick with just the iPod because it does have its limitations. You do not want to stick with just the graphing calculator. But if you mix the two together, certainly you are going to come out with a better product. Reaching more students, reaching more of what they can do.

> **Student:** Because some people like using laptops and it helps them, and some people do not, so then they should be able to have the option of whether they want to use them.

> **Student:** I would probably just want a balance of both. For notes, I definitely want to write them now, but for certain things I would also want to use the laptop.

Clearly, as Interactive Whiteboards (IWBs) become more popular at the elementary and secondary school levels in Ontario, these devices will need to be considered within a broader perspective of technology enhancements for learning. These IWBs will no doubt form a standard part of a technology-rich mathematics classroom in the 21st Century, becoming one of the multiple tools from which students and teachers will be able to select in order to solve rich problems.

Multiple Video Representations of Mathematics Skills and Solutions

The creation of multiple video representations of mathematics skills and solutions was another idea that some students shared with us during the interviews. Part of this likely had to do with the fact that the coordinator had authored the videos used in the iPod classroom, whereas some students indicated they would have preferred to hear their own teacher on the video using his usual math vocabulary. In other words, having teachers within a school or board actively involved in

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posting at least some instructional videos might be considered a viable goal for the local board. This type of reflective practice may indeed hold promise for secondary school teachers who would need to focus on key aspects of their lessons/units in order to selectively create videos. Further, there is also no reason why students could not be involved in the production of these videos, either as actors or writers/directors.

Whether the mathematics classroom involves IWBs, iPods, iPads, smaller portable laptop PCs, cellphones, or any number of other new technologies, what must remain clear is the need for adequate teacher preparation, a balance between mental/traditional techniques and those which are more technology-dependent, and a willingness of teachers, students, and administrators to continually reflect and revisit pedagogical/technological decisions based, wherever possible, on actual classroom experiences and on evidence of increased student learning and achievement.

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Chapter 3

An Innovative Practice: A Collaborative Pen-Based Tablet PC Workspace for Foundational Mathematics

Carol Carruthers

Abstract

College students in foundational mathematics programs require innovative instruction using methodologies that resonate with their personal learning style. In our classroom, an online workspace is created using a class set of tablet PC's and collaborative software. Once in a session, teacher developed course materials are projected to the front of the classroom as well as simultaneously broadcasted to individual student screens. Using the PC's stylus, students solve problems with digital ink to create an electronic notebook that is saved to a college server, available anytime via the internet. Final notes are an accurate compilation of teacher framework, student response, and class correction. Class internet searches of their e-textbook, interactive learning objects, applets, tutorials, or videos provide links that are embedded into the notebook for additional reference. Features of the software such as polling, request status or the ability to submit work electronically, give the instructor an immediate indication of student understanding. Students solve problems in onscreen groups where collaborative results are sent to the teacher for addition to the class notes. With the software feature of shared control a teacher can give one or many students ownership of the common workspace. As the temporary teacher, students present their solution to the class, thus strengthening their confidence. Early findings indicate that this student-centered tablet PC environment improves attendance, performance, group interaction, note taking skills and enjoyment, which students suggest results in their increased engagement and enhanced learning.

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Background

Some students entering post-secondary mathematics courses may be considered marginalized for several reasons: previous credentials do not meet entry standards, cultural or language barriers make comprehension difficult, extended absence from formal education requires relearning of basic skills, previous early abandonment due to rigor of subject matter, or, finally, the individual's lack of self-confidence in his or her own mathematical ability. Without a strong foundation on which to build knowledge, students who are weak in mathematics may be at a significant disadvantage if career interests lie in applied science or engineering technology fields. A six year research study by the College Mathematics Project (CMP) has demonstrated that close to one-third of college students in Ontario either fail first semester mathematics or obtain only a weak grade of D (Orpwood, Schollen, Assiri, & Marinelli-Henriques, 2009). As math is essential for technological disciplines, these results are predictive of student graduation rates in their chosen field. Many higher education institutions have implemented foundational learning programs. At our college, a common developmental mathematics course is offered for students taking technology disciplines that range from applied science to electrical engineering technology. Students with low scores on the Canadian Achievement Test (CAT3) are required to take this foundational level to promote success in their diploma mathematics subjects. Extending this practice to other disciplines a one-year certificate in Applied Science and Technology (AST) provides marginalized students a bridge to diploma technician and technology programs. The aim, in addition to strengthening mathematics and science, is to improve skills in communication, critical thinking, and problem solving.

Lecture pedagogy in many college mathematics classrooms follows a traditional pattern: teachers write problems and solutions on the board which students copy with pencil into paper notebooks. In disciplines that have a high demand to cover content, much classroom time is devoted to information transfer, with little remaining for interaction and class discussion (Thede, 2006). For marginalized students, this lecture technique may not always be successful. Foundational teams seek to find innovative approaches for teaching and learning mathematics to increase student interest. Their objective is to develop methodologies

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that more closely parallel the profile of today's college student who prefers to be immersed in their learning.

The use of enhanced technology is sought as a catalyst to re-engage students. I used a progression of teaching formats that included chalk on blackboard, dry erase markers on white board, acetates on overhead projector, and paper notes displayed via a document camera. A necessary first step was the conversion of personal hand-written notes to typed Word Documents (e-notes) which are saved as electronic files. Typing of mathematical formulation is time consuming and tedious, however, software programs such as MathType are advantageous. At this college, foundational courses span a range of disciplines, schools, and campuses. These e-notes are a summation of the wealth of knowledge available from multiple disciplines and experiences of both new and seasoned teachers. This material provides the scaffolding which individual instructors tailor to suit the technical requirements of their discipline, while at the same time ensuring all students receive a common level of mathematics understanding. Once generated, e-notes have several benefits for both teacher and student. For the teacher, they are easy to update, correct, share, and post to the learning management system (LMS – Blackboard Learn). E-notes in final form are copied to acetate for use with an overhead projector or to paper for direct electronic projection (document camera). Prior to class, notes are posted to the LMS for students to download and print. In class, students use this printout as a workbook to collect relevant materials, write in solutions, and annotate from class discussion.

The use of these audio-visual devices affords the teacher a familiar method to present complex ideas in note form with hand-written solution. As everyone is working from the same documentation, time is saved by not transferring and copying, so more can be devoted to discussion and comprehension. For developmental mathematics students, some improvement is gained by having clear notes presented in an organized fashion. Drawbacks to this methodology include the time required for initial preparation by the teacher. Further, the student must take greater responsibility for printing materials prior to class as well as bear the additional cost. In class, the student transfers solutions from projector screen to notes, with the associated inherent error caused by poor visual quality and misinterpretation. Some consideration was given to simply packaging materials in a workbook format; however, this would reduce the dynamic nature of teaching and the ability to modify

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materials on a day-to-day basis. Although a valuable first step, this asynchronous transfer of information can only provide a more efficient mechanism of delivering content. While this methodology requires students to be more responsible for pre-class preparation, the additional budget strain of printing notes becomes prohibitive for some. Further, it falls short of the goals of the teaching team to increase engagement, enhance student learning and ability to apply concepts, and to demonstrate an innovative teaching process throughout the college/math/technology/outreach community.

In September 2007, I changed delivery methods from using the document camera to a pen-based tablet PC. A document camera gives the desired presentation so that the lecturer can view and interact directly with the class. However, to make notation large enough for students to read mathematical symbolism clearly (e.g. exponents), the teacher is often writing 'off-the-screen' causing student frustration. By using a pen-based tablet PC, the teacher faces the students, the use of pen annotation is convenient, and all notes stay on the projected screen.

Workspace Design

Within our research group, tablet PC's were employed in two different formats: use by the teacher only and use by both teacher and students (a tablet laboratory).

The use of a tablet PC by the instructor was a modification of the traditional lecture classroom. After making final adjustment to the note framework, the teacher printed the Word Document into the Microsoft Journal Note Writer. This Journal note was 'drag and dropped' into the Windows Journal; a format that allowed for pen (stylus) annotation on a tablet PC (Figure 1). In preparation for writing, the PC monitor can be swiveled and snapped into place, similar to writing on a pad of paper.

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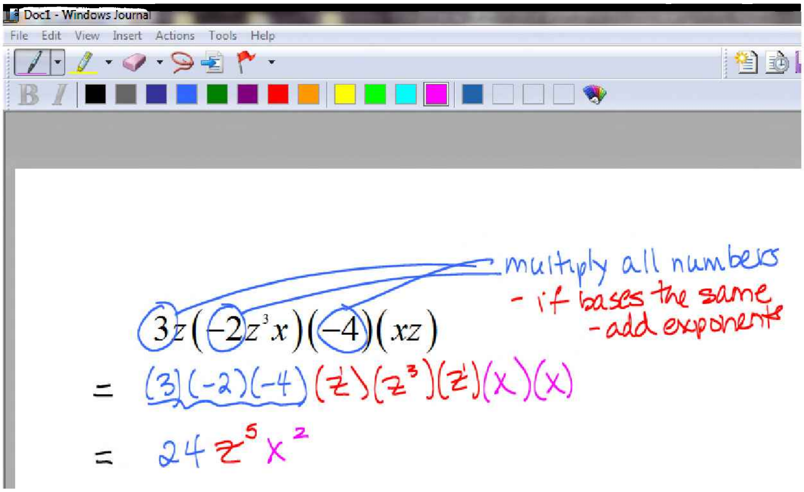

Figure 1. Annotated note in Windows Journal

As the lecture progressed, the instructor worked into the note framework using a rainbow assortment of virtual pens. Student attention was focused on key points by using highlighters or by enlarging pen width to that of a felt tipped marker. Mistakes were erased by simply reversing the pen (similar to that of a pencil eraser), or by tapping the eraser icon to remove entire sections. All desirable tools were available with the touch of the pen to the task ribbon. Notes were saved as a Windows Journal Note (.jnt), however, students could not open this file type without a pen-based tablet. It was learned that if notes were exported as a Web Archive (.mht) file, most students could open them on any PC. With an update to Office 2007, it is now possible to use the stylus directly in a Word Document by activating the Review tab and using the 'Start Inking' option. In this edit mode, all familiar inking tools are available (Figure 2) and notes can be saved as Word Documents or published as a PDF document.

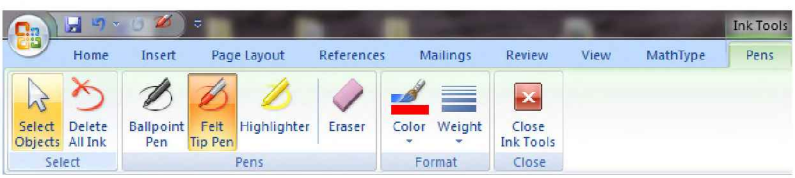

Figure 2. A view of the Review task ribbon in Word Document (Office 2007)

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A further benefit of using a tablet PC while teaching is the flexibility that computerization affords. In a traditional classroom, working space is limited by the size of the writing surface; notes must be erased before new work can be added. Using a tablet PC, instructors can insert additional space (pages in the Journal) to provide 'just in time' clarification to satisfy needs of questioning students. An accurate record of the detailed step-by step progression is retained. For a student missing a concept, return to previous information is possible by scrolling backward with the pen which acts as mouse. Finally, teaching with a computer provides the ability to reference the online textbook, integrate web based resources such as learning objects, applets, and videos to demonstrate application and context for enhanced student understanding. Class work is saved for future reference by the instructor or posted to the LMS for reference by students. With greater familiarity of the capability that pen-based computing provides, additional support to expand this environment for both teachers and students became immediately apparent.

This paper describes a strategy to create an online workspace using pen-based computing, collaborative software, and the internet for two-way communication between teachers and students.

In September 2008, the team received a Hewlett-Packard Higher Education Technology for Teaching Grant. This grant gave us twenty-one HP Compaq 2710 pen-based tablet PC's: a number sufficient to develop a pilot research study investigating the use of enhanced technology and its affect on student engagement. With the smaller enrollment requirement afforded by the newly developed Applied Science and Technology (AST) program, students in mathematics and science courses provided excellent candidates for this pilot. The intended classroom was upgraded to a tablet laboratory, giving users access to both power and hardwire internet cabling. In this two year study, tablets were used in the classroom an average of 8-16 hours per week. The laboratory maintains its multipurpose configuration; when tablets are not in use they are re-charged in a secured storage area.

The advantage of utilizing pen-based computing technology for both teacher and students to learn mathematics and science was realized. Students found some initial awkwardness writing on screen with the stylus, however, they quickly became proficient. Some inexperienced users found the right-click mouse button on the stylus barrel to be a

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nuisance if inadvertently activated by finger pressure. Excitement ensued as students discovered the spectrum of colours and thicknesses possible using digital pen or highlighter; an audible 'cool' resounded with pen reversal to erase.

Many colleges and universities in Ontario require students to purchase or rent a laptop; few have investigated the importance of a networked classroom (Roschelle, Penuel, & Abrahamson, 2004). With the receipt of the HP Grant, our laboratory had the hardware required to offer integration of handwritten notes. To fully realize an online learning community, a crucial advancement is to connect teachers and students, giving both access to a shared workspace for real-time synchronous interaction (Berque, 2006). At the onset of the project, only two software programs were found that supported this type of collaboration: Classroom Presenter and DyKnow Vision. Although Classroom Presenter is free software that provides the desired connection between tablet screens, its presentation format is based on the conversion of Power Point slides (Anderson et al., 2006). An issue for our research group was that lecture notes had been created as Word Documents. Due to timing constraints on classroom implementation, DyKnow Vision became the software of choice. DyKnow is a fee-for-license package and is compatible with both Word and Power Point. This collaborative software gave the required two-way communication deemed necessary to develop a learner centered, interactive workspace while using the pen-based tablet PC's (Chidanandan et al., 2007).

Finally, using a tablet PC while teaching allows for a seamless integration to an infinite variety of internet resources (Appendix). For foundational mathematics students, relevancy of subject material is a component of willingness to learn (Carruthers, 2010). If a class internet search results in a URL, interactive learning object, applet, tutorial, or video of interest, it can be embedded into the DyKnow notebook for reference by students. For example, teachers use the interactive applet GeoGebra to quickly demonstrate a graphing concept like the point of intersection between two linear equations. Using the software's selection tool, this professionally formatted graph can be inserted into a blank panel of the notebook to give illustration and a rubric against which students can measure their own efforts. From the internet, they can open their online textbook to read relevant passages and supplement their notes with questions. Book publishers encourage the use of the 'snipping tool' (Windows 7) allowing students the ability to copy and paste indi-

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vidual questions from the textbook into their personal notes. Students assign their own homework and determine how many questions they need to practice. The use of pen-based computing, collaborative software, and the internet affords students a variety of options to investigate challenging mathematics concepts, make connections to science applications, and thereby enable individuals to choose the approach best suited for their own learning (Carruthers, 2010).

With the success noted in the experimental group of the AST pilot study, the next logical step was to make the same experience available to a more common college class size - forty students. In September 2010, an extension of this study was undertaken to determine the effect of larger class size on synchronous, interactive communication. In order for all students to have equal access to the technology, the college provided a second tablet laboratory, outfitted with 40 HP Elitebook 2730 PC's. This tablet laboratory is a completely mobile, wireless facility; it has not been retrofitted with individual power and network cabling. The classroom retains its multifunctional capability of being both a tablet and a non-tablet room. Tablets are in use for more than 35 hours per week and are shared across various math, science, and English disciplines. When not in use, tablets are re-charged in a secured adjacent area. In January 2011, a third 21 tablet PC laboratory was created at another campus of our college.

Classroom Example

The following section illustrates how the features of DyKnow software, installed on the pen-based tablet PC's, are used in the classroom. This software is designed to provide a workspace in which students can interact, chat, share ideas, form groups, and create collaborative solutions.

Classroom administration: A DyKnow administrator is required to set up class sections and bulk load students by user ID and password. Access to saved workbooks is maintained on the college server with each student having a personal account. The teacher must download a virtual printer – the DyKnow Writer - to his or her tablet PC for conversion of Word Documents to a DyKnow compatible form. At the beginning of class, the instructor logs into the software and opens a session. The software gives the option of choosing between a blank screen or the previously created DyKnow notes. Students log into the

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software and are instantaneously prompted to join the session. While sitting at their tablet PC, the teacher's note framework is viewed directly on the screen in front of them. Students unable to attend class can remotely join over the internet using a client they download to their personal home computer.

Panels - Once in a session, notes are made available to students by pushing 'panels' to each student screen. The teacher moderates the progress and speed of the lecture by releasing panels individually or in multiples, allowing students to work at their own pace. A panel can take on multiple forms: a scaffolding of notes previously prepared by the teacher, a blank screen with lined or graph paper inserted for further clarification, diagrams to be labeled, a webpage linked for student browsing, etc.

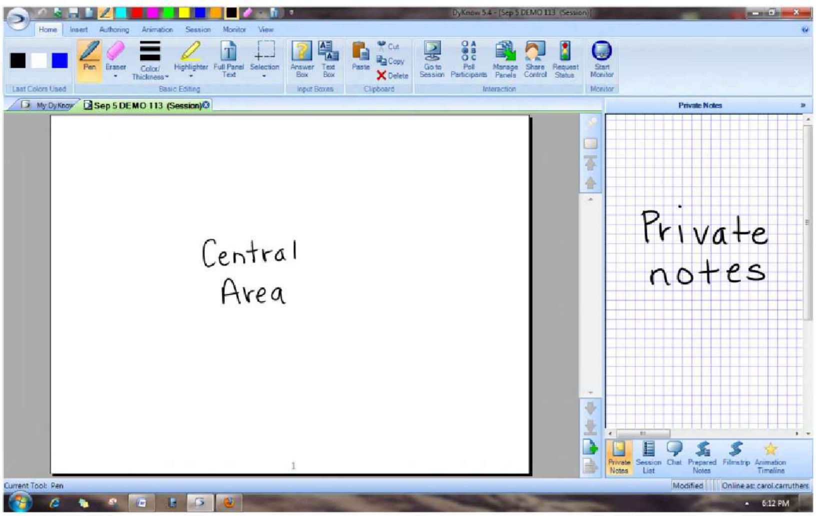

Figure 3. DyKnow Vision Panel layout showing workspace layout.

Panels, illustrated in Figure 3, have two main areas, each with a differing function. Lecture content provided by the teacher is viewed in the central area. Teachers use this space to introduce topics and provide examples of solution, which are projected on all screens. All information the session leader writes is retained in the notebook. Students

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are then requested to write into the note framework, working through problems and giving solution to demonstrate their understanding. Individual students work is only visible on their personal screen unless they choose one of the sharing features of the software (or swivel their screen for other students to view). As both the teacher and students share this central screen, care must be taken not to write on each other's work. In the initial phases of this pilot, careful consideration was given to note formatting to ensure everyone had the creative space required for personal annotation. Only the student has write-on capability in the second area, called the private notes. As page size can be enlarged using corner expansion arrows, they use this section of the screen like scrap paper. Each note is pinned to the individual panel; with advancement, a fresh private note appears. Some students use the private notes to highlight the key concepts of each panel. For studying purposes, the private notes can be used to quickly recall necessary information. Users can 'undock' this task pane and using the pen, drag it to any location on the screen giving more flexibility if extra working room is required.

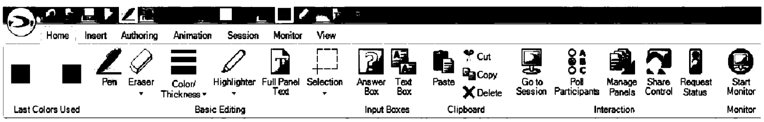

Figure 4. The task ribbon showing DyKnow interactive features.

A **task ribbon**, in Figure 4, runs across the top of each panel with annotation, collaboration and interactive features clearly indicated. These features are activated by touching the icon with a pen. Students in mathematics classes are familiar with taking notes with a pencil, the transition to pen based computing is unremarkable for most (Backon, 2006). Students comment that they especially like these features as they can individualize their learning depending on specific need (Carruthers, 2010). Visual learners draw pictures and flowcharts to improve their understanding of concepts. Both teacher and students enhance key ideas with highlighters or multiple pen colours to draw attention to important details (Figure 5). Notes come alive, making studying enjoyable and more closely matching the student's personal approach to learning.

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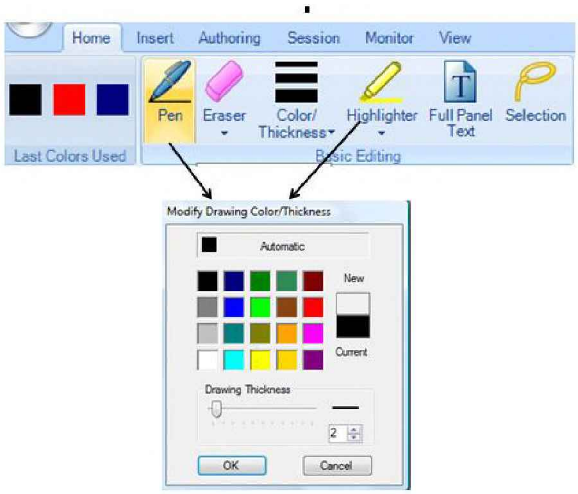

Figure 5. The pen/highlighter colour and thickness options.

Student Interactive Features of DyKnow

Polling - A question is sent to individual student screens to gather response. Questions have the form of multiple choice or true/false and responses are collected and tallied on the teacher's screen (Figure 6).

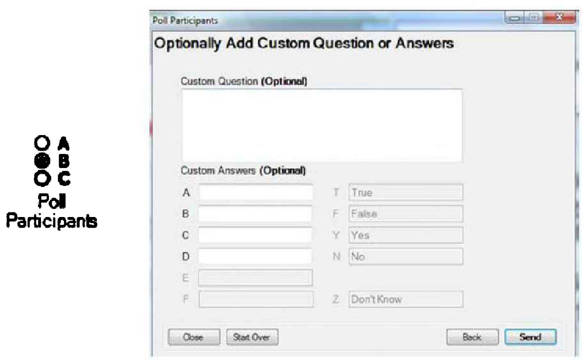

Figure 6. Interactive polling features used to clarify understanding.

The anonymous, grouped results are displayed in a choice of formats: data, histogram, pie chart or table. The result is captured and can be appended to the class set of notes (Figure 7).

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2) Which do you use your phone the most for:
a) Calls
b) Texting
c) Emails
d) Gaming
e) Social media

Answer	Count	Percent
A	10	37
B	15	56
C	1	4
D	1	4
N/A	0	0

Figure 7. Result of a class poll.

Due to the ease of application, this polling feature can be used in several different methods, depending on desired outcome. At the beginning of a class, polling gives insight into a students' prior knowledge. For example, as seen in Figure 7, the concept of percent can be demonstrated by collecting data on student phone use which forms a relevant argument as to why this learning is important. During the lecture, polling gives immediate feedback of student comprehension. For example, questioning about problem solution allows teachers to modify their planned lesson based on student response. If students all choose the correct answer, then additional rote work may not be necessary. If responses are weak, panels can be inserted into the notebook to supplement learning. I use a multiple choice method in which students call out their answers to the question. Common responses are used as A-D answers and the class polled to vote on their favorite. All answers are taken without correction and individual students are not identified. Dialogue centers on types of errors made and lessons learned from making these mistakes. Finally, polling can be used at the end of a lecture to summarize or test key points. In all cases, it is possible for the teacher to review individual responses - the poll can be saved and investigated in more detail later.

Chat – During a session, the chat feature can be activated giving students the opportunity to interact by typing comments to each other or asking questions of the instructor. Students logged in from home can use this option to join the conversation. Some use the chat window to ask questions when they prefer not to raise their hand.

Replay – Once logged into the DyKnow software, students can 'replay' the pen strokes for any panel (Berque, 2006). With replay, students can watch solutions being revealed in a pen-stroke by pen-stroke fashion instead of having the complete solutions presented in an inexplicable manner (similar to those found in textbooks). This may be ad-

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vantageous to weaker students who prefer to know how each step is obtained.

Teacher Interactive Features of DyKnow

Request status – From the teacher panel a message can be sent to individual screens, requesting students to choose a stoplight colour – red, yellow, or green (Figure 8).

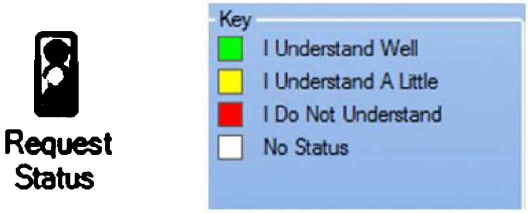

Figure 8. Teacher gauges student learning using Request Status.

From the information collected, grouped responses are displayed as data and an approximation of student comprehension can be determined. If individuals don't understand they select a warning colour to alert the teacher. This method is anonymous and students are more likely to honestly express their concern rather than simply nodding their heads in agreement. With increasing student familiarity of the software it is noted that students will send responses without being asked. Using this backchannel conversation, the teacher gains a clear and immediate indication of student comprehension before advancing to a new topic.

Submit panels – The teacher can request and/or retrieve panels from students logged into the session either in the classroom or participating remotely. Teachers collect these submissions in a saved file to be reviewed, annotated, or used for assessment. During class time, a quick scan of submitted panels gives the teacher an instant gauge of knowledge. In addition, students will often submit panels during the lecture for teacher consideration just to ensure they are on the right track. During a quiet moment in class, the teacher can annotate and return work immediately or save for later review. Exchanged panels are electronically returned to the individual student's DyKnow file, embedded by date and chosen file name. The two-way communication developed by exchanging panels gives students greater confidence to ask questions (Carruthers, 2010). As students in foundational mathematics

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courses like to have their work reviewed frequently, regular use of this software feature is extremely important (Figure 9).

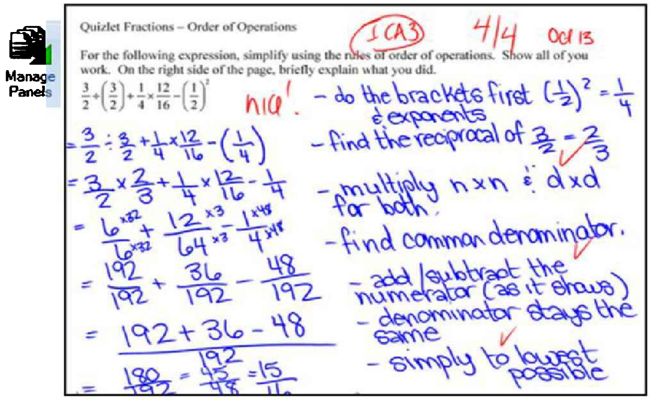

Figure 9. Students send panels for teacher review and annotation.

Collaborative Features of DyKnow Software

Share Control – With DyKnow software, the teacher can give a student or many students control of the teaching screen, without the necessity of leaving their seats.

This feature, called 'share control', allows students to become the temporary teacher (Figure 10). By activating this icon, a class list drops down onto the teacher screen. Instructors can choose to pass control to single, many, or all students. The designated student(s) with control now write on screen and their work is projected to the front screen and becomes part of the common notebook - temporarily taking on the teacher's role. At the end of this exercise, the feature is deactivated and the teacher resumes sole control of the main screen. Students take pride in contributing to the general knowledge of the entire class (Carruthers, 2010). This interactive sharing of work results in increased class discussion and debate. Share control is particularly helpful for test take-up, practicing large numbers of problems and in modeling. For test take-up, the teacher can import test questions into the class notebook, open share control, and invite all students to attempt solutions. The author of an individual solution cannot be identified, as everyone is writing on the panels. Once share control is removed, the teacher can comment

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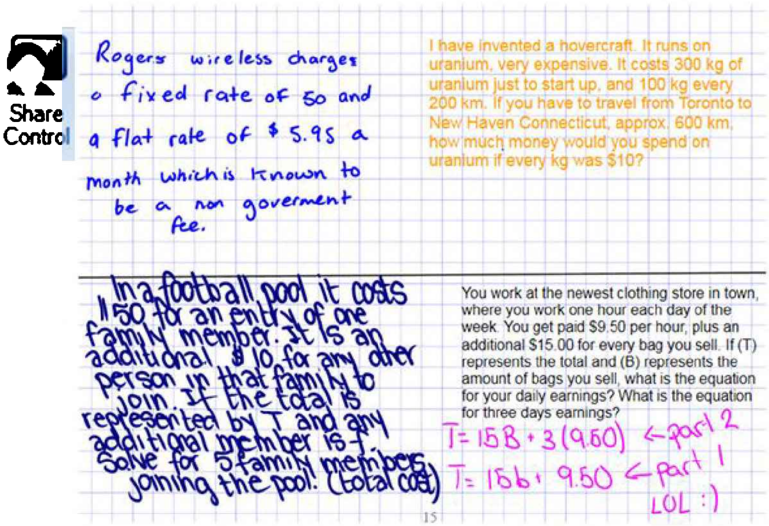

Figure 10. Student work gathered by using the Share Control feature.

and annotate about the provided solutions. Struggling students become an active and engaged part of this take-up methodology without fear of peer evaluation. Modeling is a new technique made possible with the share control feature of the DyKnow software. Previously arranged notes are organized into tables with the same question appearing in two columns labeled ‘my work’ and ‘shared work’. Students are given the opportunity to solve problems independently or in groups on the ‘my work’ side. After a suitable time allowance, the share control element of the software is opened. Students write their solutions into the shared work portion of the table, which is evaluated and corrected by the class. In this way, students have their own answer as well as the class corrected work in a side by side comparison. With inspection of each line, participants contrast their thinking process to that of the class, thereby determining for themselves where conceptual (and possibility habitual) errors are made. This technique encourages students to self-correct and model alternate approaches to problem solving which may more closely resonate with their personal learning. It is important to note that students decide their level of participation and work is shared anonymously unless authors choose to self-identify - for example, one student always wrote in pink so others could find her work easily. This provides the opportunity of contributing to entire class understanding without

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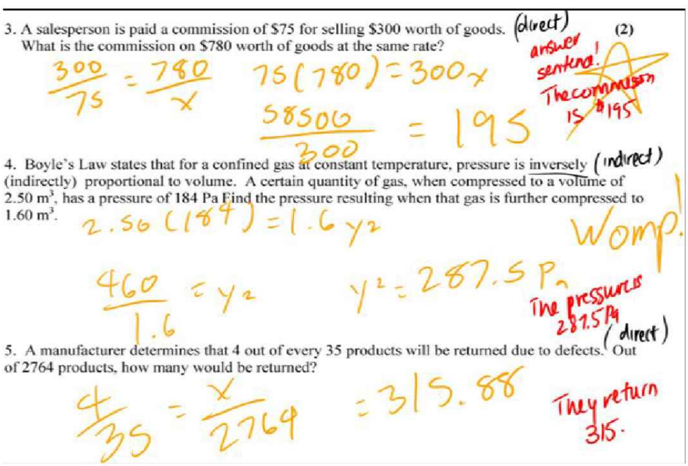
the risk of peer identification, should mistakes be made. This enabling of all students to be a teacher necessitates that the instructor assume more of a facilitator role. From a teaching point of view, there must be a willingness to step aside and let the class move in a self-directed, community-learner, multi-teacher mode.

Group Work –With the DyKnow software, students can be organized into online groups (Berque, Bonebright, Dart, Koch, & O'Banion, 2007). Each member of the group shares a screen which only responds to pen input from the designated members. Students determine how they will manage the group – either by each taking a problem and solving independently, or the entire group working step by step on one problem at a time. One member, on behalf of all, submits the group panels to the teacher. Group work can be appended to the notebook for entire class use or saved for marking and return.

Feedback - The use of tablets in class creates an environment in which instructors can provide a dynamic teaching style to capture students' attention. Student notes are generated in a collaborative fashion, which reduces the possibility of having misunderstanding embedded and practiced. For example, during a test take up exercise, one student proudly displays his response, and gives himself a gold star (Figure 11).


3. A salesperson is paid a commission of $75 for selling $300 worth of goods. (direct) (2)
What is the commission on $780 worth of goods at the same rate?
$\frac{300}{75} = \frac{780}{x}$ $75(780) = 300x$ $\frac{58500}{300} = 195$
answer sentence! The commission is $195

4. Boyle's Law states that for a confined gas at constant temperature, pressure is inversely (indirect) (indirectly) proportional to volume. A certain quantity of gas, when compressed to a volume of 2.50 m³, has a pressure of 184 Pa Find the pressure resulting when that gas is further compressed to 1.60 m³.
$2.56(184) = 1.6y_2$ Womp!
$\frac{460}{1.6} = y_2$ $y_2 = 287.5$ Pa
The pressure is 287.5 Pa

5. A manufacturer determines that 4 out of every 35 products will be returned due to defects. (direct) Out of 2764 products, how many would be returned?
$\frac{4}{35} = \frac{x}{2764} = 315.88$
They return 315.

Figure 11. Test take up using Share Control.

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Students move through exercises at a pace that matches their learning preference. During class time, students have the option of receiving direct instruction from the teacher, or indirectly, by sending a panel for immediate review, by comparing answers in the 'chat', or by viewing the group members' screen - all features of the software. The ease with which a teacher can switch from independent to group, teacher exemplar to self-directed customized learning, gives the toolbox needed to provide a unique learning opportunity for each and every student.

The Final Product - A Notebook of Panels

Notebooks are a synthesis of teacher provided scaffolding, student and teacher annotation and class collaboration (Figure 12). When saved as a DyKnow file, they are chronologically date stamped, labeled by topic information, and maintained on the college server – in a form we call a virtual binder. To view notes, students log into the DyKnow client they have downloaded onto their personal web-enabled PC. Some students save their notebooks from their classroom tablet PC to a USB storage device which allows for review without the need of an internet connection. Access to the DyKnow server and their virtual binder is also available at college's libraries or computer laboratories.

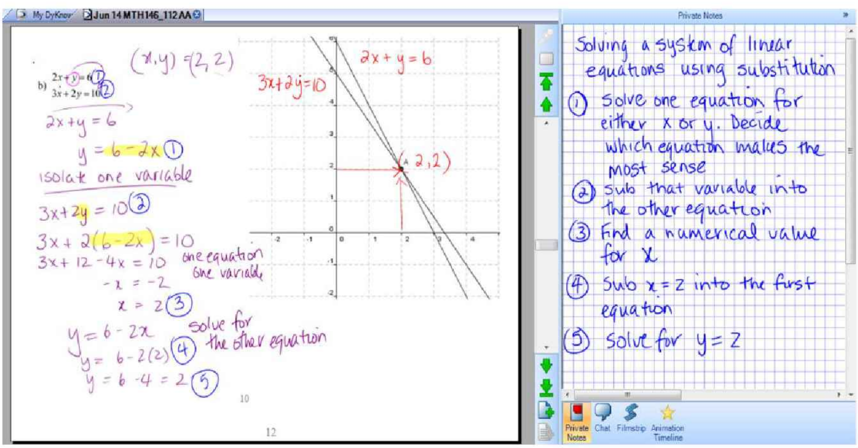

Figure 12. Hand written solution, GeoGebra confirmation graph, and stepwise explanation (in private notes).

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Teacher/Student Impression

The classroom use of the tablet PCs and collaborative software results in an innovation in teaching methodology. Lectures take the form of 1) interactive focus activity, 2) concept development, 3) collaborative practice, 4) application, and 5) reflection of learning. As some lag time occurs while students log in, the instructor often begins with an interactive focus activity. By projecting onto the front screen and using the links feature of the collaborative software, students are directed to an internet learning object, applet, or video and are encouraged to investigate individually, in pairs or in groups, in preparation for discussion of findings. While researching, students will often go to a search engine to seek meaning of unfamiliar terms or phrases. The embedded links on panels are retained as part of the class notebook which students can view in more detail later on their personal computers. Once all students are logged in, the instructor can un-tether themselves from the projector and freely move about the room with their workspace on their arm – as can students in this mobile, wireless environment. Open-ended questioning and class collaboration enhance the learning experience derived from the focus activity. The teacher captures this concept development using their pen on screen; all students have this learning experience accurately recorded in their notebook. Without the concern of copying page after page of notes, students can focus on the teacher's process used for solving, add notes that arise from discussion and annotate with pen colours and highlighter. Multiple features of the DyKnow software are employed to ensure that practice is collaborative and learning outcomes are met in unique and interesting ways. Students are placed in online groups so answers that are difficult to achieve independently can be developed through cooperation. The group submits their work to the teacher so it can be appended to the class notebook. In this 'wiki-like' arrangement of sharing ideas, the information gathered becomes greater than any work that could possibly be achieved by individual students. Further, the teacher can 'share control' of the session. While sitting at their own tablet, single, groups, or all students can be selected to demonstrate their approach to problem solving with control of the class screen. Students comment that this functionality gives them the opportunity of becoming the temporary teacher, requiring them to take greater responsibility for their own work. They quickly understand why an answer that uses poor form, undecipherable writing or skips steps is not desirable to others viewing their work. After using

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this model, students indicate that this technique increases pride in their work and makes them more likely to continue their efforts outside of the classroom. To enhance the learning benefits of the share control feature, the concept of modeling (as previously discussed) allows students a parallel comparison of their solution to that of their peers. This focuses students on the necessity of correct process, not just the final answer. The ability to send panels to the instructor enables a backchannel conversation to occur with students that prefer to have the teacher review their solutions on a regular basis. The 'send panel' feature can also be used to share application questions and request direct student input to lecture topics. For example, students are asked to design and solve questions of interest from their own experiences and send them to the instructor. These new panels are collected and appended to class notes, providing multiple examples of student generated problems – by students/for students. In addition, by 'sharing control' with all, several pages of solved problems can be quickly generated, corrected, and appended to the notebook for studying purposes. By using embedded links, teachers provide an application experience which results in a clear demonstration of how the learned math skill will be used in future discipline-specific situations. Reflection of learning outcomes is evaluated throughout the lecture by using multiple DyKnow tools. For example, the teacher can 'request status' of students, polling to quickly determine comprehension, and if required, additional panels inserted to supplement discussion. These methodologies are used to either elicit conversation or develop reflective practice. This online, collaborative workspace is then saved to the college serve, and becomes a part of the students' virtual binder. The current study is not intended to quantify the impact of this altered teaching style on student perception, but to qualitatively assess whether students feel it enhances or impedes their learning.

The combination of the tablet PCs and collaborative software provides a workspace for students to engage in two-way synchronous communication. This learner-centered environment results in student perception of enhanced engagement, supported by increased attendance and retention in the pilot study course. Survey data and student focus group comments: "…it isn't like any other class when you sit there and just listening to a teacher speak on and on…you're actually interacting with others", and "…tablet…gave you an opportunity to participate… without the embarrassment of public speaking" or "my participation …have grown significantly because of the confidence it brings when

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I'm comfortable of where I am" help teachers to comprehend the benefits of a personal learning community and its impact on student confidence and understanding of the subject material. Student performance increased measurably and comments: "Makes me perform better since it is a new and interesting way to approach classes, I tend to pay attention more and thus doing better in work and tests", "…I actually did my work and it was done right and organized…" and "…It made learning…much easier, more enjoyable and improved my performance…" allow teachers to conclude that this active learning environment results in improved student awareness and encouragement of necessary critical thinking skills. "The tablets were useful because I can see what the teacher is writing, students can share answers anonymously, we can correct each others work". With a greater emphasis placed on class development of notes, students are required to take responsibility for their learning and thus determine for themselves where weaknesses and errors occur. Few negative comments were received in the pilot study, but included "…they (tablets) could be a very distracting because it (takes) your mind away off lectures…when lectures becomes boring I tend to go on websites which is the worst thing I can do …" One solution I have employed is to provide structured 'social media breaks'. During first day activities, rules regarding appropriate use of phones or popular websites are decided as a group. Use of social media in the classroom is not prohibited; however, non-disruptive use is discussed and encouraged. In a networked classroom, faculty must be amenable to adaptation in teaching philosophy and a shift in thinking as students determine the pace of classroom delivery and contribute to the teaching role.

Conclusion

This paper examines the use of pen-based computing and collaborative software to provide an interactive environment for teaching and learning mathematics. To date, testing has primarily focused on foundational subjects. Early indications demonstrate increased retention and success of students. More data is required to determine statistical significance. The ongoing study with a larger class size is allowing researchers to gain this insight in coming years. Preliminary research strongly indicates that students find this interactive environment increases their participation, performance and enjoyment of mathematics classes. With the two-way features of the software, students suggest they have become a strong

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community of learners. Finally, the collaborative ability afforded by the tablet/DyKnow interface requires instructors to modify their teaching style to capitalize on the real-time, synchronous interaction provided by this technology.

As the development of this wireless, mobile laboratory is the first of its kind at the college, this research provides much needed understanding of connectivity requirements. The software chosen uses a large amount of broadband with parameters necessitating access for forty students synchronously; this study tests the limits of our existing wireless capability. Although the software gives the potential for group interaction and feedback, some activities must be tailored to fit the dynamics of a larger classroom size. For example, forty students will often have DyKnow, Blackboard, the Internet, and the online textbook open during a class session. Currently, creative methods are being sought to maintain the continuous connectivity that the software demands. Further advances at the college include the opening of two additional tablet classrooms. One laboratory seats forty students and addresses the technological implications of larger class sizes that are reflective of a more standard class enrollment. The second is located at another campus where students are pursuing careers in different fields of technology. Its purpose is to determine if deployment of this teaching methodology can be expanded to other disciplines, in particular where gender may have a greater influence. In September 2011, all three tablet labs are operating simultaneously, and with over 350 students using DyKnow software, the limits will continue to be tested. Currently, administration and technical services are assessing this and other technologies to develop a strategic plan for future college endeavors. Through outreach activities and presentations at several national and international events, other Ontario colleges are monitoring this research to determine its applicability to their classroom environment.

Appendix

Internet References of Interest:

http://www.freerice.com/category - A site for practicing repetitive skill while providing humanitarian aid.
http://www.cnn.com/TECH/space/9909/30/mars.metric.02/ - Gives students a real-life example of the importance of units and conversions.

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http://www.mathopenref.com/angle.html - Foundational students have a previous mathematics understanding that reflects their life experiences. This site allows for stratified learning outcomes and combined with a list of expectations, encourages students to gain information at their own level.
http://www.youtube.com/watch?v=yvKu2T9Kovo – Teachers can develop explanatory videos depending on student need. To extend classroom time, videos are used in the flipped classroom, where students watch before class, answer preliminary questions, and come to class prepared for discussion. Students can use video in multiple ways – as a refresher, for discussion, or watch multiple times to gain proficiency in math language and understanding.

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Chapter 4

Teaching Mathematics Using GeoGebra: Integrating Pedagogy and Content in Teacher Education

Lingguo Bu, Frackson Mumba, Mary Wright, and Harvey Henson

Abstract

GeoGebra provides a resourceful dynamic learning environment for mathematics educators to integrate mathematical content and pedagogical strategies for the purpose of teaching mathematics for understanding. Dynamic models built with GeoGebra invite mathematics educators to conduct purposeful and iterative reflections on their teaching practice and seek new ways to connect, extend, and enrich their instructional activities. In this chapter, we reflect on our field experience with GeoGebra-integrated mathematics teacher development and propose a comprehensive framework for conceptualizing the pedagogical uses of GeoGebra in mathematics teacher education. We further elaborate on a vignette of inservice teachers' problem solving under the perspective of mathematical modeling. We argue that GeoGebra-integrated mathematical explorations provide a powerful platform for mathematics teachers to develop content-rich pedagogical strategies for mathematical instruction. We focus on the use of GeoGebra because of our specific project activities. The findings are equally informative for the use of other similar contemporary technologies in mathematics education.

Keywords: Pedagogical reflection, instructional design, teacher education, professional development

Introduction

GeoGebra (www.geogebra.org) provides an accessible mathematics learning environment that integrates multiple mathematical representa-

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tions, computational utilities, and documentation tools. Within GeoGebra, not only are the various representations automatically linked, they can also be dynamically manipulated to illustrate or simulate the dynamic nature of mathematical ideas. In alignment with our growing knowledge about mathematical understanding and its complexity in terms of the necessity for utilizing multiple representations and dynamic mental models (Bu, Spector, & Haciomeroglu, 2011; Carpenter & Lehrer, 1999; Goldin, 2003; Hiebert & Carpenter, 1992; Moreno-Armella, Hegedus, & Kaput, 2008; Seel, 2003), GeoGebra stands as an equitably accessible digital environment that appeals to mathematics educators and students at all levels as they strive to make sense of and develop a deep understanding of mathematics. Furthermore, from a tool use perspective (Vygotsky, 1978), while GeoGebra is, in most cases, used initially by mathematics educators as a technical tool to support teaching and learning, it gradually evolves, first, into an psychological tool or an instrument that facilitates a teacher' instructional plans and strategies, and, further, into a pedagogical tool that facilitates a teacher' classroom practice in many aspects of mathematics teaching. In this paper, we reflect on our own experience with the integration of GeoGebra in both preservice mathematics teacher education and inservice professional development programs across a period of three years. We assume the role of teacher educators, upholding the position that teacher educators themselves are reflective learners in both preservice and inservice programs. We believe that the emerging GeoGebra user community is indeed a group of mathematics teachers and students who are not only actively inventing and experimenting with new ways of mathematics teaching, but are themselves learning or relearning mathematics. As Freire (1998) argues convincingly, "There is, in fact, no teaching without learning. One requires the other.... Whoever teaches learns in the act of teaching, and whoever learns teaches in the act of learning" (p. 31).

As a focus, we discuss the pedagogical roles of GeoGebra in our preservice and professional development field work. The term pedagogy refers to the ways of teaching under a certain theoretical framework for teaching and learning. In what follows, we present an overview of a model-centered framework for teaching mathematics using GeoGebra, followed by (1) an illustrative example from our work with preservice teachers, (2) a preliminary framework for conceptualizing the pedagogical roles of GeoGebra, and (3) an inservice teachers' problem

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solving vignette. Our goal is to highlight the multidimensional roles of GeoGebra in integrating worthwhile mathematical tasks and pedagogical strategies in teacher education as laid out in recent mathematics teaching and learning standards (National Council of Teachers of Mathematics [NCTM], 1991, 2000).

Theoretical Framework

To foster mathematics teachers' growth in teaching mathematics with new digital technologies, we situated our instructional design and its enactment in the theoretical framework of multiple representations (Goldin, 2003) and Model-Facilitated Learning (MFL) (de Jong & van Joolingen, 2008; Milrad, Spector, & Davidsen, 2003; Seel, 2003, 2004). The theory of multiple representations has served as the foundation for the reform-based conception of mathematical understanding (Hiebert & Carpenter, 1992). A mathematical representation refers to both the cognitive processes and the external product of mathematical reasoning, taking on such forms as graphs, algebraic expressions, and various informal diagrams or tables. From the learning perspective, the use of multiple representations contributes to the resolution of intrinsic ambiguities within the representation system (Goldin, 2003), leading to the ultimate reification of a mathematical concept as a mathematical object (Sfard, 1991, 1994). Furthermore, recent conceptions of mathematical understanding have placed much emphasis on a learner's ability to use and navigate through multiple representations, which, theoretically speaking, indicates a learner's knowledge of the mathematical processes and the corresponding conceptual structure in the form of dynamic conceptual and mental models (Doerr & Lesh, 2003; Gravemeijer, 2008; Seel, 2003).

However, effective instructional design in mathematics requires more than multiple representations. In developing instructional sequences, we were further informed by the Model-Facilitated Learning (MFL) framework (de Jong & van Joolingen, 2008; Milrad et al., 2003), which seeks to promote meaningful learning and deep understanding by fostering learners' development of a holistic view of a complex problem situation. The MFL framework consists of modeling tools, multiple representations, and system dynamics methods that allow learners to build models and/or interact with existing models as part of their effort to understand the structure and the dynamics of the problem situation. MFL recommends that learning be situated in a sequence of activities

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of graduated complexity, progressing from concrete manipulations to abstract representations while learners are engaged in increasingly complex problem solving. Through the use of multiple representational tools, MFL further maintains the transparency of the underlying mathematical model that drives the behavior of a problem simulation.

Furthermore, mathematics teaching using technology involves a deep understanding of the intersection of mathematics, technology, and pedagogy, which culminates in a form of knowledge for teaching in the digital age: Technological Pedagogical Content Knowledge (TPACK) (Mishra & Koehler, 2006; Niess, 2005, 2008). MFL stands as a promising instructional design framework to inform the development of teachers' TPACK in a model-centered perspective (Bu, Spector, & Haciomeroglu, 2011; Doerr & Lesh, 2003; Seel, 2003).

Worthwhile Mathematical Tasks

Worthwhile mathematical tasks invite students "to reason about mathematical ideas, to make connections, and to formulate, grapple with, and solve problems" (NCTM, 1991, p. 32), and in such mathematical practices, to develop skills and a disposition toward mathematics as a worthwhile field of sense-making and ongoing inquiry. Boaler (2002) compares two schools with contrasting teaching practices and finds compelling evidence that students who had opportunities to solve genuine open-ended problems, as a group, develop far more productive mathematical attitudes and problem solving skills and identify with real-life users of mathematics. Accordingly, worthwhile mathematical tasks aligned with elementary (K-8) mathematical standards are at the core of our GeoGebra-integrated courses and programs at Southern Illinois University such as Science, Mathematics and Action Research for Teachers (SMART, http://www.smart.siu.edu). In selecting or designing learning tasks, we seek open-ended realistic problems that are familiar to our teacher participants and yet provide unexpected solutions or cognitive conflicts. Many problems are directly provided or reviewed by professional mathematicians. Our overarching goal is to challenge the traditional views of mathematics held by the majority of our teacher participants and help them develop insights into mathematical problem solving through engagement in non-trivial mathematical problem solving and modeling.

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In most cases, we expect our teacher participants to develop profound understandings of elementary mathematical ideas (Ma, 1999; Wu, 2009). In accordance with the Dynamic Principle in mathematics teacher development, we agree that "the knowledge that teachers need should move from understanding relationships that are static to those which are dynamic" (Doerr & Lesh, 2003, p. 135). Mathematical examples include problems such as (1) Given the diagonal of a rectangle, how could you rebuild a/the rectangle? (2) If the class measures 20 circular shapes of various sizes, how could you make sense of the relationships between the circumference and the diameter? (3) A circle is given with no indication of the center, how can you locate its center? All the problems call for mathematical concept play (Davis, 2008), by which teachers analyze mathematical concepts and examine personal conceptions and the cultural evolutions of mathematical ideas. In the rectangle problem, for instance, our preservice teachers came with an unconventional definition of a rectangle: a quadrilateral whose two diagonals bisect each other and are congruent. One preservice teacher, in fact, imagined rotating the given diagonal around its midpoint to obtain a rectangle, thus relating a rectangle to a circle. Such ideas can be readily externalized and communicated with GeoGebra under minimal instructor guidance. There are, indeed, infinitely many such rectangles. There are, of course, other equally valid definitions of a rectangle. As a class, our preservice teachers explored or rather played with the conceptual richness of a rectangle while seeking a dynamic construction.

As a focal point, we invite our readers to think about the problem in Figure 1 and how GeoGebra may change or enhance their instructional strategies and practice when the problem is used with a group of elementary (K-8) mathematics teachers.

GeoGebra-Inspired Pedagogical Reflections on the Treasure Hunting Problem

Being experienced users of GeoGebra, we are familiar with the potentials of GeoGebra for teaching the Treasure Problem in the classroom. First, we note that the primary goal of this task is for our participants to experience genuine mathematical problem solving and what it means to teach mathematical problem solving. Second, the problem can be solved in a paper-and-pencil approach, which, of course, has its limitations in addressing what-if and what-if-not questions (Brown & Walter, 2005) because of its physical constraints in exploring the consequences

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or the lack of consequences of certain problem conditions. In our teaching practice, we found that it is indeed helpful to have participants try to solve the problem on paper as a way to understand or even to question the problem and to identify key ideas involved in the problem.

As shown in Figure 1, some treasure was buried on the beach at a location determined by a palm tree and a boulder. To find it, start from the palm tree (*P*), walk toward the sea for some distance to some point *W*. Mark *W*. Then, go back to the palm tree, at a right angle to *PW* (clockwise), walk the same distance toward land and mark a point *X*. Then, go to the boulder (*B*), at a right angle to *BW* (counter-clockwise), walk the length of *BW* to the land, mark a point *Y*. Find the mid-point *T* between *X* and Y, which is location of the treasure!

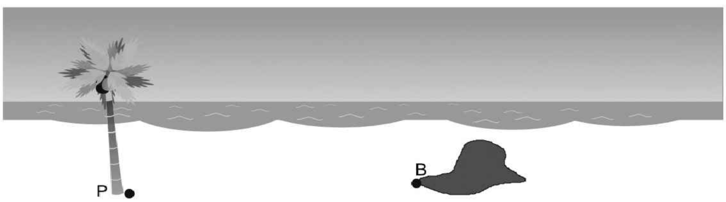

Figure 1: A treasure hunting problem.

When GeoGebra is brought into the teaching process, however, a variety of pedagogical possibilities arise. First, from students' perspective, students can be engaged in mathematical modeling, problem exploration, and open-ended questioning. They can diagnose their working models for the problem and make adjustments. They can also claim ownership on their mathematical construction, including, in most cases, the good mistakes or the false starts that are natural part of teaching and learning mathematics.

Second, from a task perspective, the problem takes on new dimensions when GeoGebra is used in instruction. The problem itself becomes an example from a large, in fact, infinite collection or problem space of similar problems. At many stages of problem solving, there exist alternative pathways. For instance, to make a 90-degree turn and walk the

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same distance could be accomplished by (a) the 90-degree rotation of a segment, or (b) the construction of a 90-degree angle followed by a segment of given length, or (c) using perpendicular lines and circles (see Figure 2). The whole problem could be used to invent another related problem: *If I were to place the treasure for others to locate, how can I position it properly?*

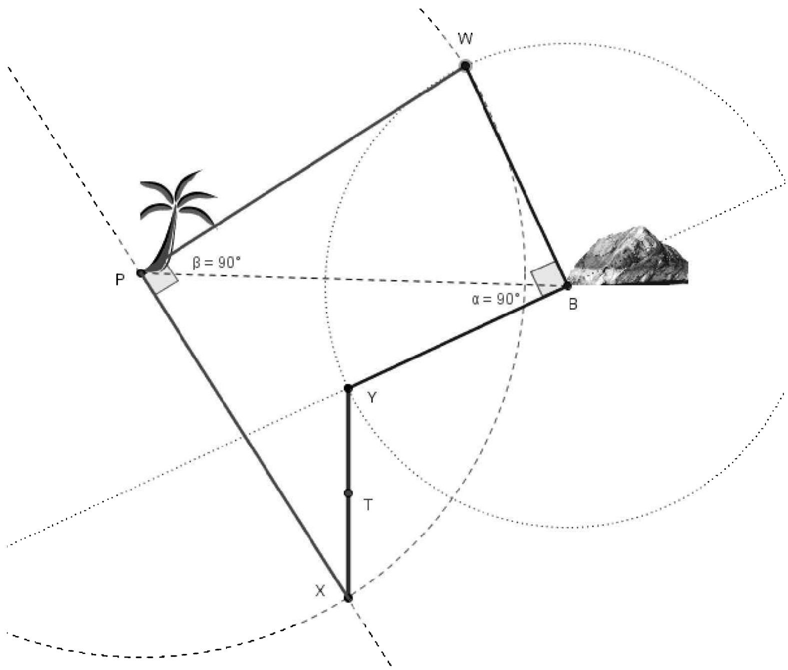

Figure 2: A GeoGebra model for the treasure hunting problem.

Third, from a *teacher educator*'s perspective, the instructor can design partially completed worksheets to scaffold and accommodate the diverse needs of students and allow various entry points to the problem. GeoGebra is also a tool of assessment, allowing the teacher to look into student teachers' thinking processes through an examination of the construction protocol or, better yet, a step-by-step onscreen replay of

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the student's construction. More importantly, the instructor can create his or her own instructional materials in the form of applets and post them online in support of his/her own reflection and participation in the local and global community of mathematics educators, including the students. For example, in the case of the treasure problem, the two right angles or one of them could be completed in advance for certain students who may have initial difficulty coordinating the complex relations in the problem.

Fourth, GeoGebra enhances the *learning environment* with its multiple representations, computational utilities, documentation tools, and web-friendly features which extend the scope of teaching and learning beyond the walls of the classroom. With the treasure problem, both students and instructors can share their dynamic constructions online through the course website or elsewhere. Such dynamic solutions can be further demonstrated, exported, or modified in support of other learning objectives. Figure 3 shows the underlying mathematical structure of the treasure problem and where the treasure should be buried to set up the problem. The dynamic construction can further be used to make conjectures about the problem, the properties of the midpoint T, and the multiple congruent triangles. In particular, what-if and what-if-not questions (Brown & Walter, 2005) can be readily explored in a dynamic construction. For example, about the treasure problem, one can ask, "what if I walk toward the land first?" or "what if I do not make 90-degree turns at the tree or boulder?" A formal proof may be the next step, if so desired.

In Spring 2011, sixteen preservice elementary school teachers investigated the treasure hunting problem using a paper-and-pencil method, followed by a GeoGebra-based exploration. In two weeks, they went from a static, paper-and-pencil model to a dynamic model as illustrated in Figure 3. On an anonymous post-survey, 75% of the preservice teachers indicated that they felt that problems such as the treasure hunting one, when explored with GeoGebra, would help build their confidence in teaching and doing mathematics.

Based on the above example and similar cases in our use of GeoGebra in teacher preparation and professional development programs, we classify the pedagogical roles of GeoGebra under the theory of model-centered learning and instruction (see Table 1), as a way to make sense of the instructional uses of GeoGebra and to generate new ideas for

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teaching mathematics teachers to use technology. The framework is intended as a lens for teacher educators to examine and further enhance the use of GeoGebra or similar dynamic learning technologies in teacher education. In reality, all the factors will not come into play at the same time. However, over time, the use of GeoGebra may have impact on all dimensions of the complex enterprise of mathematics education. In what follows, we present a GeoGebra-integrated online learning vignette, using data collected from an inservice professional development project.

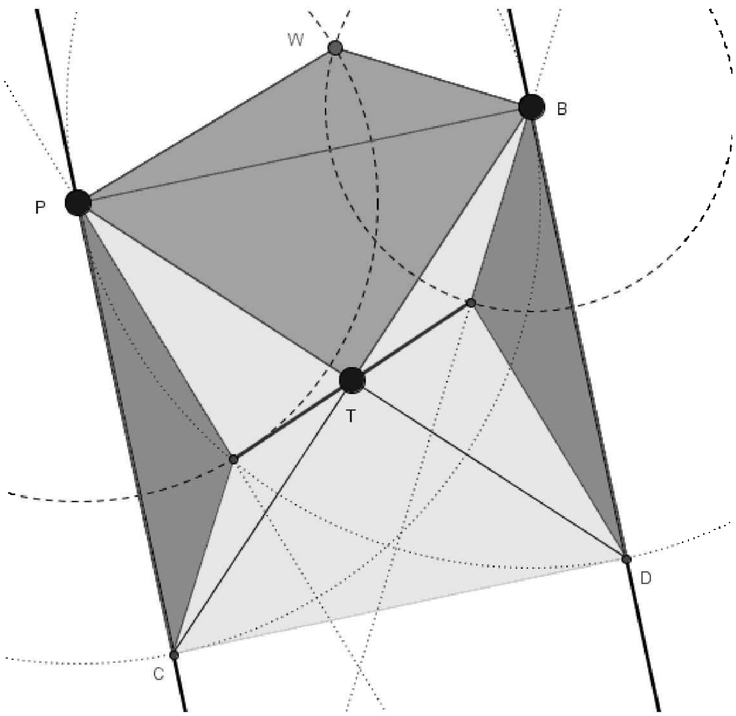

Figure 3: Where should the treasure be buried if one were to set up the problem? The GeoGebra model shows that when W moves, the square BPCD and point T do not move at all.

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Table 1: A Framework for the Pedagogical Roles of GeoGebra in Mathematics Teacher Education

Dimension	Pedagogical Roles of GeoGebra

Students	Engaging students in exploring, asking *what-if* and *what-if-not* questions (Brown & Walter, 2005)
	Supporting students' decision-making
	    confirming, conjecturing, drawing conclusions, tweaking problems, or inventing new problems
	Diagnosing students' working models for math ideas
	    revealing one's thinking
	    identifying gaps in knowledge
	    making suggestions for improvement
	Promoting ownership of mathematics learning

Tasks	Extending the problem/example space
	    changing initial conditions
	    finding singular or special cases
	Enabling alternative pathways to problem solving
	    promoting model-based solutions
	    supporting both informal arguments and formal proofs

Educators	Supporting reflections on the part of the teacher and teacher educators'
	    action, reflection, autonomy, networking (Llinares & Krainer, 2006)
	Scaffolding to accommodate diverse student needs and entry points
	Networking with the local and international community
	    using existing resources
	    contributing instructional resources to the community
	Relearning about mathematics and present-day students and (student) teachers
	Promoting ownership of instructional modules, lessons, and artifacts
	Assessing students' understanding of mathematics
	    using model-based assessment
	    focusing on the processes of problem solving

Environment	Providing multiple representations and simulations
	Providing computational utilities
	Providing documentation and organization utilities
	    graphs, images
	    dynamic worksheets
	    intermediate learning objects
	    interactive whiteboard
	Reaching beyond the space and time of the class
	    networking with peers in mathematics education
	    promoting teacher reflection
	    supporting student reflection
	Generating artifacts for documentation and research

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An Inservice PD Vignette: The Sliding Ladder

Inservice teachers' professional development (PD) has become a vital component of the ongoing educational reform, driven by accountability, diversity, and especially the constant advances of educational technologies and innovative instruction paradigms (Darling-Hammond et al., 2008). There is a growing need for sustainable PD programs in rural areas, where classroom teachers are facing both curricular challenges and a lack of affordable resources in many subject areas, particularly in mathematics and sciences. To meet the professional needs of these full-time classroom teachers, we have been implementing a state-funded project titled Science and Mathematics Action Research for Teachers (SMART), blending PD summer institutes and year-long graduate-level courses through online learning supported by the Blackboard System® and open-source web technologies, including GeoGebra. The first cohort of 27 teachers graduated with master's degree in Mathematics and Science Teaching in December 2010. The second cohort is ongoing at the time of writing.

In Spring 2010, GeoGebra was first integrated into a mathematics content and methods graduate course on problem solving. During the online course work, GeoGebra played a variety of pedagogical roles: providing for dynamic demonstration, computation, graphing, modeling, exploration, alternative solutions, online mathematical communication, and connections with school children. In this section, we provide a detailed vignette about the inservice elementary teachers' experience with GeoGebra-based mathematics learning. Our purpose is to provide an empirical justification for the preceding discussion about the pedagogical roles of GeoGebra in mathematics teacher education and its many ramifications in professional development courses.

We posted two geometry problems for online discussion during the second week so that we could document the teachers' thinking processes and their experiences with GeoGebra. The first problem is about the construction of conic section curves using a circle and a point that could be within, on, or outside the circle (see Figure 4). The second problem situation is a falling ladder (see Figure 5), and we asked our teacher participants to build a dynamic model to characterize the path of a certain point on the ladder as it slides away from the wall. The teachers were directed to try it out on GeoGebra and share their observations and ask questions in an online forum. In the following sections,

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we focus on the ladder problem, referring occasionally to the circle problem.

Segment *AB* determines the location and the size of the circle. *Q* is a point in the plane; *P* is a point on the circle. Using your imagination and features of GeoGebra, describe a few things you would like to explore with this problem.

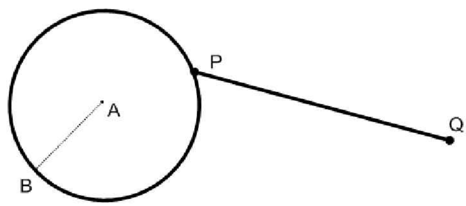

Figure 4: Exploration: A circle, a point, and a segment.

As shown below, a ladder of a certain length, say 10 feet, is placed against the wall. Joe, the painter, is working on the ladder at a position as indicated. Suddenly, the ladder starts sliding away from the wall. Can you find the curve that represents the movement of Joe in the accident?

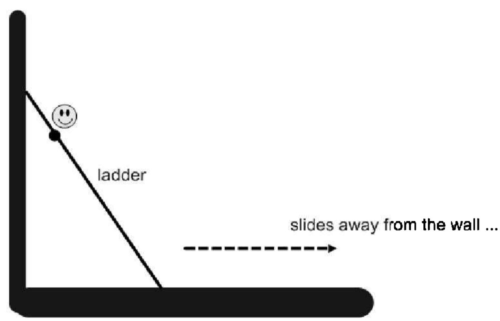

Figure 5: Modeling a ladder sliding away from the wall.

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Learning about the Problem

The ladder problem is not a very difficult one, but does call for some insightful observation with regard to the ladder length, which stays constant during the fall. The purpose of the assignment was to provide an opportunity for our teacher participants to further explore the features of GeoGebra and their connections to problem solving, and, in particular, to understand the interdependent relationships in a dynamic model (e.g., the perpendicular relationship between the wall and the ground) and how to manage the degree of freedom and constraints in dynamic modeling (e.g., the relative positioning of the ground and the length of the ladder) (Jones, 1996; Laborde, Kynigos, Hollebrands, & Strässer, 2006). The problem was posted online in the beginning of the second week of the course and was soon greeted with a series of questions in the discussion forum. As anticipated, all the teachers struggled to keep a constant length for the ladder in GeoGebra. Jane (pseudonym) posted the first message:

> I have been showing them [my students] GeoGebra as I am attempting to learn it; so we drew out the problem on there, but were stuck when we were trying to move the whole segment that represented the ladder. We could only figure out how to move one point at a time. Has anyone figured out how to make a segment fixed so the length of it won't change when you are moving it?

Other teachers posted similar questions and made suggestions for solving this problem. Charlie (pseudonym) commented, "I have had the same problem of only being able to move one point at a time and not [the] whole segment." He made references to linear equations, including such ideas as slopes and x- and y-intercepts, and compared the ladder problem to a previous one discussed in class. Still he was not able to find a way out. He wrote, "Still not sure how to do this for the specific line in the ladder problem, but I think it is a place to start."

Seeking Alternative Models and Working with Children

There was a bit of frustration building up in the online discussion forum. However, from an instructional perspective, the teachers had made significant progress toward a solution, because they had uncovered one of the critical components of a reasonable solution: how to fix the length of the ladder? It is worth noting that in a traditional setting,

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the ladder length would not have been a problem at all, although it is an important implicit factor of the situation. Indeed, many of our teachers resorted to physical modeling during the initial stage. Since our teachers had daily access to their children, they further invited their children into the exploration, which was one of the goals of our overall project. Jane left a message about team work with her children:

> My students and I explored the ladder problem together. They really enjoy helping me with my homework. They came up with the idea to create a paper ladder that we could put up on the white board and drew the ground and the wall. We then put a hole in the paper ladder that represented Joe. Using 3 people (1 holding the marker through the hole, and the other 2 making the ladder fall) we were able to create the curve that Joe would make. The kids loved it so much we made different ladders that were taller and shorter and moved Joe around on the ladders to compare the curves.

Jane's initiative was well received among other teachers, who decided to conduct similar experiments with their children. Karla (pseudonym) remarked in her message, "What a great idea! I will have to try this problem with my classes." Casey (pseudonym) also agreed, "This sounds like a great idea. I plan to do this with my students. I also plan on showing my students the GeoGebra program. I think that they will really enjoy manipulating and creating, especially on the smart board." Abbie (pseudonym) further reported her work in detail and indicated she would seek a GeoGebra-based solution:

> I tried this method with my students as well. They really enjoyed helping me. We discussed the possibilities of the curve before doing the hands-on method. I had them draw their predictions of the curve before we did it as a whole class. Some of them got it pretty close, while others were completely off. However, after doing it as a class,... it makes so much sense now. I am going to work on creating the problem in GeoGebra tonight to show them tomorrow. Hopefully I can figure it out. I think that they will love GeoGebra.

Requesting Instructional Support

To model the falling ladder with GeoGebra, however, our teacher participants were asking for prompts. It was clear to the instructors that

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the class was ready to move on. So we posted a video demonstration online, showing one possible approach to keeping the ladder length a constant. As shown in Figure 6, either endpoint of the ladder could be free, but not both at the same time. Therefore, given the ladder length, we could use a GeoGebra circle tool to locate the other point of the ladder. If we keep point Top free on the wall, then point Bot is the intersection of the floor and the circle which is centered at point Top with a radius equal to the ladder length. To allow for further exploration, we also introduced the idea of a slider to change the relatively constant length of the ladder. There are, as a general rule, alternative methods to keep the ladder length fixed such as using the Pythagorean Theorem.

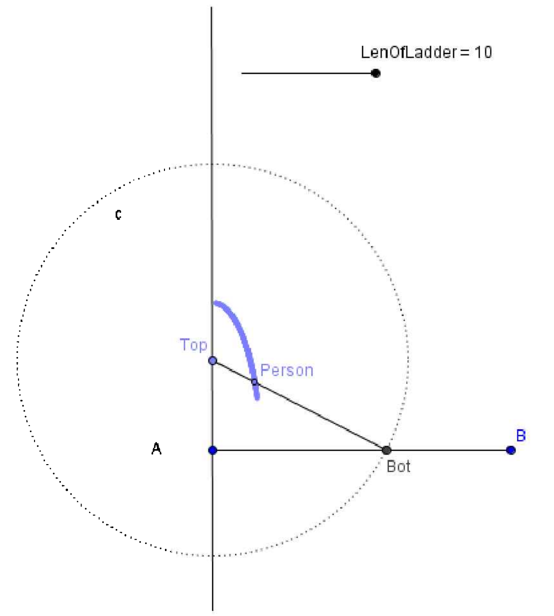

Figure 6: The ladder problem is modeled using GeoGebra by strategically fixing the ladder length.

The majority of the class came to see the meaning of the proposed strategy, the role of the circle and the need to specify the length of the ladder. The problem was thus solved. In retrospect, we have to point out that problem solving in dynamic mathematics involves as a key

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component a detailed analysis of the dependent relations in the problem situation with respect to the technology affordances (Bu & Haciomeroglu, 2010; Jones, 1996; Laborde et al., 2006). This has proven to be a time-consuming process consisting of trial-and-errors, attempts of different strategies, and deep reflections. During that process, GeoGebra itself provides instant modeling feedback in response to learners' manipulation, playing the role of a cognitive partner in problem solving (Salomon, Perkins, & Globerson, 1991). For example, many teachers initially constructed a drawing of the wall and the floor, where the wall and the floor were not mathematically perpendicular but merely looked as such. While, in a traditional setting, these implicit relationships do not pose significant obstacles, they can readily show themselves, when being dragged, in a dynamic environment. As can be seen in the teachers' discussions about the ladder length, GeoGebra constructions help learners diagnose their mathematical conceptions and make explicit the mathematical relations that are implicit and/or critical in a problem situation.

Engaging in Reflections and Further Work with Children

Teachers are reflective professionals, who are inclined to think back and forth and ask extended questions with each other or their students about a problem situation. Once the ladder problem was successfully modeled with GeoGebra, they further shared their experience with their students and reflected on the implications for their instructional practice. Helen (pseudonym) posted the following message about her experiment with children:

> I used the ladder problem as a 5-minute warm-up problem on Friday. Some students became quite interested in the problem; others seemed to think it was useless and had nothing to do with math. Most students seemed to think the person would fall in a straight path instead of a curved path. I placed students in groups, one side of the room for the straight path and one side of the room for the curved path. Most students started on the straight path side. I allow them to switch sides at any point. As I demonstrated the problem using GeoGebra, students started to change over to the curved path side of the room. There were still a few unsure students until I used the tracing tool. At this point they were all convinced and had joined the

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> curved path side of the room. They are currently working on ideas for ways to demonstrate this activity without using a computer and without bringing an actual ladder into the classroom.

Toward the end of that week, many teachers were looking back at their own experiences with the problems and the GeoGebra tools, and making connections with their own teaching. Kayla (pseudonym) posted a message about the way the class explored problems using GeoGebra:

> I love how this problem [the circle and conic curves problem] was presented, asking us to use our imaginations. Many of us posted responses and commented freely on what steps we took to solve the problem and why we tried what we did. I am going to pose questions like that to my students, asking them to use their imagination to solve math problems. It takes the pressure off of being correct and allows the student to work through solutions without fearing the wrong answer.

Ariel (pseudonym) further commented on the overall experience during the second week and made references to the way mathematicians might be solving problems:

> I have really found GeoGebra an interesting tool to use. I really enjoy the program and think I have figured several things out. The ladder problem seemed a bit easier to figure out than the circle problem. I have been told for so long that in mathematics, I am given a problem to solve. Not many times have I been told to look at something and play around with the different parts. However, this is probably what modern mathematicians need to do to figure out new concepts and ideas.

Seeking Support from Community

When working with GeoGebra in problem solving, teachers will naturally encounter problems unique to the dynamic mathematics software. When that occurs, GeoGebra, being the common tool at hand, provides an anchor for teachers to seek support from each other and further relate their concerns and experiences to their students' mathematical learning. The following thread of messages also occurred during the second week, when teachers were solving the two problems assigned (all names are pseudonyms). The concern was more of a technical nature than mathematical. Note that Jamie's question about "hiding part

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of a line" will need a technical solution: locate some points on the line, hide the line, and reconstruct a segment using those points on the line.

> Mary: I found GeoGebra kind of confusing. How do I get rid of lines that I do not need? I made the design, but I have lines that I cannot figure out how to hide them.
>
> Linda: Mary, try right-clicking, then "hide grid". I think that is what I remember from the tutorial, but I'm not sure.
>
> Jamie: I am having the same problem. It is probably something really simple.
>
> Charlie: If you right click on the line, one of the items in the menu that appears is "show object," select it, and it will hide the object. Select it a second time and it will show it again.
>
> Jamie: But what if I just want to hide PART of a line? That is where I ran into trouble.
>
> Charlie: Not sure about that. Maybe try drawing a segment over the portion you want to keep, then hiding the original line.

Reaching Out to the Community

Elsewhere, we reported on our inservice teachers' attitudes toward GeoGebra-integrated mathematical activities (Bu, Mumba, Henson, Wright, & Alghazo, 2010). Our data show that the majority of the teachers agree or strongly agree that GeoGebra helps them make connections among mathematical ideas, reach out to more students, learn with students, and provide instant feedback to them. More than 90% of the teachers in our study believed that their students generally find GeoGebra appealing in mathematical activities. As one teacher wrote to a free-response question on a post-survey, "my students really enjoyed seeing the things that we could create in GeoGebra. One student has an aunt and an uncle who are math teachers and they contacted me about the program because she was so excited about it. I really think that this is a great way to get kids into math!"

In summary, our inservice teachers benefited significantly from the use of GeoGebra in various ways, ranging from personal mathematical exploration, improved attitudes toward mathematics and mathematics teaching, to enhanced pedagogical reflections. These changes are well

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aligned with the emphases of the ongoing mathematics education reform, including the integration of technology. Figure 7 represents a summary of our inservice teachers' problem solving experience with the ladder problem. The structure remains consistent across other non-trivial GeoGebra-based problem situations for the duration of the course work. One unique component is the involvement of children in teachers' learning processes. Furthermore, the position of instructional support is important. The presence of GeoGebra allows teachers to self-diagnose and self-regulate much of the initial exploration of the problem space in spite of some necessary emotional reactions.

Nonetheless, instructional support is critical for teachers to make the cognitive advance. It is only after the initial self-diagnosis and cognitive disequilibrium that teachers are given instructional prompts, which guide them to construct a possible solution and to pose more questions about the problem situation, their prior knowledge, and their thoughts about teaching mathematics. We believe that children are naturally inclined to use technology to explore mathematical problems. By explicitly engaging children in the process of professional development, teachers can better try out and strengthen their emergent ideas of mathematics and pedagogical initiatives, including the use of GeoGebra. Our findings support the use of GeoGebra in PD programs that seek to enhance inservice teachers' understanding of focal ideas of mathematics, the nature of mathematics, and empower them with the pedagogical tools to enact changes in their teaching practice. GeoGebra plays multidimensional roles in setting up a dynamic learning environment with rich resources, engaging educators and children, and redefining and extending learning tasks.

Conclusion

GeoGebra provides a rich set of resources for teacher educators to enhance and reflect on their mathematical practices in both preservice and inservice mathematics teacher development. In proposing a tentative framework for the roles of GeoGebra as a pedagogical tool to integrate content and pedagogy in mathematics teacher development, we are informed by the basic principles of model-centered learning and instruction, aiming to critically reflect on our own teaching experience and our use of new dynamic learning technologies in teacher education.

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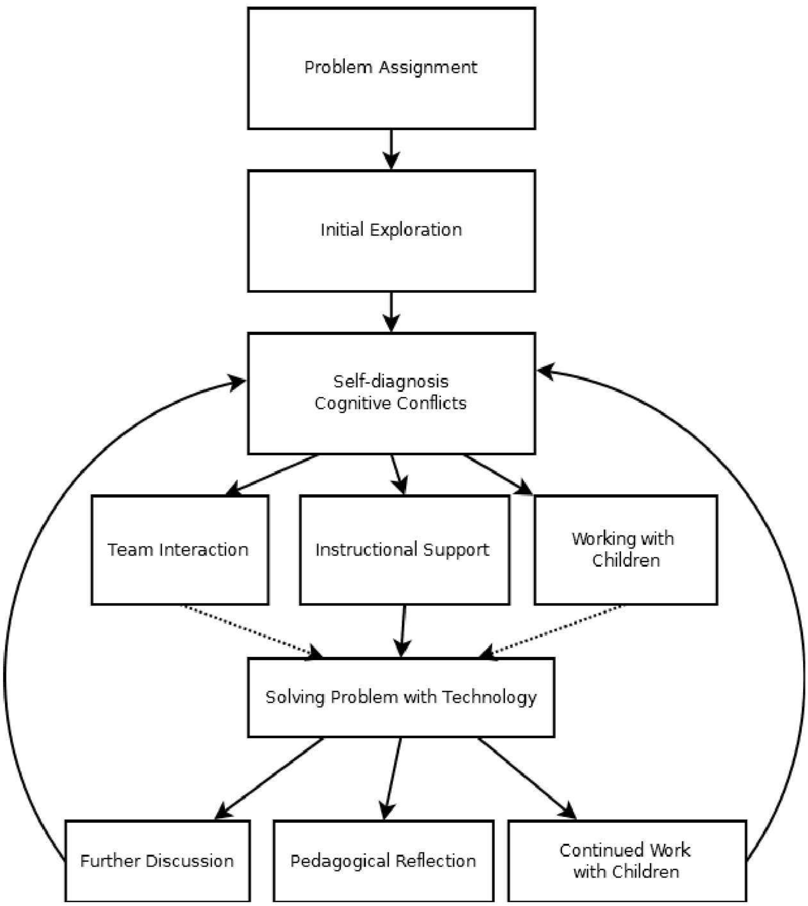

Figure 7: A summary of inservice teachers solving problems with GeoGebra.

In making the difference between the categories over our instructional practice, we have tried to make sense of our own experience (cf. Shulman, 2002). Indeed, as teacher educators who learned much of the mathematics decades ago, we were relearning or even diagnosing our own understanding of the mathematics we think we have confidence in

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doing and teaching. GeoGebra stands as a tool for thought and reflection for mathematics teacher educators in terms of student learning, teacher learning, and our own learning as the world changes and brings us new tools, new opportunities, and new challenges. GeoGebra provides a learning environment where mathematics content and pedagogy are deeply intertwined in the practice of mathematics teaching and learning. By virtue of its very nature, GeoGebra stands as a learning environment where mathematics educators can document, critique, and study the formation of Technological Pedagogical Content Knowledge (TPACK) (Mishra & Koehler, 2006; Niess, 2005, 2008), which is critical to effective mathematical teaching in the current information society. There is an increasing need for the field to document students', teachers', even teacher educators' and mathematicians' use of GeoGebra in concrete non-trivial mathematical problem solving at all level of mathematics in support of theory development and large-scale quantitative program evaluations.

In closing, we quote Shulman's (2002) view about design and invite all mathematics educators and technology enthusiasts to reflect on GeoGebra-based mathematical modeling and instructional design: "Design is a matter of exercising understanding, as well as applying skills, under a variety of constraints and contingencies" (p. 41). We hope that the international community of GeoGebra users will continue to engage in innovative design initiatives in software development, mathematics tasks, mathematics curricula, teacher development programs, and local and global communities of mathematical practice, integrating technology, pedagogy, and mathematical content in the new era of mathematics education.

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Chapter 5

Kinematic Geometry with GeoGebra

Petra Surynková

Introduction

This chapter deals with the applications of dynamic system GeoGebra in the teaching and learning of geometry. The main field of our interest is study of classical, kinematic, and descriptive geometry – geometric constructions, projections, geometry of curves and surfaces - geometry of the Euclidean plane and space. Our aim is to increase the interest of students in studying geometry at secondary schools and colleges. One possible approach of improvement in studying geometry is the integration of computer software in the teaching process. This way seems to be interesting, attractive, and motivational for students. Indeed the usage of computers in education is very current (at least in the Czech Republic).

In this chapter, we will illustrate the advantages of dynamic geometry system based on examples from the field of kinematic geometry in the plane. For further understanding, it is necessary to explain the theoretical part of kinematic geometry in the plane. We will introduce main definitions and theorems and concentrate on creation of planar curves by special motions of geometric objects. The usage of software will be shown using examples. We will describe determinations of special motions in GeoGebra and show the usage of dynamic tools in this software using two examples in detail. We will focus on geometric motion determined by envelopes and paths and motion determined by fixed and moving centrodes. We will precisely describe the tools and commands which we used.

The tasks related to the kinematic geometry can be very difficult for students if they solve them without any computer software. It can be very exacting to imagine the result of geometric motions; for that reason the dynamic systems such as GeoGebra bring new possibilities.

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Students may use GeoGebra in regular schooling as well as home schooling and e-learning. We have web pages with database of examples of geometric motions in the plane and provide access to this database for our students, so the students have dynamic worksheets at their disposal. In dynamic worksheets some parameters can be changed. Of course, students also create some examples and tasks themselves.

The responses from students using GeoGebra to learn geometry are very positive. Students are satisfied because the computer software is very motivational and attractive for them. Kinematic geometry in the plane is more understandable and geometry in general becomes a modern discipline. GeoGebra can be a suitable teaching aid. The outputs from GeoGebra can be used in publications and also for e-learning.

The rest of this paper is organized as follows: The section "Geometry" is devoted to geometry in general: its importance, current problems of the unpopularity and the difficulty of studying geometry and the integration of computers in the teaching process. In the section "Kinematic geometry", we will describe the theoretical part of kinematic geometry and discuss main definitions and theorems. In the section "Special motions in the plane", we will introduce some definitions and examples of special motions in the plane. The main focus of the section "Constructions with GeoGebra" is the description of two examples in detail from the field of kinematic motions which are solved with the support of GeoGebra. In conclusion, we will discuss the advantages of using GeoGebra and the responses from our undergraduate students.

Geometry

Geometry, the study of properties and relations of geometric figures, is an important and essential branch of mathematics. Geometry can be conceived as an independent discipline having many branches – Euclidean geometry, differential geometry, algebraic geometry, topology, non-Euclidian geometry and so on.

Geometry is useful for learning other branches of mathematics, and it can also be used in a wide range of scientific and technical disciplines. Some scientific branches require direct knowledge of geometry.

The main field of our interest is the study of classical, kinematic, and descriptive geometry – geometric constructions, projections, geometry of curves and surfaces. Mainly it is geometry which allows the represen-

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tation of three dimensional objects in two dimensions. That is, we are dealing with geometry of the Euclidean plane and space.

The Study of Geometry

The study of geometry can be very difficult. This branch of mathematics is not popular among students. For example, drawings (the results of geometric projections of some 3D object) are sometimes very difficult to understand. For that reason, geometric problems must be provided with clear examples. Intuitive understanding plays a major role in geometry. With the aid of visual imagination, we can illuminate the problems of geometry. In many cases, it is possible to show the geometric outline of the methods of investigation and proof without entering needlessly into details. The problem can be more understandable without strict definitions and actual calculations. Such intuition has a great value not only for researchers, but also for anyone who wishes to study and appreciate the results of research in geometry. If we understand the main principles of a problem then we can use formal definitions. According to Hilbert (1999), the study of geometry develops logical reasoning and deductive thinking which helps us expand both mentally and mathematically.

The current predominant view among students and the general public is that classical descriptive geometry is not important or useful. Drawings of classical and descriptive geometry can be replaced by the outputs of modern computer software. Of course, computers can help us solve geometric problems and increase the efficiency of our work but we still have to know the basic principles and rules in geometry.

Geometry would be easier for students if they had encountered classical geometry, constructions, and geometric proofs early in their schooling. Sometimes students of technical specializations experience geometry only at college. That is too late. We work mainly with undergraduate students, so what can be done to make college geometry more comprehensible? How to increase the interest of students in studying classical geometry at secondary schools and colleges? This is the main concern of this chapter.

The Usage of Computers in the Teaching Process

Our aim is to increase the interest of students in studying geometry in secondary schools and colleges. One possible approach is the integra-

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tion of mathematical and modeling software in the teaching and learning processes. This way seems to be interesting, attractive, and motivational for students. Indeed, the usage of computers in education is comparatively new. Computers influence our everyday life including geometry. We have to follow the general trend.

Currently, computer-aided design is commonly used in the process of design, design documentation, construction, and manufacturing processes. There exists a wide range of software and environments which provide the user with input tools for modeling, drawing, documentation, and design. These software and environments can be used to design curves and geometric objects in the plane, and curves, surfaces, and solids in the space. More than just shapes of geometric objects can be affected in this software. In modern modeling software, we can also work with rotations and other transformations; we can change the view of a designed object. Some software provides for dynamic modeling. These tools can help in better understanding the geometric situations mainly in the space. For example, we can see spatial geometric objects from another view so it how it looks can be clearer. Technical and engineering drawings must contain material information and the methods of construction. Computer-aided design is used in numerous fields such as industry, engineering, science, and many others. The particular use of computers varies according to the profession of the user and the type of software.

These modern methods, which are widespread in various branches, can be useful in teaching too. We should help students to improve their skills for their future employment. If we use computers in the teaching and learning processes, we still put emphasis on the understanding of the principles used in geometry. Software can work automatically but this is not desirable in the teaching and learning of geometry.

I have experiences in teaching classical, descriptive, kinematic, and computational geometry at universities such as Charles University in Prague, at the Faculty of Mathematics and Physics, and Czech Technical University in Prague, at the Faculty of Architecture. College mathematics, and especially geometry, is very difficult for many students. It is necessary to motivate and to develop an interest in geometry.

In my lessons, I use computer software for visualization, for the proving of geometric problems in the plane and in the space, for the demonstration of the application of geometry in practice, for the crea-

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tions of study materials for home schooling and e-learning, and for the transformation geometric problems into algebraic form. I work, for example, with Rhinoceros - NURBS modeling for Windows (Rhino), Cabri II Plus, Cabri 3D, MATLAB, Maple, and GeoGebra.

I use GeoGebra for the creation of study materials, which can help my students understand geometric problems in intuitive and natural ways. Moreover, I show special constructions applied in descriptive geometry problems and, due to included functions and tools, students can discover proofs more easily. In this paper, we will demonstrate the advantages of using a dynamic and interactive mathematics learning environment on examples from the field of kinematic geometry.

Good geometric imagination and perception are very important for understanding constructions in geometry. It is not possible to memorize the constructions; we have to understand geometric problems. Let us start with the theoretical part of kinematic geometry. This is necessary for further understanding because we will explain important terms. Then we will introduce definitions and examples of curves which are created by special motion of geometric objects.

Kinematic Geometry

Kinematic geometry in the plane is a branch of geometry, which deals with the geometric properties of objects, which are created by motion of moving a plane. These are geometric motions without regard to the cause of the motion, velocity, and acceleration. Theoretical kinematics is a large subject and it is not possible to treat it completely in this chapter. We will restrict our discussion to essential basics.

We consider an unbounded infinite plane, which contains geometric elements (points and curves). We treat such elements as points, straight lines, circles, and line segments, and study the geometric properties as they move in the plane. We focus on aspects of transformation geometry. We consider those transformations in the Euclidean plane such that all distances remain fixed during the motion.

The Determinations of Geometric Motion

Let Σ be *the moving plane* which slides over *the fixed plane* Π. The moving plane Σ contains curves and points which remain identical throughout the movement. In the fixed plane there are generated the

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paths and the envelopes. The *path* (often called *roulette*) is a curve in the fixed plane Π which is described by moving a point of the moving plane Σ, it is the locus of a point in the moving plane. The geometrical *envelope* of a family of curves in the moving plane is a curve, which at each of its points, is tangent to a curve of the family. Any curve of the moving plane Σ describes a curve, its envelope, in the fixed plane Π. Let $\Sigma^1, \Sigma^2, \Sigma^3,$ … denote the sequence of positions of the moving plane Σ. The positions in the moving plane of points $A, B, C,$ … will be $A^1, B^1, C^1,$ …, when Σ is at $\Sigma^1,$ and $A^2, B^2, C^2,$ …, when Σ is at Σ^2. It is analogical for curves. Figure 1 shows an example of the motion of the moving plane.

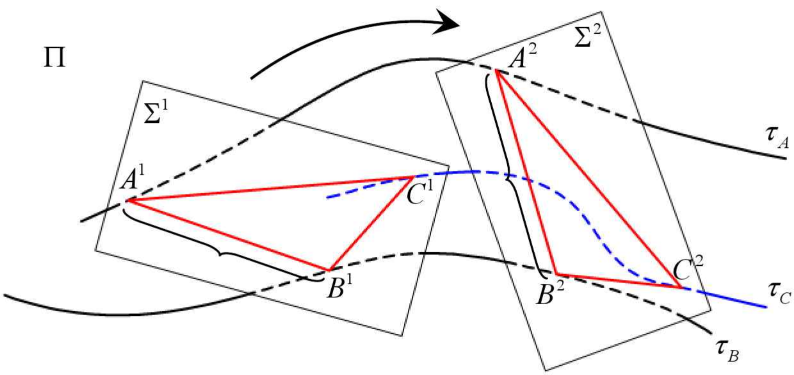

Figure 1: An illustration of the moving plane Σ containing points A, B, C (with line segments), which slides over the fixed plane Π. Points A, B, C describe curves τ_A, τ_B, τ_C - the paths. τ_A, τ_B are given, τ_C is obtained by moving. Indices denote positions of the moving plane. All distances remain fixed.

Let us discuss the determination of the motion in the plane. There are several possibilities how to define the motion:

a) **The motion is completely determined by paths τ_A and τ_B of two points A and B (end points of segment line).** These points are in the moving plane, the paths are in the fixed plane.

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The following equations are satisfied: $|A^1B^1| = |A^2B^2| = |A^3B^3| =$ … See Figure 2.

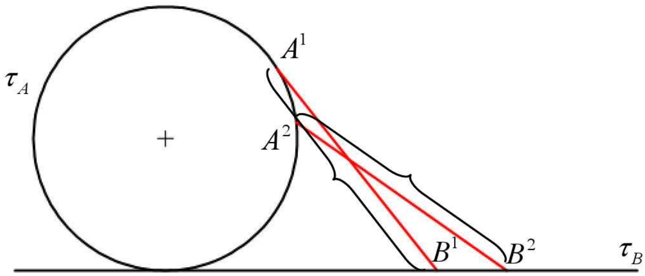

Figure 2: The motion is given by paths τ_A *and* τ_B *of two points* A *and* B .

b) The motion is completely determined by envelopes (m) and (n) of two curves m and n . These curves are in the moving plane, the envelopes are in the fixed plane. The following equations are satisfied: $|\sphericalangle m^1n^1| = |\sphericalangle m^2n^2| = |\sphericalangle m^3n^3| =$ … See Figure 3.

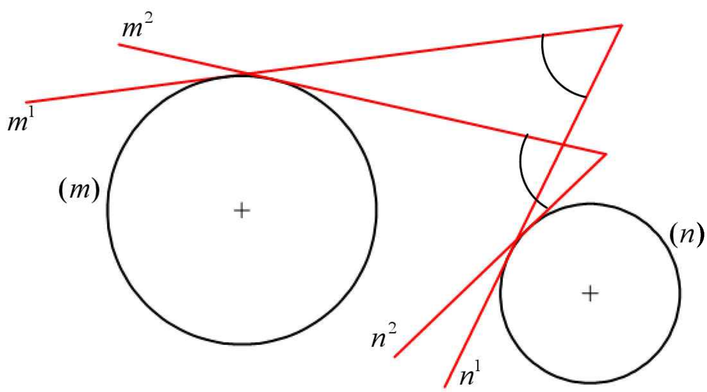

Figure 3: The motion is given by envelopes (m) *and* (n) *of two curves,* m *and* n .

c) **The motion is completely determined by envelope (m) of curve m and path τ_A of point A**. The following equations are satisfied: $|A^1m^1| = |A^2m^2| = |A^3m^3| =$ … See Figure 4.

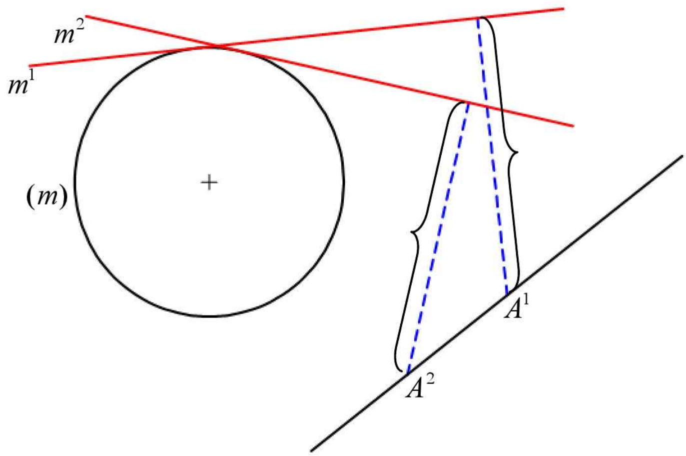

Figure 4: The motion is given by envelope (m) of curve m and path τ_A of point A.

In special cases, the envelope of a family of curves in the moving plane degenerates into point. These situations are shown in Figures 5 and 6.

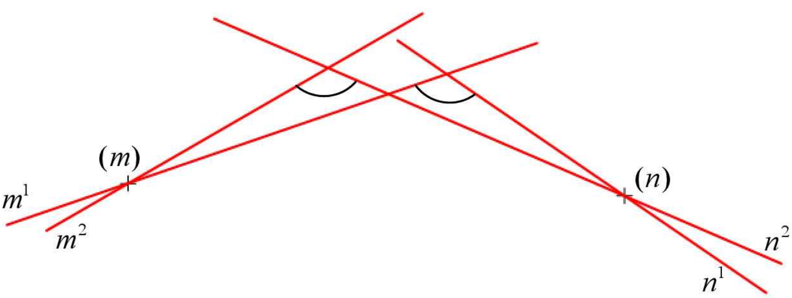

Figure 5: The motion is given by envelopes (m) and (n) of two curves m and n which degenerate into points.

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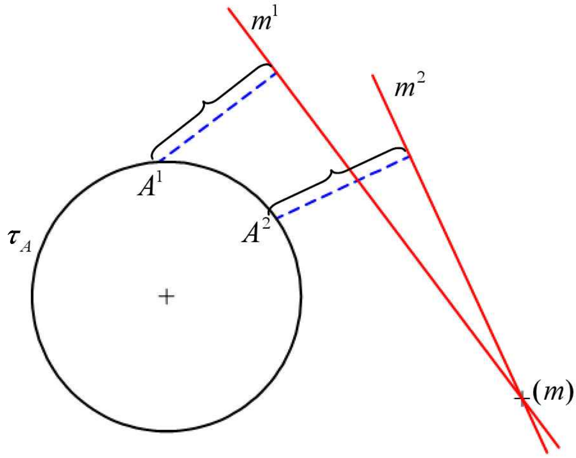

Figure 6: The motion is given by path τ_A of point A and envelope (m) of curve m which degenerates into point.

The proofs of these theorems can be found in Bottema and Roth (1979), Lockwood (1967) and Gibson (2001).

The Centrodes

We shall characterize any motion of the plane only by the initial and final positions. Of course, we can get the difference between the initial and the final position in different ways. It will be one of several tasks to find the simplest possible way of effecting any given motion.

The simplest motions in the plane are translations in which every point of the plane moves through the same distance in the same direction and every straight line remains parallel to its initial position. Another well-known type of motion is the rotation of the plane at a given angle about any given point. The direction of every straight line is changed by the given angle and the center of the rotation is the only point of the plane that remains fixed.

It is possible to prove that every motion of the plane can be carried out in one translation or one rotation. This fact considerably simplifies the study on geometric motions in the plane. More detailed information can be found in Bottema and Roth (1979).

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We can consider translations as rotations at the angle zero about an infinitely distant point. If we adopt this point of view, we may regard any motion of the plane as a rotation through some definite angle which is zero in the case of translation. Several theorems and the possibilities of compositions of simple motions are discussed in detail elsewhere (see Bottema & Roth, 1979).

Now the motion in the plane is given. We assume that Σ^i and Σ^{i+1} are two positions of the moving plane Σ . The change of position $\Sigma^i \rightarrow \Sigma^{i+1}$ is associated with a center of rotation S^i. If we consider a limiting position of Σ^i and Σ^{i+1} (the difference between Σ^i and Σ^{i+1} becomes smaller and smaller), the point S^i is called *the instantaneous center of the motion* which is related to instant i .

Definition 1. The locus of the instantaneous centers at every moment of the motion is a curve in the fixed plane. This curve is called *the fixed centrode* of the motion. ■

But in the same motion, we may also regard the plane Σ , which we had considered movable as fixed, and the plane Π , which we had considered fixed as moveable. That is, we may interchange the roles of the two planes. This motion is called *the inverse motion.* The original motion is called *the direct motion.* One motion determines the other one and the inverse of the inverse motion is the direct motion.

Definition 2. The locus of the instantaneous centers at every moment of the inverse motion of a given motion is a curve in the moving plane. This curve is called *the moving centrode* of the direct motion.

A more detailed study shows that the motion is completely determined by the form of the two centrodes. At each instant of the motion, the two curves are tangent to the other at the instantaneous center and there is no slipping. The motion is obtained by rolling (without slipping) the moving centrode in the moving plane on the fixed centrode in the fixed plane. If we interchange the roles of the centrodes, we get the inverse motion. From the fact that the centrodes roll on each other without slipping, it follows that the arc bounded by any two points on the fixed centrode has the same length as the arc bounded by the corresponding points on the moving centrode. There is one more possibility how to define the motion:

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d) **The motion is completely determined by the fixed centrode p and the moving centrode h.** The following equations are satisfied: $\left|\widehat{S^i S^{i+1}}\right| = \left|\widehat{(S^i)(S^{i+1})}\right|$. See Figure 7.

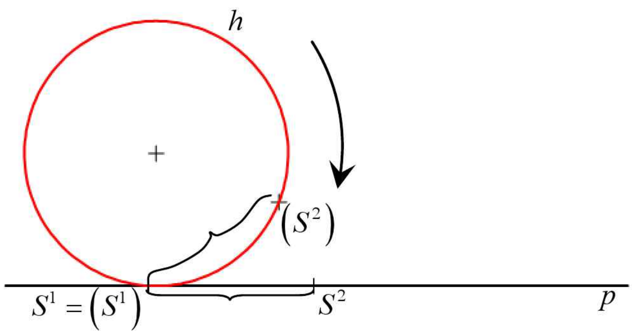

Figure 7: The motion is given by the fixed centrode p and the moving centrode h.

Geometric Constructions of the Centrodes

We shall discuss geometric construction of the centrodes. Let us consider the example which is given by paths τ_A and τ_B of two points A and B. We construct fixed and moving centrodes for this special determination.

Construction 1. The fixed centrode is the locus of the instantaneous centers at every moment of the motion. The instantaneous center S^1 ($S^2, S^3, \ldots$) is the intersection of the normal line to the path τ_A at the point A^1 ($A^2, A^3, \ldots$) and the normal line to the path τ_B at the point B^1 ($B^2, B^3, \ldots$) (see Figure 8).

Construction 2. The moving centrode is the locus of the instantaneous centers at every moment of the inverse motion to a given motion. We construct the point (S^1) ($(S^2), (S^3) \ldots$) of the moving centrode using congruence of the triangles $\triangle A^1B^1(S^1) \cong \triangle A^1B^1S^1$ ($\triangle A^1B^1(S^2) \cong \triangle A^2B^2S^2, \ldots$). The moving centrode is constructed in the position Σ^1 of the moving plane Σ (see Figure 9).

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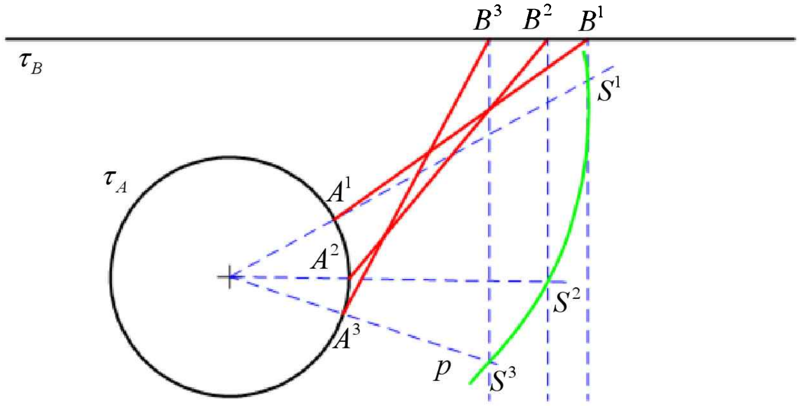

Figure 8: Geometric construction of the fixed centrode $\mathbf{p}$.

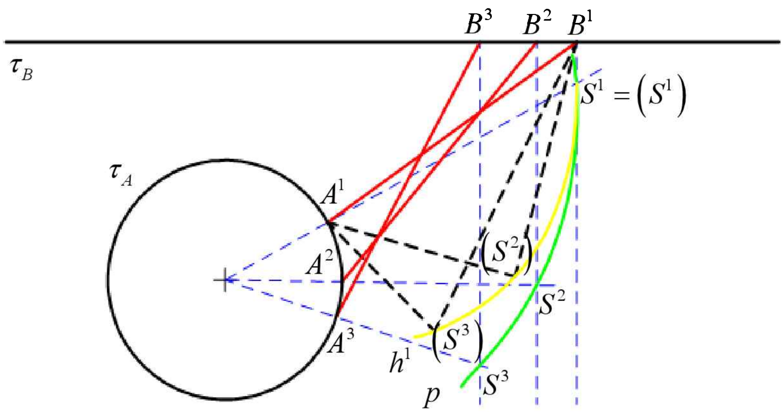

Figure 9: Geometric construction of the moving centrode $\mathbf{h^1}$.

Special Motions in the Plane

Special motions in the plane will be discussed in this section. For the creation of the outputs and images, we use GeoGebra. We have web

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pages (see http://www.surynkova.info/geogebra.php) with database of examples of motions in the plane (Surynkova, 2011). We provide the access to this database to our students, so students have dynamic worksheets at their disposal. In dynamic worksheets, some parameters can be changed, see the following figures.

Cyclical Motion

As mentioned before, a plane motion may be defined by its centrodes. The simplest example is the case where both centrodes are circles, or one centrode is circle and the second is straight line. These motions are called *cyclical motions.*

The motions are classified according to the type and relative positions of the centrodes.

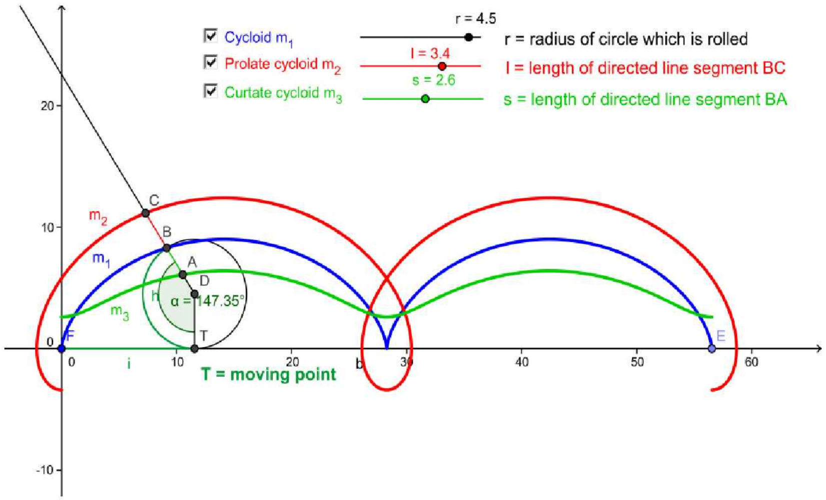

Figure 10: Cycloidal motion and examples of cycloids. Cycloid m_1, prolate cycloid m_2, curtate cycloid m_3.

We first consider the example where a circle is rolled on a straight line. That is, the moving centrode is the circle and the fixed centrode is the straight line. The paths of points which are obtained by rolling the circle on the straight line are called *cycloids, curtate cycloids or prolate cycloids*, generally *cycloid*. The motion is called *cycloidal*. The cycloid is the path of a point on the circumference of the rolling circle, the prolate cycloid is

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the path of a point outside the rolling circle and the curtate cycloid is the path of a point inside the rolling circle; see Figure 10.

The parametric equations of the cycloid are

$$x = r\alpha - v \sin \alpha$$

$$y = r - v \cos \alpha,$$

where r is the radius of the rolling circle, α is the angle at which the rolling circle has rotated (real parameter), in this example within the interval of $\langle 0, 4\pi \rangle$ and v is distance between the center of the rolling circle and a point on the path. If $v = r$, we get the cycloid, if $v > r$, we get the prolate cycloid, and if $v < r$, we get the curtate cycloid.

History and applications of cycloids are very important. More details can be found in the literature (see Rutter, 2000; Lockwood, 1967).

We can interchange the roles of the two centrodes as a straight line is rolled on a circle. The moving centrode is the straight line and the fixed centrode is the circle. The paths of points which are obtained by rolling the straight line on a circle are called *the involute of the circle*. The motion is called *involute*. The classification of paths is similar to the cycloidal motion. The paths of points which are obtained by rolling the straight line on the circle are called *involutes, curtate involutes or prolate involutes*; see Figure 11.

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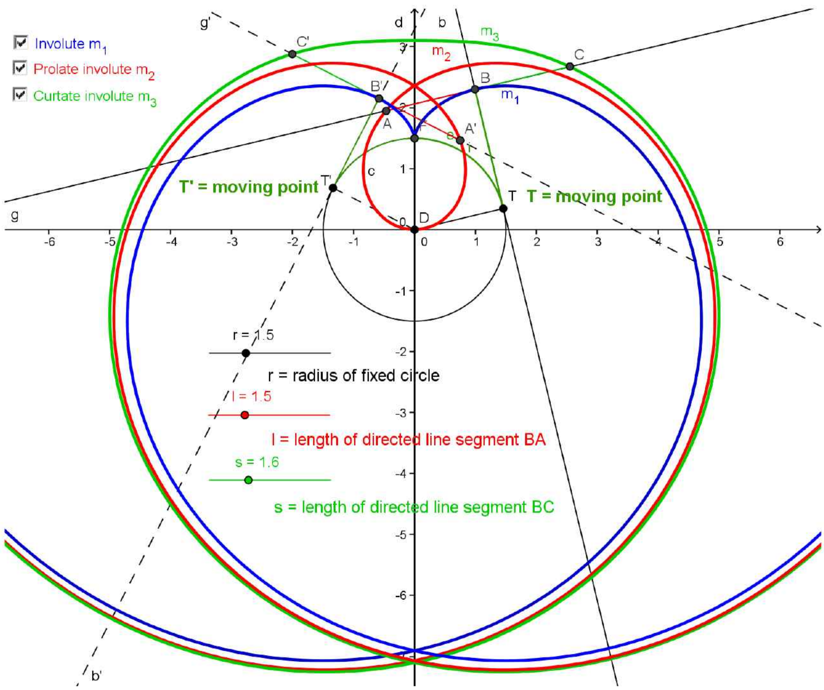

Figure 11: Involute motion and examples of involutes. Involute m_1, prolate involute m_2, curtate involute m_3.

The parametric equations of the involute of the circle are

$$x = (r - v)\sin\alpha - r\alpha\cos\alpha$$

$$y = (r - v)\cos\alpha + r\alpha\sin\alpha,$$

where r is the radius of the fixed circle, α is the angle at which the rolling straight line has rotated (real parameter), in this example within the interval of $(-2\pi, 2\pi)$ and v is directed distance from a point on the path to the rolling straight line (directed distance from the center of the fixed circle to the rolling straight line is positive). If $v = r$, we get the involute, if $v > r$, we get the curtate involute and if $v < r$, we get the prolate involute.

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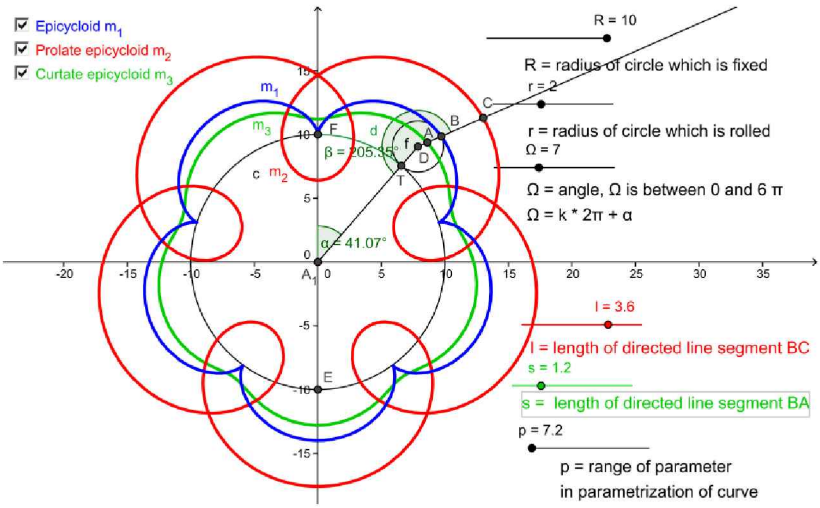

Figure 12: Epicycloidal motion and examples of epicycloids. Epicycloid m_1*, prolate epicycloid* m_2*, curtate epicycloid* m_3.

We now consider the example where a circle is rolled on a second circle. There are three cases. The paths of points, which are obtained by rolling the circle on the outside of the fixed circle, are called *epicycloids.* The motion is called *epicycloidal.* The epicycloid is the path of a point on the circumference of the rolling circle, the prolate epicycloid is the path of a point outside the rolling circle and the curtate epicycloid is the path of a point inside the rolling circle; see Figure 12.

The parametric equations of the epicycloid are

$$x = (R+r)\sin\Omega - v\sin\frac{(R+r)\Omega}{r}$$

$$y = (R+r)\cos\Omega - v\cos\frac{(R+r)\Omega}{r},$$

where R is the radius of the fixed circle, r is the radius of the rolling circle, v is distance between the center of the rolling circle and a point on the path, Ω is the parameter (see Figure 12). If $v = r$, we get the epicycloid, if $v > r$, we get the prolate epicycloid and if $v < r$, we get the curtate epicycloid.

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The paths of points, which are obtained by rolling the circle in the fixed circle, are called *hypocycloids*. The radii of the two circles cannot be equal. The motion is called *hypocycloidal*. The hypocycloid is the path of a point on the circumference of the rolling circle, the prolate hypocycloid is the path of a point outside the rolling circle and the curtate hypocycloid is the path of a point inside the rolling circle; see Figure 13.

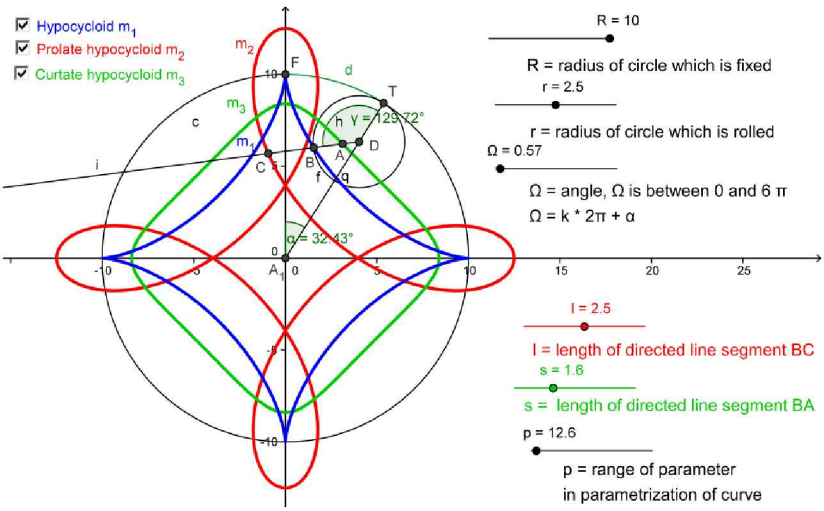

Figure 13: Hypocycloidal motion and examples of hypocycloids. Hypocycloid m_1*, prolate hypocycloid* m_2*, curtate hypocycloid* m_3*.*

The parametric equations of the hypocycloid are

$$x = (R - r)\sin\Omega - v\sin\frac{(R - r)\Omega}{r}$$

$$y = (R - r)\cos\Omega + v\cos\frac{(R - r)\Omega}{r},$$

where R is the radius of the fixed circle, r is the radius of the rolling circle, v is distance between the center of the rolling circle and a point on the path, Ω is the parameter (see Figure 13). If $v = r$, we get the hypocycloid, if $v > r$, we get the prolate hypocycloid and if $v < r$, we get the curtate hypocycloid.

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The paths of points, which are obtained by rolling the circle on the outside of the fixed circle, which is inside the rolling circle, are called *pericycloids*. These curves are same as epicycloids. The motion is called *pericycloidal*.

Dynamic natures of motion in Figures 12 and 13 is realized by changing the parameter Ω .

Elliptic and Cardioid Motion

The elliptic motion is given by paths τ_A and τ_B of two points A and B (end points of segment line), where τ_A and τ_B are straight lines. It can be proved that this motion can be defined by rolling the circle in the fixed circle. The radius of the fixed circle is double radius of the moving circle. That is the elliptic motion is special case of hypocycloidal motion. The paths of points which are obtained by this motion are *ellipses*, segment line or circle; see Figure 14.

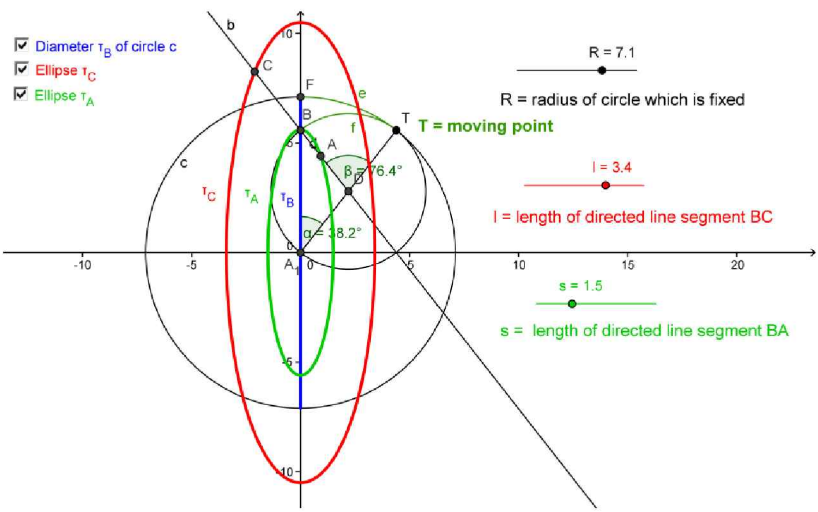

Figure 14: Elliptic motion.

We know the parametric equations of the hypocycloid, now let us consider $R = 2r$

$$x = (2r - r)\sin\Omega - v\sin\frac{(2r - r)\Omega}{r} = (r - v)\sin\Omega$$

$$y = (2r - r)\cos\Omega + v\cos\frac{(2r - r)\Omega}{r} = (r + v)\cos\Omega$$

We get the parametric equation of segment line ($r = v$), circle ($v = 0$), or ellipse (otherwise).

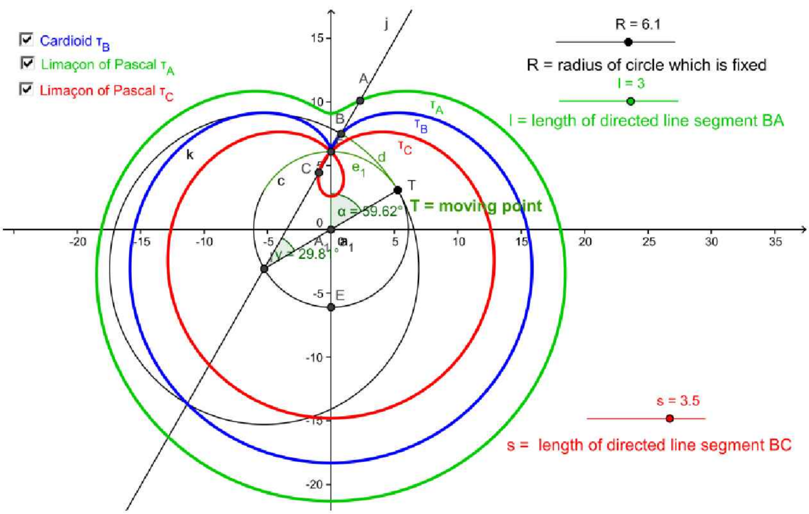

Figure 15: Cardioid motion.

The cardioid motion is the inverse motion to the elliptic motion. The paths of points, which are obtained by this motion, are *limaçon of Pascal* or circle. In special case, the limaçon of Pascal is *cardioid*. It can be proved that this motion can be defined by rolling the circle on the outside of the fixed circle, which is inside the rolling circle. That is the cardioid motion is special case of pericycloidal motion; see Figure 15.

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The paths of points, which are obtained by rolling the circle on the outside of the fixed circle, which is inside the rolling circle, are pericycloids. We know that these curves are same as epicycloids, so we do not show the parametric equations. More details can be found in Gutenmacher and Vasilyev (2004) or Carmo (1976).

Conchoid Motion

A *conchoid* is a curve derived from a fixed point O, another curve, and a length d. For every line through O that intersects the given curve at A, the two points on the line, which are at distance d from A are on the conchoid. We can get the branches of this curve by the conchoid motion which is given by path τ_A of point A and envelope (b) of straight line b, which degenerates into point. The envelope (b) is the fixed point O. See Figure 16, there are some examples of various paths.

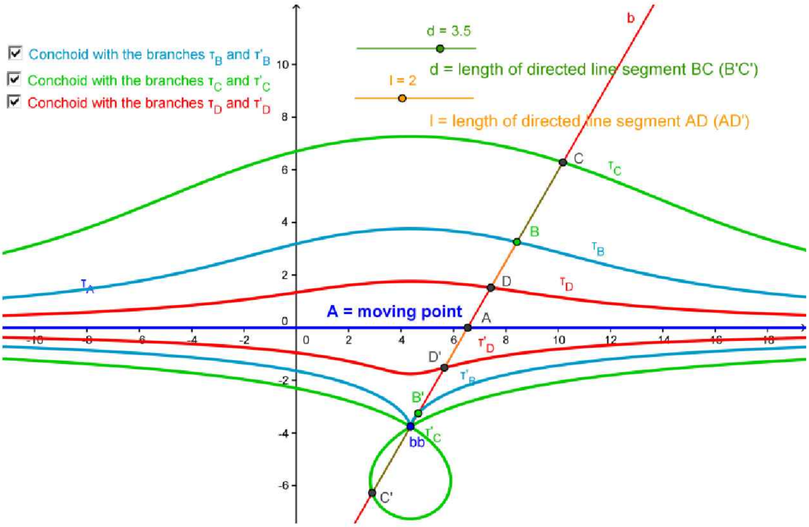

Figure 16: Conchoid motion where τ_A is a straight line.

In some publications, other terms for the paths and the special motions can be found. We work with the simplest notations.

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There exists a wide range of various special motions in the plane. We discussed only the most important and the best known. We can also study the properties of curves; we refer the reader to Abbena et al. (2006) and Pottman et al. (2007).

Constructions with GeoGebra

In this section, we will describe the details of solving two examples from the field of kinematic geometry by using GeoGebra. Firstly, we will focus on geometric motion determined by envelope and path, and secondly, on geometric motion determined by fixed and moving centrodes. We will precisely mention the tools and commands which we used.

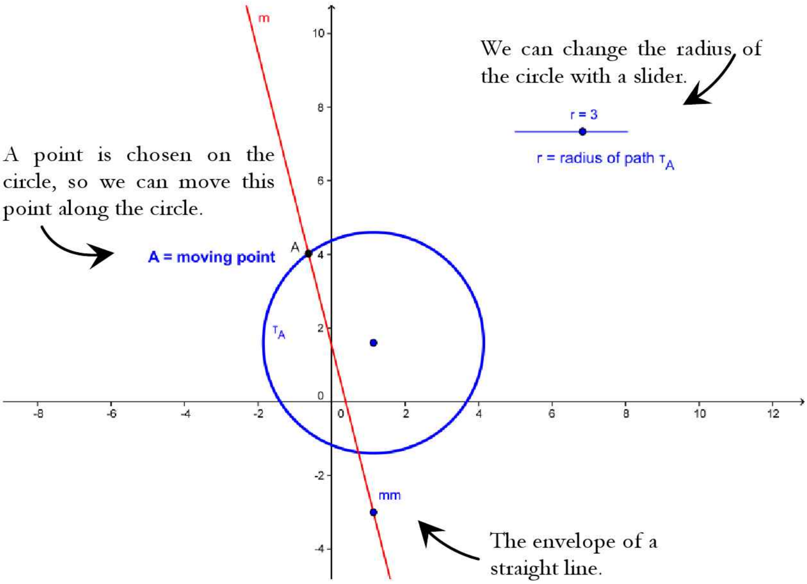

Figure 17: The determination of geometric motion. The motion is given by path τ_A *(circle) of point* A *and envelope* mm *of curve* m *(straight line) which degenerates into point.*

Example 1. The motion is determined by path τ_A of point A and envelope mm of curve m, which degenerates into point. The path τ_A of point A is a circle and the curve m is a straight line. This motion is special case of the conchoid motion; see Figure 17.

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Our task is to find the path of a given point $\mathbf{B}$, an envelope of a given straight line $\mathbf{n}$ and fixed and moving centrodes.

For the creation of the determination, we use these commands (in this order): *New Point* (the center of the circle τ_A), *Slider* (for radius r), *Circle with Center and Radius* r (the circle τ_A with center in a given point), *New Point* (point mm), *New Point* (point A on the circle τ_A), *Line through Two Points* (through points mm, A).

The simplest task is to find a path of a given point B. Point B is obtained as the intersection of straight line m and auxiliary circle with center in point A and radius l. Radius l is given by the slider. We use the command *Locus* (we choose point B and then point A) for finding the path τ_B; see Figure 18.

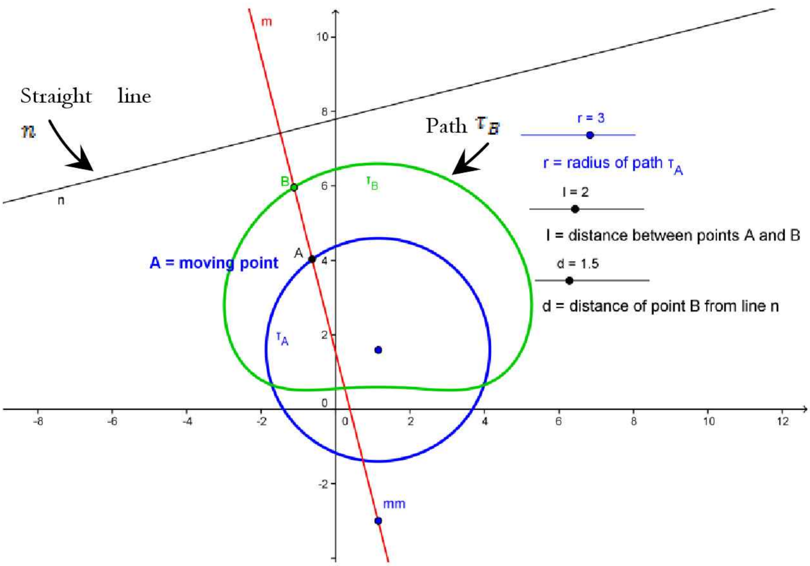

Figure 18: The path τ_B of a given point B is obtained by moving.

Straight line n is perpendicular to straight line m and the distance d of point B from straight line m is given by slider (the same construc-

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tion as the construction of point B with auxiliary circle with center in point B and radius d). For creation of a straight line n , we use the command *Perpendicular Line*; see Figure 18.

Let us construct fixed and moving centrodes. The fixed centrode is the locus of the instantaneous centers at every moment of the motion. In GeoGebra, we construct the instantaneous center S at one moment and then we use dynamic nature of the software to indicate the rest. The instantaneous center is the intersection of the normal line to the path τ_A at the point A (according to section Geometric Constructions of the Centrodes) and the perpendicular line to the straight line m through the point mm (the proof of this theorem can be found in Bottema & Roth, 1979).

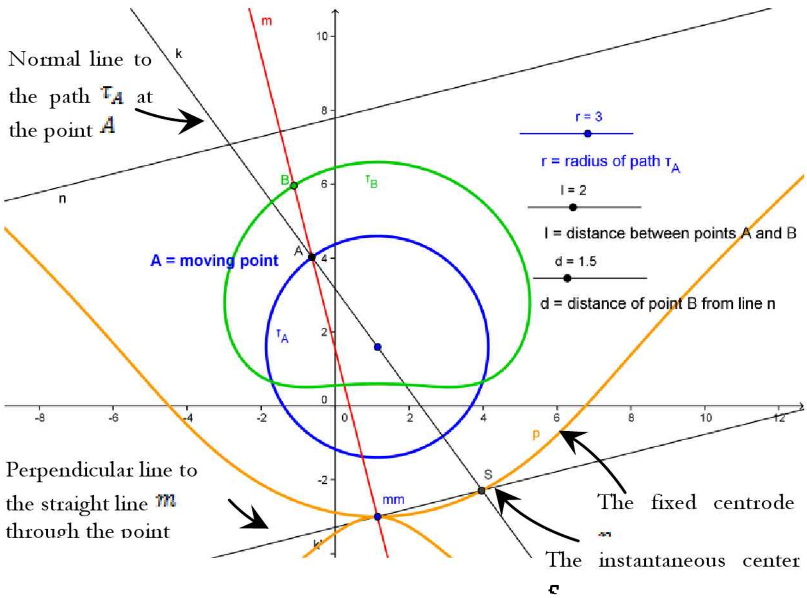

Figure 19: The fixed centrode p .

For the construction of the fixed centrode we use these commands (in this order): *Line through Two Points* (through points A and center of the circle τ_A), *Perpendicular Line* (to the straight line m through the point

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mm), *Intersect Two Objects* (point S is the intersection of two previous lines), *Locus* (we choose point S and then point A) for finding the fixed centrode p ; see Figure 19.

The moving centrode is the locus of the instantaneous centers at every moment of the inverse motion of a given motion. We construct the point of the moving centrode by using congruency of triangles as was mentioned in the section Geometric Constructions of the Centrodes.

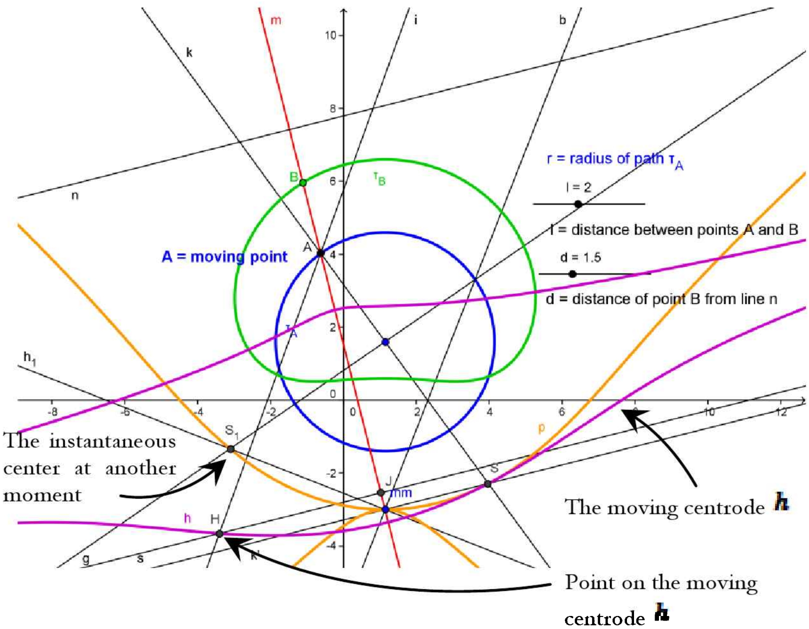

Figure 20: The moving centrode h .

For the construction of the moving centrode, we use these commands (in this order): *New Point* (point A' on the circle τ_A), *Line through Two Points* (through points A' and center of the circle τ_A), *Perpendicular Line* (to the straight line $A'mm$ through the point mm), *Intersect Two Objects* (point S_1 is the intersection of two previous lines). We move the triangle $\triangle A'S_1mm$ to the initial position with these commands: *Circle with Center and Radius* (the auxiliary circle with center in point A and radius $|A',mm|$), *Intersect Two Objects* (auxiliary point J), *Perpendicular*

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Line (to the straight line m through the point J), *Angle with Given Size* (for the construction of the triangle), *Intersect Two Objects* (point H as the vertex of the triangle), *Locus* (we choose point H and then point A') for finding the moving centrode p ; see Figure 20.

The final task is to find an envelope of a given straight line n . We construct perpendicular line to the straight line n through the point S (*Perpendicular Line*) and find the intersection N with the straight line n (*Intersect Two Objects*), then we use the command *Locus* (we choose point N and then point A) for finding the envelope nn ; see Figure 21.

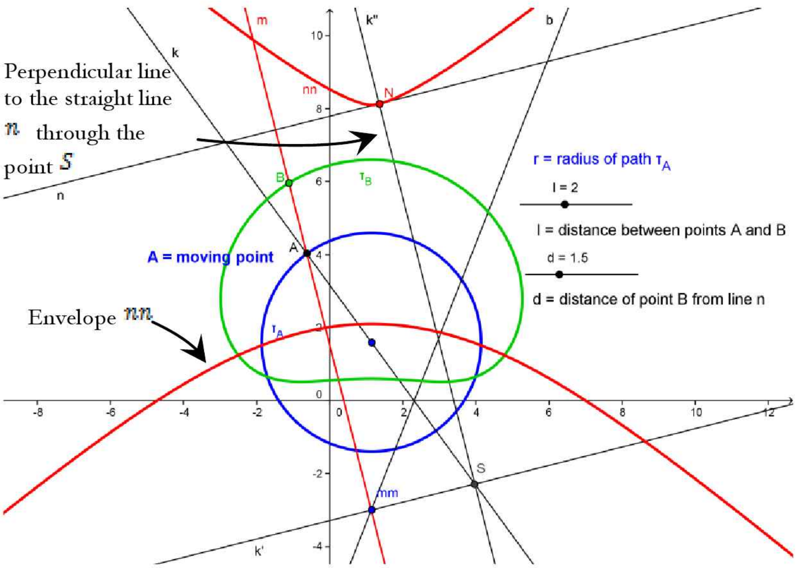

Figure 21: The envelope nn of a given straight line n .

Example 2. The motion is determined by the fixed centrode p and the moving centrode h , both curves are circles. The moving centrode h is rolling on the outside of the fixed centrode p . This motion is called epicycloidal; see Figure 22.

Our task is to determine this motion in GeoGebra.

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For the creation of the determination, we use these commands (in this order): *New Point* (the center of the circle p), *Slider* (for radius R), *Circle with Center and Radius* R (the circle p with center in a given point). We choose a point F on the circle p (for example the intersection the circle p and axis y).

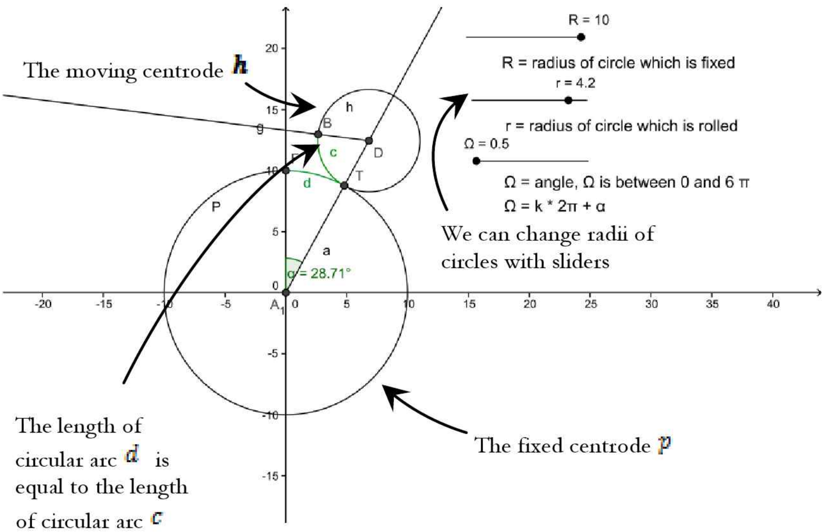

Figure 22: The determination of geometric motion. The motion is given by the fixed centrode p and the moving centrode h.

In GeoGebra, we can define the angle in radians within the interval of $\langle 0,2\pi)$ but in this special example, we have to move with point along the circle for more times and to compute the angle mostly greater than 2π. In our example, point T will be moveable. How can we do that? We define the parameter Ω - the angle between 0 and for example 6π (it means three times along the circle) and the parameter $\Omega' = 2\pi\left(\frac{\Omega}{2\pi} - floor\left(\frac{\Omega}{2\pi}\right)\right)$ (it means the number Ω' is the remainder after dividing the angle Ω by 2π). We construct point T with the help of the command *Rotate Object around Point by Angle* (we choose point F, around the center of the circle p and angle Ω'). We also

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create the circular arc d (with command *Circular Arc with Center between Two Points*) and now we can construct the moving centrode h (with auxiliary circle with center in point T and radius r). Let us construct point B on the moving centrode with the following equation $|\widehat{FT}| = |\widehat{TB}|$ at every moment. We compute the angle $|\sphericalangle TDB|$, which is equal to $\frac{\Omega R}{r}$. Same as before, we define the number $\omega' = 2\pi\left(\frac{\omega}{2\pi} - floor\left(\frac{\omega}{2\pi}\right)\right)$ and construct point B by command *Rotate Object around Point by Angle* (we choose point T , around the center of the circle h and angle ω'). (We have to be careful in choosing the direction of the rotation.) We can change the parameter Ω then point B describes its path.

We can construct the paths of various points, for example of point $\mathbf{B}$; see Figure 23. The paths of points which are obtained by rolling the circle on the outside of the fixed circle are called epicycloids; see the section Cyclical Motion.

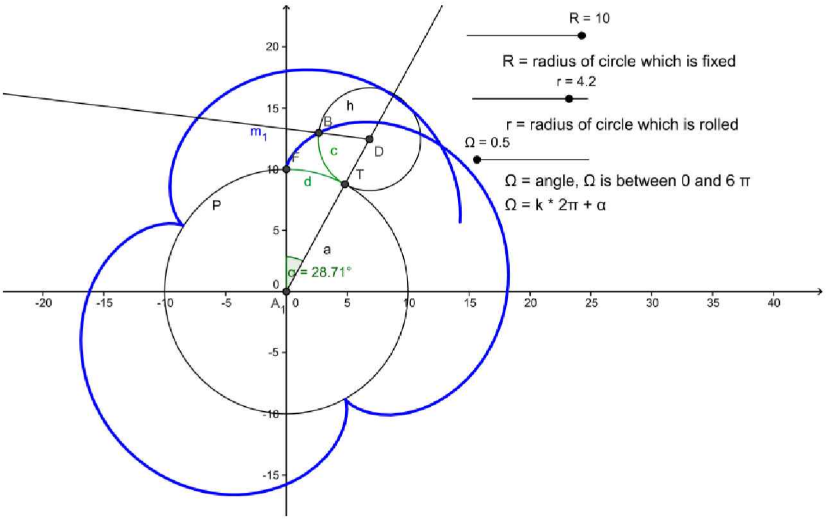

Figure 23: The path of point $\mathbf{B}$.

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Conclusion

In this chapter we discussed possible approaches on how to increase the interest of students in studying geometry at secondary schools and colleges. We demonstrated, using examples from the field of kinematic geometry, how GeoGebra can be used in the teaching process. We precisely described the tools and commands which we used.

We also used GeoGebra for creation of stepwise guides through geometric construction, which can help students understand the problem in intuitive and natural ways. Moreover, we show special constructions applied in descriptive geometry and due to included functions and tools students can discover proofs more easily. Of course, students also create some examples and tasks themselves.

We have web pages with a database of geometric tasks in GeoGebra and provide the access to this database to our students.

The feedbacks from students using GeoGebra in learning geometry are very positive. Students are satisfied because GeoGebra is very useful for them. Kinematic geometry in the plane is more understandable and geometry in general becomes modern discipline. The outputs from GeoGebra can also be used for e-learning not only for our students.

In future work, we will focus on further methods which can improve the teaching process. We also plan to extend our gallery of geometric tasks.

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Chapter 6

Supporting Students' Mathematical Thinking during Technology-Enhanced Investigations using DGS

Milan Sherman

Abstract

Research has found that the mathematical thinking that students do while engaging with instructional tasks has important consequences for student learning, but few studies have examined how technology influences students' general thinking processes apart from the learning of specific content and the role of the teacher in that process. Data were collected in three secondary mathematics classrooms for a unit of instruction via classroom observations, collection of student work, and post-lesson teacher interviews. The Mathematical Tasks Framework was used to evaluate the opportunities for mathematical thinking present in the mathematical tasks chosen by these teachers, and how those opportunities were realized or not during implementation. Teacher moves during implementation and how they contributed to students' thinking were coded and analyzed qualitatively in order to understand what teacher practices were effective or ineffective in supporting high level thinking during a technology based investigation. Results show that these teachers varied in their support of students' high level thinking during technology-based explorations, and various factors accounted for these differences. Two teachers seemed to have unrealistic expectations about what their students would be able to do when provided with technological tools, especially as it relates to the thinking demands of the task. A third teacher was more successful in supporting students' engagement with tasks at a high level. Specific examples of these practices are described and analyzed qualitatively to demonstrate their connection to students' thinking during technology-enhanced investigations.

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Introduction

The Common Core State Standards (2010), adopted by an overwhelming majority of states in the U.S., articulate eight standards of mathematical practice, which have garnered much attention in the mathematics education community. These standards include mathematical behaviors such as (a) looking for and making use of structure of the problem; (b) looking for and expressing regularity in repeated reasoning; (c) making sense of problems and persevering in solving them; and (d) reasoning abstractly and quantitatively. These standards echo the Process Standards defined by the National Council of Teachers of Mathematics (2000), which include problem solving, proof and reasoning, communication, representation, and making connections. These standards are unique in that they address students' mathematical thinking and behavior in a way that transcends and cuts across specific content areas. Research has found that the mathematical thinking that students do while engaging with instructional tasks has important consequences for student learning (Boaler, 1998; Boaler & Staples, 2008; Henningsen & Stein, 1997; Hiebert & Wearne, 1993; Stein & Lane, 1996; Stein, Grover, & Henningsen, 1996). With the advent of the Common Core State Standards, and the forthcoming associated assessments, there is hope among many mathematics educators that mathematical processes, practices, and behaviors will receive the attention in K-12 classrooms that they deserve.

Concurrently, there has been a proliferation of interest in and research on the use of technology in mathematics education (e.g., Heid & Blume, 2008; Zbiek, Heid, Blume, & Dick, 2007). This is reflected in the inclusion of the Technology Principle in NCTM's Principles and Standards (2000), as well as the bold and controversial assertion that technology is "essential to the teaching and learning of mathematics" (NCTM, 2000, p. 24). In addition, another of the Common Core standards for mathematical practice is the ability to "use appropriate tools strategically," including digital technologies. The increased attention on students' thinking and the use of technological tools in K-12 mathematics education raises the question of how the use of technology may influence or support students' general thinking processes and behaviors apart from the learning of specific content. However, little research to date has addressed this question. This chapter reports the results of a study designed to investigate the role of dynamic geometry software

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(DGS) in supporting students' mathematical thinking while engaging with classroom instructional tasks, with particular attention paid to the ways that teachers shape those opportunities during implementation.

Theoretical Framework

The Mathematical Tasks Framework

The present study investigates the question: "How do teachers shape students' opportunities for mathematical thinking while using DGS during classroom instructional tasks?" The Mathematical Tasks Framework (Stein & Smith, 1998; Stein, Smith, Henningsen, & Silver, 2009) is used to assess the type of thinking that students do. A mathematical task is defined as "a classroom activity, the purpose of which is to focus students' attention on a particular concept, idea, or skill" (Stein et al., 1996, p. 460). The Mathematical Tasks Framework includes two essential elements: the cognitive demand of a given task, and how that demand may change during classroom implementation. Cognitive demand refers to the type of thinking that is required for successful completion of a task, and is categorized as high or low. The Mathematical Tasks Framework contains two categories of tasks which are considered low level: (a) memorization tasks, which involve the learning or recalling of facts, formulas, or definitions; and (b) procedures without connections to concepts or meaning tasks, in which students apply a known procedure to produce answers without attention to why the procedures work, where they come from, or what they mean mathematically. Likewise, high level tasks are classified as procedures with connections to concepts or meaning or doing mathematics. A procedures-with-connections task is one in which students develop meaning for procedures, or use procedures for the purpose of developing deeper levels of understanding of mathematical concepts, and often include using multiple representations in order to develop meaning. Mathematics tasks generally involve problem solving and are characterized by their open-ended and non-algorithmic nature.

An important contribution of the Mathematical Tasks Framework is the identification of three phases of implementation of a mathematical task: the task as it appears in curricular materials, as set up by the teacher, and as implemented by students. These distinctions with regard to the enactment of a task are important because the cognitive demand of a task may change as it passes through each of the phases of implemen-

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tation, and ultimately these changes influence what students learn (Stein & Lane, 1996; Stein et al., 1996) (see Figure 1).

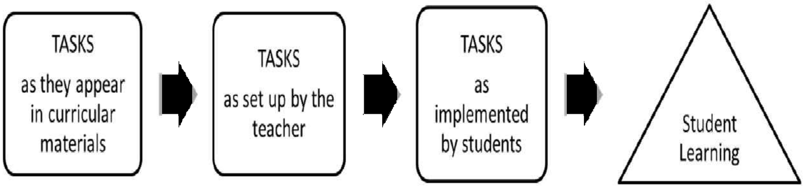

Figure 1. Visualization of the Mathematical Tasks Framework, as given in Stein & Smith (1998).

Studies have shown that the cognitive demand of classroom mathematical tasks virtually never increases from one phase to the next (Henningsen & Stein, 1997; Stein & Lane, 1996; Stein et al., 1996). Indeed, maintaining the demand of a task set up at a high level during implementation is a difficult endeavor for teachers. However, the cognitive demand of a task during the implementation by students is the most decisive in terms of the quality and depth of students' learning. Stein and Lane (1996) showed that students in classrooms where the cognitive demand was high during implementation significantly outperformed students who were not strong in problem solving, use of representations, and ability to explain solution strategies.

This research has also demonstrated that certain factors are generally present when the cognitive demand of a task set up at a high level declines or is maintained during the implementation phase (Henningsen & Stein, 1997; Stein et al., 2009). Factors that were associated with the decline or maintenance of a task during implementation are included in Appendix A. Research has shown that not all of these factors are necessary for the decline or maintenance of a task set up at a high level during implementation. Individual factors associated with maintenance may be necessary but not sufficient for the task to stay at a high level during implementation. An important question for the study described here is whether and how these factors may be related to the use of dynamic geometric software while engaging with tasks set up at a high cognitive level.

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Dynamic Geometry Software

The use of dynamic geometry software (DGS) has been found to support students' problem solving by providing feedback and allowing for successive refinements of students' solutions: "The capability of the software to incorporate knowledge and to react in a way consistent with theory impacts the student trajectory in the solving process" (Hollebrands, Laborde, & StraBer, 2008, p. 174). In the studies reviewed by Hollebrands and her colleagues, the use of DGS supported students in using multiple, linked representations in problem solving, including geometric diagrams, graphical representations, and symbolic notation. The link between representations that is automated by DGS was found to help students construct meaning for mathematical concepts, such as the idea of curvature (Hollebrands et al., 2008). Dynamic geometry software allows for students to make and test conjectures in ways that would be impossible without it, and a number of studies have demonstrated the diverse and novel ways in which the use of DGS can support students' development of deductive reasoning and proof (Kondratieva, 2012; Mariotti, 2000; Marrades & Gutierrez, 2000; Sanchez & Sacristán, 2003).

However, Hollebrands et al. (2008) warn that learning "results from the conjunction of the use of a DGS, of a careful design of the teaching/learning situation and of the tasks, of the social organization, and of the role of teacher" (p. 186). For example, Glass and Deckert (2001) note that Galindos (1997) found that students might be too willing to accept multiple examples in the form of "dragging" as proof within a dynamic geometry environment. However, Glass and Deckert (2001) hypothesize that this may be due to students working on close-ended rather than open-ended tasks: "[S]tudents may view conjectures from close-ended tasks or from given statements as automatically true and therefore may not see a need for formal reasoning" (p. 228). They claim that having students work on open-ended tasks results in the formulation of "shaky conjectures," the truth of which are in question, thus motivating the need for deductive reasoning and proof. This is an example of how the role of the teacher in designing and enacting the mathematical task can have a direct impact on the type of thinking required by the task as implemented in a technology enhanced environment.

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Method

Selection and Description of Teachers

Teachers were recruited for the study primarily based on their belief in and consistent use of technology for instruction. There was no attempt to manipulate instruction or measure an intervention. Teachers were asked to identify a unit of instruction that they had previously taught and that included the use of technology. As the focus of this article is the role of the teacher in supporting students' engagement with technology-enhanced tasks, the primary criterion for the selection of tasks for this analysis was that students engaged in student-centered tasks in which they directly manipulated the technology while working on the task. Tasks of this nature were enacted in three of the four teachers' classrooms who participated in the larger study, and thus the tasks selected for the present analysis are taken from the classrooms of those three teachers. These tasks were similar as students used DGS while investigating geometric objects and their properties. A complete list and brief description of these tasks set up at a high level using DGS is provided in Appendix B.

Ms. Jones is a third year teacher of ninth-grade integrated mathematics at an urban charter high school. The observed class consisted of 28 ninth-grade students, with a support teacher (referred to as a "paraprofessional") who was present for seven of the twelve observed tasks. Her class met for 65 minutes each day, and students each had a school-issued laptop for their individual use. The primary instructional technology that students used in this class, besides calculators, was The Geometer's Sketchpad (GSP), a dynamic geometry software program published by Key Curriculum Press.

Ms. Young is a third year teacher at a suburban high school. The observed class was an 11th grade regular inclusion geometry class of ten students (three girls and seven boys) with a special education support teacher who was present for five of the nine observations. Six students in the class had an IEP, although not all of the IEPs were academic in nature. Her school was on an alternating block schedule, meeting for 80 minutes every other day. Each student had a school issued laptop for their individual use, and used both GSP and GeoGebra.

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Ms. Lowe is a third year teacher at a small Catholic high school. The class observed was one of her sections of honors geometry because, as she put it, "my chapter 5 in my honors class has such a nice mix of technology/non-technology activities." The class consisted of 16 students (six boys and ten girls) and met for 39 minutes each day. Ms. Lowe generally took the students to a computer lab and used the dynamic geometry software program GeoGebra on desktop PCs.

These students' experience using DGS was very similar across the three classes. Ms. Jones' and Ms. Lowe's students had been introduced to these programs earlier in the academic year, but had little experience with the use of these programs. Ms. Young's class was observed toward the beginning of the academic year, and her students had no experience with these programs.

Data Collection

In order to investigate how teachers support students' engagement with high level tasks which incorporate the use of technology, data were collected for one unit of instruction, generally three to five weeks. Data collection included classroom observation fieldnotes, task artifacts, collections of student work, and post-lesson teacher interviews. The purpose of classroom observations was to determine the cognitive demand of a given task during the set up and implementation phases of enactment by documenting how the task was introduced by the teacher, what students did while working on the task, interactions between the teacher and students, including questions posed by the teacher to individual students and questions asked by students of the teacher and how he or she responded, and interactions between students. In addition, students' interactions with the technological tools employed during the task, if any, were noted, including visible actions taken by students, and discussions with the teacher or fellow students while using the technology. Student work was collected, including computer files, as it provided insight into the type of thinking students were engaged in while working on the task, and post-lesson teacher interviews provided the teachers' perspective on the thinking that students engaged in while working on the task.

Data Coding and Reliability

Each task was coded as requiring a low or high level of cognitive demand at each of the three phases of implementation: (a) curricular ma-

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terials; (b) set up by the teacher; and (c) implemented by students. In addition, each task set up at a high level was coded with respect to the factors associated with maintenance or decline, depending on whether the task was deemed to have been implemented at a high or low level.

To ensure the validity and reliability of the fieldnotes in capturing those dimensions of classroom instruction and students' thinking necessary to make evaluations of cognitive demand during set up and implementation, and factors associated with maintenance or decline, and how technology was used during set up and implementation, two reliability coders were employed. At three of the four data collection sites, a coder accompanied the researcher to lesson observations, and both the observer and researcher coded these dimensions directly from the observation prior to the generation of fieldnotes using the Task Analysis Guide (Appendix C). After fieldnotes were constructed, a second coder coded the task from the fieldnotes. The roles of these coders were exchanged on a regular basis, with each coding approximately the same number of tasks from observations and from fieldnotes.

The Task Analysis Guide (Stein et al., 2009) was used to code the cognitive demand as memorization, procedures without connections, procedures with connections, or doing mathematics, but for the purposes of the present article, only the distinction between low and high level tasks is used. Agreement for categorizing tasks as high or low was 98% with the observer, and 83% with the fieldnote coder, with all discrepancies resolved with both reliability coders.

Each task coded for reliability that was set up by the teacher at a high level of cognitive demand was coded for reliability using the list of classroom-based factors associated with the decline or maintenance of cognitive demand during implementation given in Appendix A. Another factor associated with decline was identified during coding, which is referred to as "lack of attention to students' thinking." Agreement on coding the factors associated with maintenance and decline was 80% with the lesson observer and 78% with the fieldnote coder, and all discrepancies were resolved with both coders.

Data Analysis

The present analysis focuses on tasks which were set up at a high level by the teacher. Moves made by the teacher during implementation and how they contributed to students' thinking were analyzed qualitatively

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in order to understand what teacher practices were effective or ineffective in supporting high level thinking during a technology-based investigation, with the results of the coding of the factors associated with maintenance or decline providing the sample and division of the tasks for this analysis. NVivo qualitative analysis software was used to examine the implementation of these tasks in detail and to annotate the fieldnotes with regard to the specific pedagogical moves which contributed to the maintenance or decline of these tasks. Using the constant comparative method (Glaser, 1965), similarities and differences in how these tasks were implemented, and the role of the teacher in supporting students' mathematical thinking were identified. The explication of the results is meant to shed insight into the connection between the decisions and actions of the teacher, and the thinking, behaviors, and engagement of students with the task.

Results

Overall, the results show that these teachers varied in their support of students' high level thinking during technology-based explorations and that various factors accounted for these differences. The description of the results begins with a discussion of the factors associated with the decline and maintenance of tasks set up at a high level in each of these three classrooms and is followed by a more fine-grained description of how these factors were manifested during instruction. What they show is that some factors associated with maintenance are necessary regardless of whether or not technology is used, while others are specifically related to supporting students' thinking while using technological tools.

Factors Associated with Decline

The present analysis focuses on a sample of 18 tasks that were set up at a high level using DGS by these three teachers. Ms. Lowe was the only one to implement any task at a high level, having maintained the cognitive demand of five of the nine tasks that she set up at a high level with DGS. All four tasks that Ms. Jones set up at a high level and all five tasks set up at a high level by Ms. Young, declined during implementation.

The factors associated with the decline of the cognitive demand during implementation (Stein et al., 2009) of these tasks by students are depicted in Figure 2. It shows the percent of tasks that declined when the given factor was present and are ordered from bottom to top by preva-

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lence across sites. The most prevalent factor across all sites was the teacher taking over the high level thinking demands for the students. For example, this included asking leading questions or giving students direction or hints, which eliminated the problematic aspects of the task. The second most common factor was the inappropriateness of the task for a given group of students, which took many forms in this data, but primarily referred to a lack of prior knowledge of the content, mathematical behaviors, or use of technology required by the task. Shifting the emphasis of the task from the high level thinking involved to finding the correct answer or not providing enough time for students to grapple with the demanding aspects of the task were two more factors that each of the teachers exemplified during implementation.

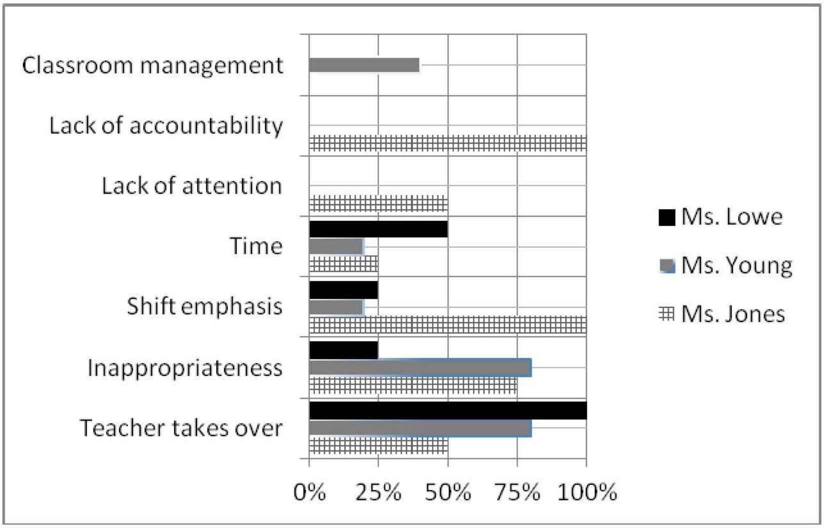

Figure 2. Percent of tasks for which a given factor associated with decline was present.

Factors Associated with Maintenance

Ms. Lowe was the only teacher to have any success with maintaining the cognitive demand at a high level. The factors associated with the maintenance of the high level thinking demands during implementation of these tasks are summarized in Figure 3. These results show that building on students' prior knowledge and providing a sufficient amount of time to engage with the task were present in all five of these tasks, and that scaffolding students' engagement with the task and con-

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sistently pressing students to justify their thinking and make meaning of their work were present in four of the tasks. A salient feature of Figure 3 is that a large number of these factors were present in many of these five tasks. Indeed, three tasks had four factors present, and two tasks had five factors present.

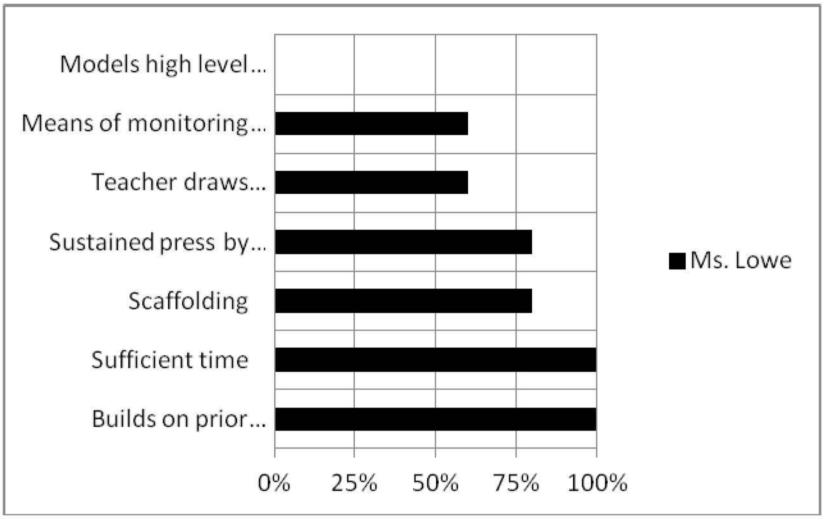

Figure 3. Percent of tasks for which a given factor associated with maintenance was present.

This summary of the factors present during implementation of tasks set up at a high level provides some insight into the way that teachers may undermine or reinforce opportunities for high level thinking and reasoning by their students. Furthermore, these results confirm the findings of previous research, which has shown that teachers have difficulty implementing tasks at a high level, and that many factors work in concert to create an environment which supports students' high level thinking while engaging with mathematical tasks (Stein et al., 1996; Henningsen & Stein, 1997). Taken together, these results confirm and underscore the complexity of maintaining the cognitive demand of tasks set up at a high level during classroom instruction. However, what they do not demonstrate is precisely how these factors were manifested in specific tasks, or how the decline or maintenance of tasks set up at a high level was related to the use of technology. A more fine-grained description of how the interactions between students, teacher, and technological tools contributed to the decline or maintenance during the enactment of a task set up at high level follows below. The purpose

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of this analysis is to better understand the role of the teacher in shaping these opportunities for students while using technology.

Technology-Related Factors

The results shown in Figures 2-3 describe the factors associated with the maintenance or decline of the high cognitive demand of tasks that are consistent with previous research (Stein et al., 1996; Henningsen & Stein, 1997). However, these results do not elaborate the role of technology in these instances. In some cases, a factor had little to do with the fact that DGS was used during the task. For example, providing students with enough time to grapple with the demanding aspects of the task seems to be a necessary but not necessarily sufficient condition for high level engagement, whether technology is used or not. However, the way that other factors were manifested during implementation was directly related to the use of DGS on these tasks. A few cases that are characteristic of the way that these factors were manifested in each of these teachers' classrooms are described below.

Technology-related factors involved in decline of task cognitive demand

The decline of the cognitive demand of tasks set up at high level during implementation was related to the use of DGS for both Ms. Jones and Ms. Young. In both cases these issues had been coded as the "inappropriateness of the task for a given group of students," as students struggled to connect the affordances of the tools provided by the DGS to the requirements of the task.

For example, in Ms. Jones class, students used GSP in order to investigate the properties of angles formed by parallel lines cut by a transversal. The idea of the task was that students would use GSP to construct a dynamic figure of parallel lines cut by a transversal, including the measures of all of the angles formed, and drag the figure dynamically in order to investigate how the various angles are related, i.e., which are congruent and which are supplementary. The task requires that students look for patterns, make generalizations and conjectures, and strategically drag their figures in order to test their conjectures. The thinking required by dragging has been described in detail elsewhere (see Hollebrands et al., 2008).

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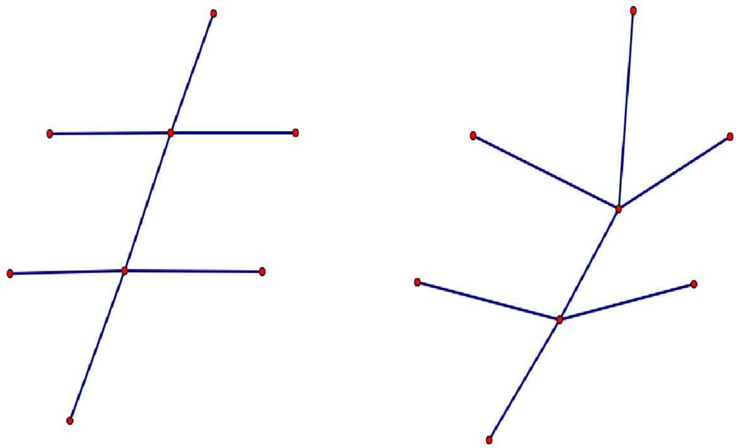

Figure 4. Ms. Jones' students' constructed what appeared to be parallel lines cut by a transversal, but which deformed when dragged.

A number of Ms. Jones' students created figures that appeared to be parallel lines cut by a transversal, but when the students dragged them they deformed to reveal that the figure was merely a collection of line segments made to appear like the intended figure (see Figure 4). Based on Ms. Jones' verbal instructions, and the directions provided in her handout, students seemed to understand what parallel lines looked like. Indeed, after dragging their figures and observing them to deform, students invariably "fixed" their figure to make it appear that the lines were parallel again. However, when students attempted to make observations and find patterns, their constructions did not provide a resource for doing so as none of the angles were congruent (although they were close due to the figure's approximation of parallel lines).

Ultimately, the issue in this example is that the students lacked the necessary prior knowledge, both of the mathematics and of the features of the technology. That is, these students did not understand what "parallel" means in a dynamic geometry environment, in the sense that, when parallel lines are constructed in a dynamic geometry environment, the parallel quality of lines will always be maintained; moving one line will result in the line parallel to it automatically mirroring the same movement in order to maintain the "parallel-ness" of the two lines. These students seem to consider "parallel" to be a contingent rather than necessary property of the lines displayed on their screen. Thus, lines are

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parallel when they look parallel. In GSP, there is a definite difference between lines constructed to be parallel and lines that are made to look parallel. Students' inability to understand those differences, and how they are represented in GSP, prevented them from encountering the mathematics that was the goal of the task or the thinking that was intended by the task. Furthermore, these students were unable to verify that the two lines are indeed parallel, either because they did not understand how to verify this property mathematically, or they did not know how to use the tools in GSP to do this. In this case, it seems that the use of DGS by her students was assumed to be unproblematic, and Ms. Jones did not anticipate the problems that her students experienced or ways to support their engagement with the task. Indeed, Ms. Jones was unaware that at least a quarter of her class had constructed the figure in this way. However, even when she was aware of her students' issues with using DGS, she failed to support their high level engagement using GSP, as evidenced by her interactions with students on another task, which involved investigating and discovering the Triangle Inequality Theorem.

Students were given a worksheet with directions for constructing a triangle in GSP and prompted to manipulate the triangle to determine whether it was possible to create a triangle in which the sum of two sides was the same or smaller than the third, and to make a conjecture about the relationship between the side lengths of any triangle. A number of students recorded on their worksheet that it was possible. Rather than inquiring how students had come to this conclusion, which would presumably have involved students manipulating their triangle while Ms. Jones observed, she asked these students to try to make specific triangles. She asked them to create a triangle with side lengths of two, three, and five if students reported that it was possible to make a triangle with the sum of two side lengths equal to the third, and a triangle with side lengths two, three, and ten if they had said it was possible to create a triangle with the sum of two side lengths less than the third. This modification of the task essentially changed the thinking required and the activity and behaviors students engaged in. The modified task did not make use of the dynamic affordances of GSP in any meaningful way and could just as easily have been accomplished using paper, a ruler, and a pencil.

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Indeed, given that some students did create triangles with side lengths of two, three, and five in GSP due to rounding error, the modified task may have been better accomplished without GSP. More importantly, the aspect of the task, which required students to manipulate and investigate, to look for patterns and make generalizations, and ultimately to make and test a conjecture that would apply to all triangles, was lost. Students' focus was now on attempting to create a specific triangle. In this case, Ms. Jones seemed to anticipate this response on the part of students, as she roamed the class and immediately gave students the modified task whenever she saw that one of them had reported that it was possible to create a triangle with the sum of two side lengths less than or equal to the third. It seems, however, that she did not anticipate how the modified task essentially changed the thinking demands of the task. Furthermore, it is impossible to know what these students had done with GSP that would have led to their erroneous conclusions, as Ms. Jones never investigated this with students. Indeed, one student drew a triangle on her paper with side lengths labeled six, six, and twelve in order to show that it was possible, prompting one to wonder if she had even attempted to use GSP for this task. In either case, Ms. Jones' students' difficulties seemed to be with using DGS as a tool to investigate properties of figures, and in these examples she seemed unable to support them in this endeavor.

Ms. Young's students' issues seemed to be less with using the technology and more with understanding the requirements of the task within which it was to be utilized. Ms. Young also had her students engage in a task investigating the angles formed by parallel lines cut by a transversal, but avoided the issues that Ms. Jones' students experienced in constructing their figures by having her students work with an applet of the figure already constructed and published on the Web. Thus, the figure was constructed correctly and could be dynamically manipulated to explore, make observations, and make and test conjectures about the relationships between the angles formed by parallel lines cut by a transversal. However, her students struggled to understand what it was that they were supposed to be doing, or how the applet might help them. Numerous students asked Ms. Young what she meant by "observation" or "conjecture," and many did not use the dynamic features of the software to make general observations. Ms. Young stated in our post-lesson interview:

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> [T]hey don't understand the word 'observations,' and neither did my Honors kids... They don't know what to write for observations. They're like, "they're both blue," (angles with special relationships had been color-coded in the applet to scaffold students' observations) or "one is blue and one is green"…"they are both 131 degrees." Which, they were on the right track, but that doesn't help when you move A, and now that angle is 107, so now you're observation is not right. (Interview, 9/27/10)

The issues that students seemed to struggle with had less to do with how to use the technology and more to do with requirements of the task, especially given that there is nothing to construct and Ms. Young demonstrates the dynamic and interactive features of the applet before students begin the task. Rather, how the affordances of the technology could be used to meet those requirements and the type of behavior that students were to engage in by investigating and exploring seemed to be the issues for her students.

Her students' lack of mathematically meaningful observations prevented them from using the technology to engage in high level mathematical thinking. In another post-lesson interview, Ms. Young says of her students:

> They're not the best observation-makers. I think they don't know what's important…We haven't done much of this observing, theorem-ing, and stuff in other classes… So it's kind of a new idea, we don't do much of that in algebra, or at least I don't. 'Look at this picture, and what do you notice?' So, it's different for them. So hopefully we'll get better at making observations, or I'll get better at…the questions I ask. But I feel that these ones are so…not basic, but, there isn't much I can say without telling them the answer. There's not a lot of leading that I can do. (Interview, 10/1/10)

Ms. Young is clear about the fact that she is asking students to do things with technology that they had very little experience with and that she struggles with supporting students' activity at a high level. She sees the task as fairly straightforward, making it difficult to support students' activity without telling them exactly what to do, and thereby taking over the cognitively demanding aspects of the task.

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Ms. Young's students experienced the same types of issues when engaging with the Triangle Inequality Theorem task in GSP as had Ms. Jones' students. Ms. Young reported that students were able to construct and measure their triangles correctly, but that they had "questions about the questions" that they were asked in the worksheet regarding making observations and conjectures. Students' inability to understand the generality of the observations they needed to make in order to make and test conjectures seemed to prevent them from using the affordances of the dynamic environment that would support that type of thinking.

The observations of these tasks suggest that teachers may have unrealistic expectations regarding what students will be able to do using technology. Understanding how mathematics is built into technological tools is a necessary condition for students to use them to engage in productive mathematical investigations that require high level thinking. Likewise, if students have never been asked to make a conjecture before, providing them with technological tools will not necessarily result in their ability to do so. While the use of technology can support students' ability to make conjectures by providing numerous examples to analyze as the basis for a conjecture and strategically manipulating objects in order to test a conjecture, it does nothing to support students' understanding of the importance of examining a variety of examples, what is mathematically meaningful to look for across those examples, how to make a mathematically precise statement as a conjecture, the importance of testing a conjecture or looking for counterexamples, or the difference between a conjecture and a proof.

These examples provide insight into the complexity of having students explore the properties of mathematical objects and the thinking implied by such an investigation. They also underscore the need for support by the teacher to scaffold students' use of DGS for engaging in the mathematical thinking and behaviors called for by these tasks. While it is clear from these examples that the students needed more support, it is unclear from these examples alone what that support may consist of, as Ms. Young admits. Ms. Lowe's classroom provides some insight into the answer to this question, as she was able to support students' high level engagement during implementation.

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Technology-related factors involved in maintaining task cognitive demand

The strategies that Ms. Lowe employed in maintaining students' high level thinking during implementation were generally related to how she supported students' use of DGS during the task. Analysis of the five tasks that were maintained at a high level revealed that several practices were particularly crucial in the maintenance of the cognitive demand during implementation of these tasks. Those practices that supported students' high level engagement with the task to students' use of DGS included carefully monitoring students' work on the computer, and maintaining a sustained press for meaning and explanation.

Carefully monitoring students' work

Carefully monitoring students' work on the computer was a crucial and a consistent factor in Ms. Lowe's enactment of these tasks. An example of this practice occurred during a task using GeoGebra to explore the circumcenter of a triangle. Ms. Lowe wanted students to notice that the three perpendicular bisectors of a triangle (lines that intersect each side of a triangle at right angles and divide the side of the triangle they intersect into two equal segments) intersect at one point, called the circumcenter, and to discover that the circumcenter is equidistant from the three vertices of the triangle, as in Figure 5. One way to do this in GeoGebra is simply to measure the distance from the circumcenter to each vertex. Instead, Ms. Lowe had students construct a circle with the circumcenter as its center and passing through one of the vertices. Since the circumcenter is equidistant from the three vertices, this circle passes through the other two vertices as well. Students must reason that, because the circle passes through the vertices, and the circumcenter is the center of the circle, then the distance from the circumcenter to each vertex is the radius of the circle, and therefore the distance to each vertex is the same. Furthermore, this relationship holds true no matter how the triangle is changed or deformed, and thus the circumcenter is equidistant from the vertices of any triangle.

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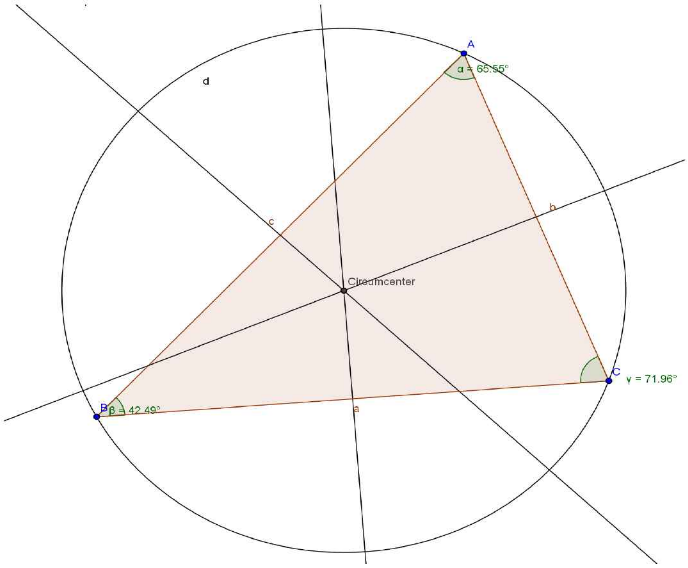

Figure 5. Representation of a circle with the circumcenter of a triangle as the center in GeoGebra.

Sustained press for meaning and explanation

Another important factor in the maintenance of high level tasks in Ms. Lowe's classroom was her insistence that students interpret their observations; making an observation was not enough to satisfy the task requirements. Ms. Lowe does not assume that as long as students have mathematically accurate and correct constructions, then the mathematical meaning or importance of that construction will be obvious. In particular, she engaged in the following practices while students used GeoGebra:

- she asks questions that require students to think about the mathematical meaning and conceptual connections embedded in the task
- she turns students' questions back to them and their construction
- she allows students to grapple with cognitively demanding aspects of the task

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- she requires student to use technology as a means of monitoring their own progress on the task.

An example of how Ms. Lowe requires students to interpret their work mathematically is taken from the circumcenter task. During the post-lesson interview she said that students struggled to understand the implication of a circle with the circumcenter as the center and passing through the vertices of the triangle, i.e., that the circumcenter is equidistant from the vertices. Below are two excerpts from the fieldnotes from that task that demonstrate how Ms. Lowe presses students to make this connection:

> Talking with another student, the student tells Ms. Lowe that all three perpendicular bisectors intersect at a point which is the center of a circle. She tells the student to think about what that means, and to think about the parts of a circle.
>
> Ms. Lowe: move the triangle and show me what you're seeing. (student moves her triangle) What's it doing?
>
> Student: it stays on it.
>
> Ms. Lowe: what does that mean? What is the relationship between the circumcenter and the vertices?
>
> Student: it keeps equal distance.
>
> Ms. Lowe: what is? What is the equal distance from the center to the points?
>
> Student: the radius.
>
> Ms. Lowe: the radius is what?
>
> Student: the same.
>
> Ms. Lowe: so what does that mean?
>
> Student: that the distances are congruent. (Fieldnote, 1/27/11)

These excerpts demonstrate that Ms. Lowe requires students to interpret their observations mathematically. In the excerpt above, Ms. Lowe asks students, "what does that mean?" in response to students' observations. Ms. Lowe does not simply have students make observations, but presses them to interpret those observations mathematically and to make connections to prior knowledge. This kind of questioning is im-

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portant to the maintenance of the high level demand of the tasks and builds on the monitoring that she has done. This practice is connected to carefully monitoring students' work, as ensuring that students' constructions are accurate puts them into a position to make observations that are mathematically meaningful.

Another practice that Ms. Lowe used to press students for meaning and justification is to turn students' questions back to them. For example, while working on the midsegment triangle task using GeoGebra, Neil has made some observations but has not discovered all of the properties of midsegment triangles that Ms. Lowe had intended. She pushes him to do more:

> Ms. Lowe looks at Neil's paper and says, "there's a little more." She tells him that he labeled his triangle differently than hers, and she wants to make sure that he's seeing the things that she wants him to see.... She tells him, "look at this," referring to DF and AC, which she reminds him don't change when he "bounces" point B. Neil asks, "is that half of the whole?" and she replies, "I don't know. Is it? If you bounce A, what changes?," and Neil replies, "FE?" Ms. Lowe tells him to try it, to move A, and asks what doesn't change... She tells him to look at the measures, and Neil says, "oh, ½!" ...Neil asks, "what does that mean?" and Ms. Lowe replies, "I don't know, what does it mean?" and asks him about the other pairs. Neil says, "this is also ½ of this, and this is ½ of this, and this is ½ of this," referring to the segments and midsegments. (Fieldnote, 2/16/11)

Ms. Lowe scaffolds Neil's observations by helping him to know where to look, but she refuses to confirm them. Rather, she refers him to his construction. In this way she keeps the onus on him to make and confirm observations and conjectures, which is considered to be part of the high level aspect of this task. Thus, when answering a student's question would lower the cognitive demand, one strategy Ms. Lowe uses is to pose the question back to the student. While this may be an effective strategy for maintaining the cognitive demand in general, it is important to note the way that she leverages the technological tool in these cases by referring students back to their construction in order to investigate their own question. By reflecting students' questions back to them she is encouraging them to use the technology in this way and to interpret their observations while doing so.

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Ms. Lowe also reflected students' questions back to them when they asked a question which extended the exploration in the task. For example:

> Brian asks "if the triangle is an equilateral triangle, will the incenter be the same distance to the sides as the vertices?" Ms. Lowe says that that's a great question, and tells him that he has 9 minutes and a tool to investigate it with. (Fieldnote, 1/28/11)

While this was not a task that Ms. Lowe had prepared for students, she encourages Brian to remain engaged with the task at a high level by extending the task for him and encouraging him to continue the exploration. In fact, Brian stayed after school (the observed class was the last period of the school day) for about 30 minutes to conduct his investigation, concluding that the distance from the incenter to a vertex of the

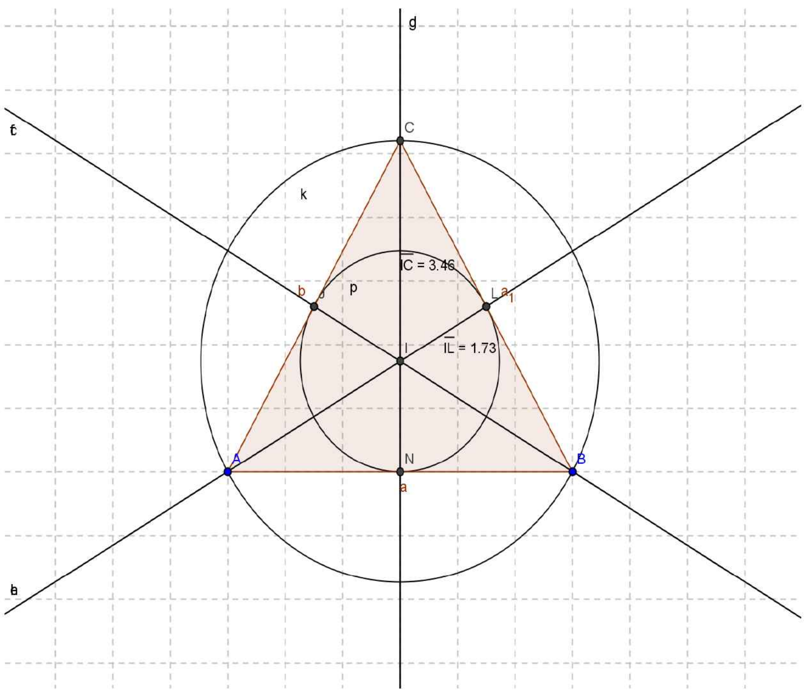

Figure 6. The figure Brian constructed in GeoGebra while investigating the location of the incenter in an equilateral triangle.

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triangle is twice the distance from the incenter to the side of the triangle and that in an equilateral triangle the incenter and the circumcenter coincide, as shown in Figure 6.

Another way in which Ms. Lowe sustains the press for meaning and explanation is by walking away from a student before he or she has come to a conclusion if she feels that the student has enough information to make progress on the task. She ensures that students have made the construction accurately and have made relevant observations that can be used to make progress on the task and then asks students to interpret those observations.

> She asks another student, "what do you think?" She tells him, "you're seeing what I want you to see. What does it mean?" The student struggles to make a generalization, perhaps unsure of what Ms. Lowe is looking for. She tells him to think about it, and then tells him to think about the parts of a circle, and she walks away. (Fieldnote, 1/27/11)

A similar example precedes the exchange above in which Ms. Lowe reflects Neil's questions back to him:

> After Neil shows Ms. Lowe what he's noticing by dragging the triangle, he asks her if that "has anything to do with it" and she says, "I think it does. What's not changing?" Neil replies, "the lengths" and Ms. Lowe says, "what else?" Neil says "the midpoints" and Ms. Lowe again replies, "what else?" and asks him to think in terms of the coordinate plane, and Nick says something about the x-axis, and then says he doesn't know. Ms. Lowe tells him to keep playing with it and walks away. (Fieldnote, 2/16/11)

By walking away, she prevents further discussion or questions from the student, which could result in lowering the cognitive demand. She is effectively telling the student, "You don't need to ask more questions, you need to think about what you've observed." Furthermore, it communicates to her students her confidence in their ability to interpret their observations and make conceptual connections for themselves.

These cases also exemplify "providing students with the means to monitor their own progress" in the sense that Ms. Lowe is referring them to the tools that they have available in order to investigate their own questions and conjectures, the approach she often used in the context of a

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sustained press for meaning or explanation. For example, the practice of reflecting students' questions back to them is often associated with students using the technology to monitor their own progress, and a concrete example of how the use of technology can help to redistribute the mathematical authority in the classroom. The potential for students to use a tool like GeoGebra to form and verify their own conjectures has important implications for students' mathematical agency and authority.

Discussion

Simon (1996) distinguishes transformational reasoning from inductive or deductive reasoning, describing it as the reasoning a learner would engage in while investigating a system. Simon (1996) defines transformational reasoning as:

> the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider not a static state, but a dynamic process by which a new state or a continuum of states are generated. (p. 201)

This sort of reasoning might be engaged in by students with or without the aid of technological tools, but the interactive and dynamic nature of DGS and the design of the tasks described here had the potential to engage students in this sort of reasoning. Indeed, while Simon asserts that transformational reasoning is not inherently high level, having the potential to range from "relatively trivial" to "extremely powerful," it seems that the transformational reasoning intended by these tasks encompasses much of the high level thinking requirements. However, while the use of DGS in all three of these classrooms was quite similar in the set up of these tasks, only Ms. Lowe's students seemed to engage in this sort of transformational reasoning during the implementation phase.

It would be an oversimplification of the matter to explain this difference by simply noting that Ms. Lowe's students understood how to use DGS, while Ms. Jones' and Ms. Young's students did not. First, this begs the question of how or why Ms. Lowe's students knew how to use

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DGS, given that they did not have more experience with it than the students in the other two classes. Furthermore, knowing what buttons to push is merely a necessary condition for students to engage in high level thinking and reasoning while using it. The real challenge is coordinating the affordances of the technological tools with the high level requirements of the task, i.e., using DGS to engage in transformational reasoning. Ultimately, students must construct mathematical meaning for the tools they use if they are to use them in support of high level thinking.

The process by which learners construct meaning for tools, technological or otherwise, has been described as instrumental genesis (Drijvers & Trouche, 2008; Guin & Trouche, 1999). According to this idea, an important, even essential, part of the process of constructing meaning for a tool is that learners must construct mathematical meaning with the tool, i.e., use technological tools in the context of meaningful mathematical activity. However, this need to simultaneously construct meaning for a tool while using it results in a complex process that can be difficult to support or foster. For example, Ms. Jones' students failed to understand the difference between lines constructed to be parallel and lines that looked parallel in GSP, even though they had been provided with instructions for doing so. Likewise, Ms. Young's students used accurate pre-constructed figures but still did not know how to use them to make high level observations and conjectures. On the other hand, there is evidence that Ms. Lowe's students were able to use DGS to engage in transformational reasoning. A poignant example of this is Brian's question regarding the relationship between the circumcenter and the incenter in an equilateral triangle, and his use of GeoGebra to investigate it.

The practices exhibited by Ms. Lowe provide insight into ways that teachers can foster students' instrumental genesis. In particular, the following pedagogical moves provide empirical support for the idea that promoting the process of instrumental genesis for technological tools can, and perhaps should, be done in the context of using them: (a) monitoring and scaffolding students' use of the DGS; (b) requiring that students interpret their observations mathematically, pressing them for explanations and justifications which made reference to the figures they had constructed using DGS; and (c) requiring that students use the DGS to monitor their own progress on the task. Given the increased attention to student thinking and reasoning promoted by current stand-

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ards, including the strategic and appropriate use of tools, further work is needed to identify a more complete set of teacher practices for assisting students in constructing meaning for technological tools and using these tools to support students' transformational reasoning in mathematics.

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Appendix A. Classroom-based factors associated with maintenance or decline (Stein & Smith, 1998)

Factors Associated with the Decline of High-Level Cognitive Demand Tasks	Factors Associated with the Maintenance of High-Level Cognitive Demand Tasks
• Problematic aspects of the task become routinized (e.g., students press the teacher to reduce the complexity of the task by specifying explicit procedures or steps to perform; the teacher "takes over" the thinking and reasoning and tells students how to do the problem). • The teacher shifts the emphasis from meaning, concepts, or understanding to the correctness or completeness of the answer. • Not enough time is provided to wrestle with the demanding aspects of the task or too much time is allowed and students drift into off-task behavior.	• The teacher scaffolds student thinking and reasoning. • Students are provided with means of monitoring their own progress. • Teacher or capable student models high-level performance. • Sustained press for justifications, explanations, and/or meaning through teacher questioning, comments, and/or feedback. • Tasks build on students' prior knowledge. • Teacher draws frequent conceptual connections.

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• Classroom management problems prevent sustained engagement in high-level cognitive activities. • Inappropriateness of task for given group of students (e.g., students do not engage in high-level cognitive activities due to lack of interest, motivation, or prior knowledge need to perform; task expectations not clear enough to put students in the right cognitive space). • Students are not held accountable for high-level products or processes (e.g., although asked to explain their thinking, unclear or incorrect student explanations are accepted; students are given the impression that their work will not "count" toward a grade). • Other:	• Sufficient time to explore (not too little, not too much). • Other:

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Appendix B. Brief description of the tasks set up at a high level using DGS as part of student-centered task.

Teacher	Technology	Task Description
Ms. Jones	GSP	Teacher leads the class in using GSP in order to determine how a line segment connecting two sides of a triangle can create a similar triangle within the given triangle
Ms. Jones	GSP	Students use GSP to explore the relationship between the lengths of the sides of a triangle, i.e., the Triangle Inequality Theorem
Ms. Jones	GSP	Students use GSP to individually explore the relationship between the angles formed by parallel lines cut by a transversal and between the angles formed by intersecting lines
Ms. Jones	GSP	Students use GSP to discover that trig ratios (sine, cosine, and tangent) depend only on the angles and not on the side lengths
Ms. Young	GeoGebra applet	Students use dynamic GeoGebra applet to discover angle relationships formed by parallel lines cut by a transversal
Ms. Young	GeoGebra applet	Students use a dynamic GeoGebra applet to discover that the sum of the interior angles of a triangle equals 180°

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Ms. Young	GeoGebra	Students use GeoGebra to construct a triangle and an exterior angle to discover that the sum of the two remote interior angles is equal to the exterior angle
Ms. Young	GSP	Students use GSP to explore the relationship between the lengths of the sides of a triangle, i.e., the Triangle Inequality Theorem
Ms. Lowe	GeoGebra	Students use GeoGebra to explore the properties of the perpendicular bisector and circumcenter of a triangle
Ms. Lowe	GeoGebra	Students use GeoGebra to explore the properties of the angle bisector and incenter of a triangle.
Ms. Lowe	GeoGebra	Students use GeoGebra to explore properties of altitudes and the orthocenter, and to use their results to solve for the coordinates of the orthocenter of a triangle analytically
Ms. Lowe	GeoGebra	Students use GeoGebra to explore properties of medians and the centroid of a triangle, and to discover the relationship between the median segments
Ms. Lowe	GeoGebra	Students use GeoGebra to explore the properties of the midsegments of a triangle

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Appendix C. The Task Analysis Guide (Stein & Smith, 1998)

Low Level Cognitive Demand Tasks
Memorization Tasks • Involve either producing previously learned facts, rule, formulas, or definitions or committing facts, rule, formulas, or definitions to memory. • Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • Are not ambiguous – such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated. • Have no connection to the concepts or meaning that underlay the facts, rules, formulas, or definitions being learned or reproduced.
Procedures without Connections Tasks • Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task. • Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • Have no connection to the concepts or meaning that underlies the procedure being used.

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• Are focused on producing correct answers rather than developing mathematical understanding. • Require no explanations or explanations that focus solely on describing the procedure that was used.
High Level Cognitive Demand Tasks
Procedures with Connections Tasks • Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Student need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
Doing Mathematics Tasks • Require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example).

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- Require students to explore and to understand the nature of mathematical concepts, processes, or relationships.
- Demand self-monitoring or self-regulation of one's own cognitive processes.
- Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.
- Require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

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About the Author

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Chapter 7

How Can Dynamic Geometry Environments Assist the Learning of Geometrical Proofs at the University Level?

Margo Kondratieva

Abstract

This paper describes an experiment in teaching Euclidean geometry which was undertaken by the author at a Canadian University. The approach combined the methodology of Basic Geometric Configurations (BGC) with the introduction of a dynamic geometry environment (DGE). Basic Geometric Configuration is a geometrical drawing that depicts a statement along with auxiliary elements pertinent to its proof. Benefits of using BGC in teaching geometry were enhanced by employment of corresponding applets produced with dynamic geometry software. Several problems that highlight geometrical invariance observable in DGE are presented. Responses of 13 students who had taken the course indicate a potential of this practice and suggest directions for further research.

Introduction

Many North American universities have an undergraduate course in *Euclidean Geometry*. This subject is essential for students pursuing engineering and science degrees as well as for those preparing to teach in high school. The course usually places great emphasis on proofs and deductive reasoning, and hence presents a challenge especially for students with weak geometrical backgrounds and insufficient retention of high school knowledge in plane geometry. It is also apparent that students often fail to make sense of the many geometrical facts and have a tendency to memorize most of the material without either internalizing it or constructing in their own way (Gagatsis & Demetriadou, 2001). As

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a result, many students struggle instead of enjoying a subject that is essentially vibrant and hands-on.

Teaching of geometry may include various approaches ranging from the sequence of technique-followed-by-applications, two-column proofs outlining intermediate statements and reasons, formal derivation of statements from axioms and postulates to compass-and-straightedge constructions, geometrical experimentations and 'genetic' methods (see e.g. Safuanov, 2007) taking into account historical, logical, epistemological, psychological, and socio-cultural aspects as well as natural development of knowledge and progress of the learner.

While the instructors in Euclidean Geometry may consider incorporating dynamic geometry environments (DGE) hoping that students will appreciate the beauty of the subject when given an opportunity to experiment and explore, it is unlikely that a DGE is going to enhance the learning of geometry just by its presence. A thoughtful design of classroom activities and homework practices is required.

This chapter resulted from a small action research project aiming to address this need while teaching a course in Euclidean Geometry in a Canadian university (Kondratieva, 2011c). The paper starts with a brief discussion of existing literature regarding challenges and various approaches to teach proofs in geometry. The role of DGE in connection with educational research on the development of deductive thinking is highlighted in the next section. With this in mind, the following section describes a blended approach based on the use of basic geometric configurations (BGC) with construction of interactive applets in DGE. Then I give examples of several geometrical problems which were reshaped by the presence of DGE. The problems are used to illustrate various advantages that the focus on invariance observation in a DGE may have for the learner. These problems also represent well the approach which was undertaken in teaching the course. Students' reflections to this new teaching initiative are given in the last section.

The Challenges and Approaches to Teaching Geometrical Proofs

This paper concerns teaching geometry at the undergraduate university level. The first geometry course aims to introduce students to the axiomatic world of Euclidean Geometry and to improve their ability to

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reason logically and compose rigorous proofs. Equipped with these abilities, students sharpen their precision of thinking in terms of definitions, theorems, and abstract mathematical concepts, which is vital for learning in all branches of mathematics (Balacheff, 2010; Rav, 1999). At the same time, by learning to prove, students also may enrich their repertoire of techniques and approaches applicable in a variety of problem solving situations (Hanna & Barbeau, 2010).

However, learning to prove and think rigorously presents a major challenge for students and consequently requires particular attention of mathematics educators at all levels. The problem is that "[p]upils fail to appreciate the critical distinction between empirical and deductive arguments and in general show a preference for the use of empirical argument over deductive reasoning." As well, "proof is not 'used' as a part of problem-solving and is widely regarded by students as an irrelevant, 'added-on' activity" (Hoyles & Jones 1998, p. 121; see also Coe & Ruthven, 1994).

Naturally, the process of cognitive development requires a long time before learners start to conceptualize their empirical experiences and symbolic exercises in terms of formal objects and operations and before their thinking becomes hypothetical and fully abstract (Tall et al., 2012). By encountering various properties of an object and establishing (and proving!) relations of implications between these properties, one comes to a mental construction of 'crystalline concepts' corresponding to the objects of study. This scenario "offers the possibility of [constructing/developing] increasingly complex and connected knowledge structures" (Tall, 2011, p. 6) The process of cognitive growth in this direction requires, beside time and effort on behalf of the learner, specific "mathematical activities that could facilitate the learning of mathematical proof" (Balacheff, 2010, p. 133). There is a need for finding a productive way for incorporating experimentation and proving so that "*proofs do not replace measurements but make them more intelligent*" (Janhke, 2007, p. 83, italics in the original). At the same time, it was observed that between the Platonic world of Euclidean geometry and the learners' world of physical practice and experiences (such as drawing or paper folding) there exists an "experimental-theoretical gap" (Lopez-Real & Leung, 2006, p. 667) which originates from a fundamentally different nature of those two worlds. These differences may also be expressed in terms of dichotomies characterizing the focus in teaching geometry: intuition-deduction, construction-proof, and spatial-numerical (Laborde, 1985). Indeed, methodology based on inductive reasoning, constructions using compass and ruler and spatial

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perception differs significantly from formal deductive proofs or coordinate and algebraic approaches in teaching geometry.

Thus the instructors are confronted with the following dilemma: the development of deductive thinking may be their goal (as is in this undergraduate course) but the elements of formal thinking should grow from the learner's intuition and prior experience and cannot be simply imposed on a learner in their final form (Freudenthal, 1971). Another issue is that the learners who are exclusively exposed to formal procedural approach in teaching geometry (such as two column proof) often experience 'epistemological anxiety' (Wilensky, 1993) resulting from not being able to understand the meaning and purpose of the actions they perform, even if they receive high marks for their performance. 'Making meaning' is also regarded by Sfard (2003) as one of the main learners' needs along with challenge, relevance, social interaction, and the sense of belonging to the learning community.

This dilemma, which was confronted by the author, may be resolved by making careful distinction between visual appearances and structural organizations of geometrical images (see also Kondratieva, 2011d). When learning proofs in geometry, one would benefit from "problem situations calling for an interaction between visual methods and geometrical methods" (Laborde 1998, p. 114). Leading students to focus their attention on significant elements of a geometrical construction and interrelation between these elements may help them to transform 'messy drawings' into 'figural concepts' that were defined by Fischbein (1993) as "mental entities which possess simultaneously conceptual and figural characteristics" (p.143). While geometry largely relies on pictorial materials, figures support visual thinking only if a learner grasps the mathematical structure they represent (Arnheim, 1969). Understanding in geometry "cannot be achieved just through visual evidence as understanding requires restructuring the system of conceptions and ideas. Proof based on theoretical arguments becomes a means to understand." (Laborde, 2000, p. 155). Indeed, proofs can help a learner to verify certain empirical observations and explain them at both informal and formal deduction levels (De Villiers, 1990). Yet, another important function of proof is the "*systematization* of various known results into a deductive system of axioms, definitions and theorems" (p. 20, italics in the original) that allows one to unify already known results, organize them logically and often "provides new

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perspectives and/or is more economical, elegant and powerful than existing ones" (p. 21).

Different approaches to teaching geometry exist, and Jahnke (2007), for example, proposes that "inventing hypotheses and testing their consequences is more productive … than forming elaborate chain of deductions" (p. 79). However, these practices also require students to possess a certain mathematical background and must be directed towards a unifying mathematical framework. Students "should build a small network of theorems based on empirical evidence" and become accustomed to "*hypothetico-deductive method* which is fundamental for scientific thinking" (p. 83, italics in the original). Similarly, Tanguay and Grenier (2010) suggest that "current curricular trends, promulgating proving processes based on experimentation and conjectures, will lead to an effective learning of proof, with proof attaining its full meaning in the learners' understanding only if these processes are set within a genuine process of building 'small theories'. " The conjectures should be formed and viewed as part of "hypothetico-deductive networks, which would then be confronted with the initial experimentations" (p. 41). As well, the "*incorporation* of well-known facts into a new framework" will call for a proof functioning as means of "*construction* of an empirical theory" (Hanna, 2000, p. 8, italics in the original).

Proofs in Dynamic Geometry Environments

Intuition required in the process of making conjectures and inventing hypotheses develops through students' experiences not only in formal logical manipulations but also in experimental explorations of objects and ideas (De Villiers, 1990). Constructions with a compass and straightedge were traditionally used for building students' geometrical intuition. Since the time when first dynamic geometry environments (DGE) such as Cabri or The Geometer's Sketchpad became available for students, educators started to look at various possibilities these systems can offer to the learner of geometry. At first the systems were used to produce accurate and nicely looking geometrical drawings. Soon after, it was realized that the *dragging* operation available in these environments contains a much larger potential than just creating a number of static cases of a certain geometrical property. An observed 'motion dependency', that is, a continuous transformation of a figure in response to change of certain parameters by dragging some base points of the figure, can be interpreted by a learner and transformed into a

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logical conditional relation within a mathematical context (Mariotti, 2006). Dragging allows one "to 'see' mathematical properties so easily" that some educators feared that this fact "might reduce or even kill any need for proof and thus any learning of how to develop the proof" (Laborde, 2000, p. 151). Consequently this can widen the gap between the inductive nature of experimental geometry (enhanced by this dynamic feature) and deductive nature of Euclidean geometry (Mason, 1991). Considering students' inclination to exclusively experimental verification of geometrical statements, this scenario of narrow-minded use of DGE in schools still presents a real threat to a proof-oriented curriculum. However, the hope is that certain pedagogical approaches which incorporate DGE in the right way will facilitate learning of proofs and help students to produce logical links between various properties of a dynamic drawing. The optimism is supported by the fact that DGE give a new dynamic meaning to static statements of Euclidean geometry. For example, the phrase "any point on a circle" can be interpreted as a physical action of dragging a point along the circle. Thus theorems can be illustrated by constructing dynamic drawings satisfying the set of conditions listed in the theorem and then observing the facts listed in the concluding part of the theorem. For example, one can *observe* that "three side perpendicular bisectors of any acute triangle always intersect at one point which lies inside the triangle" by drawing an acute triangle with perpendicular bisectors to its sides and then dragging the vertices of this triangle so that it remains acute. Such an experimentation-observation process may help the student to notice the implicative 'if-then' structure (as opposed to 'and'-structure when properties are viewed simultaneously with no grasp of cause-effect relationship between them) of the statement and perhaps to memorize the statement better. However, students may never fully understand the reason of the observed phenomenon.

Prefabricated drawings allowing students to drag points in a constrained domain and observe the result of such dragging, are known as *robust constructions* (Healy, 2000). They were found very useful specifically for the purpose of illustrating geometrical statements and letting students to make a clear distinction between premises and conclusions of these theorems. But it is another type of dynamic drawings that currently gives hope to mathematics educators in connections with learning to do proofs. *Soft constructions* (Healy, 2000) allow a learner to conjecture the region for dragging of an element of a drawing that leads to

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production of the desired property. For example we ask the student "under what condition do the three perpendicular bisectors intersect at a point lying inside the triangle?" The student then experiments by dragging vertices of a triangle and makes a conjecture based on the experimental evidence that "the triangle must be acute". This type of dragging is called "maintaining" since it purposely maintains the property of interest (Baccaglini-Frank & Mariotti, 2010). In contrast with the robust construction described above, in a soft model the students conjecture the conditions themselves as they can also explore the case when the triangle is not acute. This wider context for exploration may also lead to additional by-product conjectures such as "the intersection point of perpendicular bisectors lies precisely on the side of the triangle when the triangle is right-angled". These and other features of soft constructions identified through classroom research hold a promise that "maintaining dragging" may produce argumentation leading to both the conjecture and proof, and thus help to bridge the worlds of experimental and theoretical geometries (Baccaglini-Frank & Mariotti, 2010).

Dynamic Geometry and Basic Geometric Configurations

There are various ways of employment of DGE in mathematics classrooms. In a more traditional approach, DGE is used as "a convenient parallel to paper and pencil; to provide accurate static figures and generate measurement data; to highlight invariant properties through their visual salience under dragging" (Ruthven et al., 2007, p. 299). Some instructors may teach the same problems and theorems by encouraging students "to consider geometrical relationship inductively before being exposed to deductive proof" (Lampert, 1993, p. 150). In this approach, largely supported by a DGE, a textbook or a worksheet can provide an important structuring resource for lesson activity.

In a more progressive, constructivist, scenario teachers accept that "learning might take place in computer-based situations without reference to a paper-and-pencil environment" (Laborde, 2001, p. 311) and without regarding a book or teacher as a main source of information. In this case students assume a greater ownership for their study when they learn fact by experimenting with dynamical figures, making observations, conjecturing, and trying to explain their findings. At the same time, left alone in a DGE the students may undergo various pitfalls such as (1) invent their own terminology or assign different meaning to

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standard terminology; (2) stick with certain techniques that proved useful and keep returning to them despite availability of a better alternatives; (3) focus on procedures rather than on analysis of geometrical structure; and (4) may not appreciate the significance of invariance (Jones, 1999). Thus students need teacher's intervention in their practice with a DGE in order to avoid such pitfalls.

Context of This Study

For this study the content was strictly shaped by the textbook (Shawyer, 2010) but at the same time students were encouraged to use several other resources including lecture notes and experimenting in the DGE (GeoGebra). All three sources of geometrical ideas were welcomed by the instructor. In fact, the goal of this study was to find to what extent and in what ways will the students use this freedom in their learning.

My traditional approach to teaching Euclidean Geometry emphasizes the use of *basic geometric configurations* (BGC) - fundamental geometric facts expressed in drawing (Kondratieva, 2011a). Such drawings contain auxiliary elements and labels (e.g., for equal angles, equal segments, perpendicular and parallel lines) that allow remembering the statements along with the ideas of their proofs. For example, Figure 1 (middle) shows a BGC corresponding to the fact that "locus of the right-angle vertex of a right triangle with hypotenuse FG is the circle with diameter FG". For comparison, Figure 1 (left) illustrates possible visualization of this fact in a DGE by setting a segment AB of a fixed length, lines AD and BC to be perpendicular, and then dragging point D with *Trace* of point C turned on.

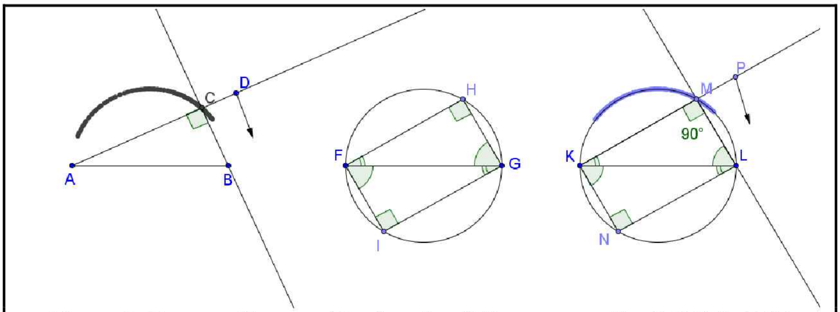

Figure 1. Property illustrated by dragging (left), corresponding BGC (middle) and its version in DGE (right).

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Ideally, BGC are visualizations that correspond to *crystalline concepts* (Tall, 2011), since they represent complexities and connections between geometrical objects in a structural way. As such, BGC are the stepping stones to proving or solving geometric problems. Once BGC are identified by a learner as a part of a more complex structure, the learner can activate one of the implications decoded by this BGC and thus make a deductive step in her reasoning. More elaborate proofs in synthetic geometry can be decomposed in sequences of BGC. Thus, learning to use/create BGC in geometry is in a way similar to learning an alphabet of pictorial language. An employment of BGC approach calls for the following teacher's actions: (1) Asking students to explain the relations between geometrical objects they observe in a figure and the role of the auxiliary lines drawn on the original figure; (2) Constantly relating to already learned geometrical facts and focusing students' attention on the key ideas used in a particular solution; (3) Demonstrating several proofs or solutions of the same problem in order to show connections between geometry, trigonometry, and algebra; (4) Directing students' attention to the implications, converse and equivalent statements; (5) Helping students summarize their findings in the form of a mathematical statement; (6) Surprising students with an unexpected conclusion or asking them to correct errors in a flawed reasoning (Kondratieva, 2009; 2011a).

Taking into account that "computers can offer a new context for designing innovative activities to address the main problem of linkage between empirical experiments and deductive reasoning" (Osta, 1998, p. 111), I introduce my students to dynamic drawings of BGC (i.e., applets) produced in GeoGebra (GG). These applets mostly present robust constructions and allow observing the elements of the statement or proof that are invariant under dragging (see for example, Figure 1-right). The applets are used during the lecture discussions to accompany the blackboard presentation by "dynamical visual proofs, which are based on 'drawing in movement' that can be properly performed in a dynamical environment" (Gravina, 2008). The applets are linked to the webpage associated with the course and are available for students' further experimentations. This way, students become accustomed to the idea of supporting their geometrical reasoning by an interaction with the dynamic drawings.

During the course, students were asked to perform the following assignments. They were asked to create a robust dynamic drawing based on a verbal description and a static figure in the book. They were asked to recognize BGCs as a part of a proof given in the book and to illustrate this

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proof by constructing their own applets. Finally, students were asked to create their own proofs of given problems and indicate BGCs employed. In the latter case they could choose to create a related applet first and try to explain behavior of the drawing and produce a proof based on these explanations. However, they were allowed to work with pencil-and-paper only.

The novelty of this approach consists in combining the methodology of the BGC approach with the advantages offered by dynamic geometry software, in order to balance empirical and deductive practices. First, students read and analyze sample proofs and identify BGC and key ideas pertinent to the proofs. At the same time students construct interactive applets in GG with the requirement to make the constraints described in the statement indestructible by dragging. This forces them to use geometrical properties of the object they draw. Students are asked to show auxiliary lines and measurements pertinent to the idea of the proofs. Students are encouraged to invent alternative proofs to the statements they analyze and interpret with the help of GG. Students are given examples of all these activities in class. They discuss BGC with their teacher using both static and dynamic drawings. As the semester evolves, the students are provided with fewer hints for problems and are asked to continue building GG applets and experiment with them in order to find their own solutions. In this way students gradually adopt the Euclidean (synthetic) geometry tradition of proofs and learn to recognize and apply BGC. The students learn to observe and explain individual empirical facts, then build, and check their 'small theories' based on many dynamic and static drawings.

In a DGE "a critical point of the solving processes is the visual recognition of a geometrical invariant by the students, which allows them to move to geometry" (Laborde, 1998, p. 120). The next section explains several possible advantages for students' learning that are related to the phenomenon of invariance with respect to dragging.

Case-invariant solutions in a DGE

When doing proofs on a case by case basis on paper, very often we need to come up with different ideas and techniques in each particular case. However, in a DGE a smooth visual transition between different cases is often available. For example, one may easily pass from the case of obtuse triangle to the case of acute triangle by dragging a vertex of

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this triangle. In this section we are interested in solutions which are valid in all possible cases of a given problem and their case-invariance is observable by dragging. We discuss four examples of problems from Euclidean geometry and their case-invariant solutions produced in DGE.

However, in each of our examples the discussion of a case-invariant solution has a slightly different emphasis. In the first example we demonstrate the importance of consideration of special cases which could be much simpler to handle than the general case. The key contraction that was found in a special case of our problem suggested the solution to the original problem taken in full generality. In the second example we highlight the advantage of the Trace function available in DGE and how its use may generate the insight in the solution which is valid in various situations. The third example illustrates the possibility to learn some additional geometrical facts useful for proving other statements while looking at different cases of a theorem's proof. The last example shows that working with various cases of a problem in DGE allows one to deeply understand certain geometrical notions (such as area) and make connections with other branches of mathematical knowledge.

Case-invariant Solution Originated from a Simple Special Case

Very often it is relatively straightforward for students to make an applet with required conditions, and the property of interest becomes easily observable by dragging base points of a drawing. At the same time the dragging does not produce a significant insight in students towards a possible proof of the observed fact. It is of interest to discuss what can be done in such a case in order to help the learners to generate some useful ideas. One thing may be to drag the points to produce special cases of the general situation. Special cases may suggest a way helpful for a generalization (Polya, 1945). They also may present an interesting problem by itself. Consider the following example.

Problem 1 (Shawyer, 2010, p. 133). *Suppose that ABCD is a convex cyclic quadrilateral. Let points P, U, Q, V be the midpoints of arcs AB, BC, CD, and DA respectively. Prove that PQ is orthogonal to UV* (see Figure 2).

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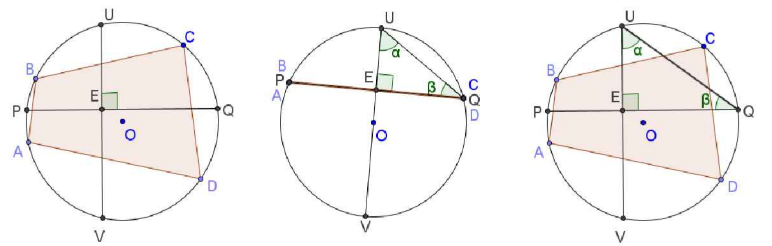

Figure 2. Problem 1: general case, special case, and the idea of the proof.

Once the applet is built, the statement can be illustrated by dragging vertices A, B, C, D and observing the angle UEQ. In particular, we can drag the points to produce a special case, when points A, B, and P coincide, as well as points C, D and Q (see Figure 2, middle). The good thing is that we are now dealing with only 4 points, P, U, Q, and V such that arcs PU and UQ are equal as well as arcs QV and VP are equal. Denoting their angle measurement in degrees by x and y respectively, we obtain $2x+2y=360^0$, and thus $x+y=180^0$. Notice that we have inscribed angle PQU subtended by arc PU measured x degrees. Similarly, inscribed angle QUV is subtended by arc QV measured y degrees. Recalling that an inscribed angle is exactly half of the corresponding central angle, or equivalently, of the corresponding arc measure, we conclude that $\angle PQU + \angle QUV = (x+y)/2 = 90^0$. Thus triangle UEQ is a right triangle, and the statement is proved. In considering this special case, we introduced in the construction an auxiliary segment UQ and focused on inscribed angles α and β. Our key algebraic observation used the fact that certain pairs of arcs are equal and altogether they constitute a full circle. Now by dragging points A and D back to produce a general case we observe that these ideas remain useful. Once again, we focus on triangle UEQ and angles α and β. We denote equal arcs as follows: $AP = PB = a$, $BU = UC = b$, $CQ = QD = c$, $DV = VA = d$. In this case $\alpha = (QD + DV)/2 = (c+d)/2$ and

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$\beta = (PB + BU)/2 = (a+b)/2$. On the other hand we can see that $2a + 2b + 2c + 2d = 360^0$, and thus $\alpha + \beta = 360^0 / 4 = 90^0$, which completes the proof in the general case.

Note that the solution to this problem outlined in Shawyer (2010) is based on the geometry of complex numbers and this presents an opportunity to compare the two approaches and to strengthen the various mathematical connections. If the special case of the problem is discussed in class, students will have a good chance to discover the general case solution and complete this assignment at home on their own.

Case-invariant Solution Suggested by the Locus Observation in DGE.

Our second example is a problem where students must first find the locus of points and then explain their answer.

Problem 2 (Shawyer, 2010, p. 133). *Let C be a circle and P be any fixed point. Consider the collection of all lines on P that intersect C. Suppose that typical such line meets the circle at points A and B. Find the locus of mid-point of AB.*

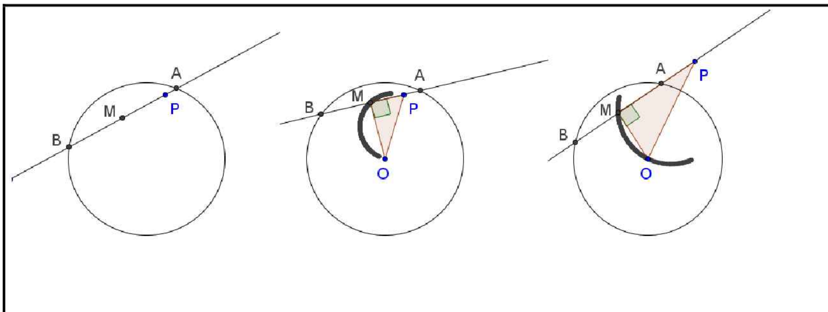

Figure 3. Problem 2: static drawing (left), possible cases with 'Trace On' and the idea of the proof (middle and right).

A static figure (Figure 3, left) can be easily drawn for this problem, but students do not find it very helpful. By dragging with the *Trace* function of the midpoint M turned on we observe that the locus forms a circle with diameter PO, where O is the center of circle C (see Figure 3, middle). As we drag P inside or outside of the circle C, we observe that M follows the arc of the circle with diameter PO (Figure 3, right). The explanation of this fact comes from recognition of two basic geometric

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configurations learned in the course. First, MO is orthogonal to AB because it is a radius-chord property. Second, since PMO is a right triangle, M lies on the circle with the diameter equal to the hypotenuse of this triangle. Once again, this idea works in all cases regardless of whether P lies inside or outside the circle C.

It was found during our teaching experiment that seeing the locus as a result of the dragging action triggered in the majority of students the recognition of the 'inscribed right triangle' configuration and consequently generated insight into the proof (see section "Aha Moments" for more details).

Case-invariant Proof and Notice of Additional Geometrical Facts

Our next example refers to the Six Point Circle theorem. The theorem states that *in any triangle the midpoints of the sides and feet of the altitudes lie on a circle*. One possible proof is based on the following approach. One needs to recognise that the quadrilateral formed by the three midpoints and one foot is an isosceles trapezoid and thus is cyclic.

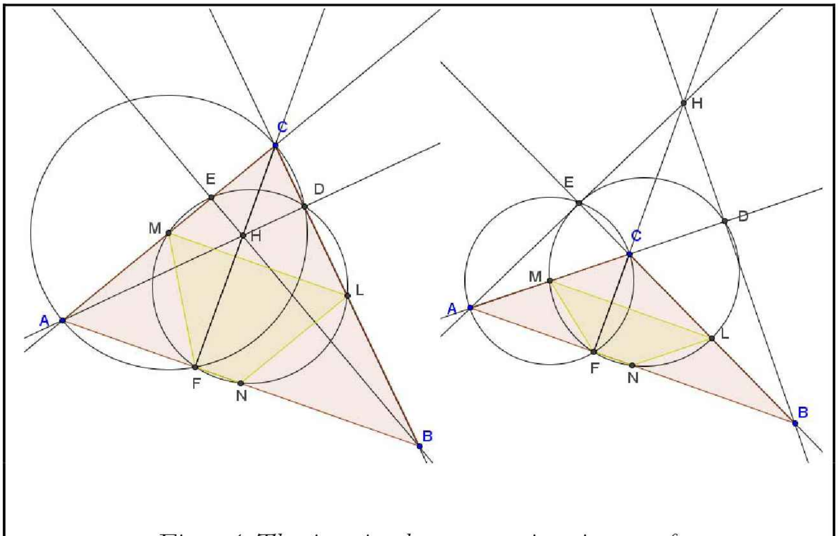

Figure 4. The six point theorem: case-invariant proof.

In Kondratieva (2011c) we discuss a possible classroom scenario regarding the development of a proof based on Figure 4. The cases of an

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acute or obtuse triangle may differ in certain details such as the position of the orthocenter H with respect to the given triangle ABC. At the same time, the fact that MLNF is an isosceles trapezoid and the explanation of this fact survive the transformation from one case to another as shown on Figure 4. Again, several BGCs are present in this proof. First, ML is parallel to FN because a middle line, that is, the segment connecting two side midpoints, is parallel to the third side of the triangle. Second, $|NL|=|AC|/2$ due to another middle line property. Finally, $|MF|=|AC|/2$ because AFC is a right triangle with hypotenuse AC, and thus F lies on a circle with diameter AC and center M. We found that it may be useful to discuss one case with the whole class and then ask students to consider the second case on their own paying attention to the details that remain and those that change. It is remarkable that playing with case-invariant solutions-applets students start to notice other geometrical properties which are not employed in the proof of the original statement but could be useful elsewhere. From this particular applet some students noted that if H is the orthocenter in ABC then C is an orthocenter in ABH, the fact that became important for developing their own proof of the Nine Points Circle theorem also studied in the course (see also section Aha Moments).

Case-invariant Solutions Helpful for Making Mathematical Connections

The last example comes from the fact that was first observed in a DGE and led to the following problem (DeVilliers, 2010). Recall that *a parallelo-hexagon* is a hexagon with three pairs of opposite sides being parallel and equal.

Problem 3. *Let ABCDEF be a parallelo-hexagon, and let points G, H, I, J be the midpoints of the sides AB, CD, DE, and FA respectively. The problem is to show that Area(ABCDEF) =2 Area(GHIJ).*

A parallelo-hexagon has several interesting properties. It is a natural generalization of a parallelogram. Each of its main diagonals AD, BE, and CF cuts the parallelo-hexagon in two congruent quadrilaterals. All three main diagonals intersect at one point, call it the center. A parallelo-hexagon remains invariant under rotation around its center by 180 degrees. Justification of these properties along with construction of corresponding applets can be assigned as a preliminary exercise to the students, before they attempt solving Problem 3.

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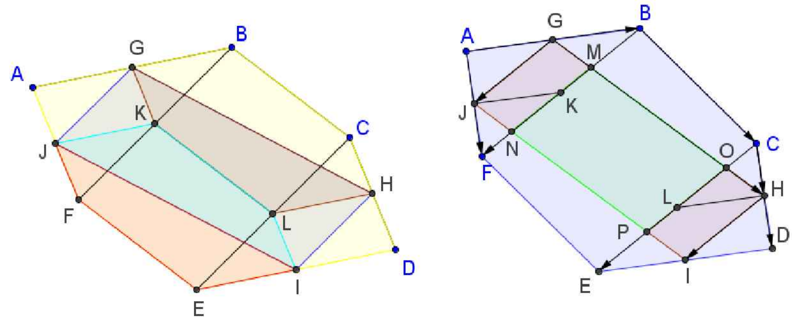

Figure 5. Two approaches to proof that area of GHIJ is half of the area of the parallelo-hexagon.

One possible solution of Problem 3 is based on the following area decomposition (see Figure 5, left). Let K and L be the midpoints of the diagonals BF and CE respectively. Then GKJ is the medial triangle of ABF, that is, the triangle formed by joining the midpoints of the sides of the original triangle ABF. Thus GKJ and GAJ are congruent and their areas are equal. Similarly, we obtain that areas of HLI and HDI are equal.

Now observe that GBCHLK is also a parallelo-hexagon and GH is the diagonal that divides its in two congruent quadrilaterals GBCH and HLKG, which consequently are of equal areas. Similarly, areas of JKLI and IEFJ are equal. Thus we have [GHIJ]=[GKJ]+[GHLK]+[HLI]+[JKLI]=[ABCDEF]/2. Here [...] denote the (geometrical) area of corresponding polygon.

It is remarkable that the area relation in Problem 3 remains the same in a non-convex case as well as in the case of self-crossing parallelo-hexagon. This fact can be easily observed by dragging vertices and comparing numerical values of the areas of interest. The problem with presented solution is that it does not survive the transformation across the cases. This particular area decomposition does not illustrate the required area relation and becomes non-informative in the case of self-intersecting hexagon. Thus here we face with the situation when we are

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looking for a case-invariant explanation of the phenomena observed in a DGE.

Let us recall the notion of 'algebraic area', or 'area with a sign' which allows us to preserve information about the orientation of the region's boundary. According to our agreement, the algebraic area of a triangle is the usual geometric area if the triangle is oriented clockwise and the negative of that area if the triangle is oriented counterclockwise. For example, Figure 6 (left) shows two congruent triangles of the same geometric area; however the algebraic area of $A_1B_1C_1$ is positive while algebraic area of ACB is negative. Note that this choice of sign is completely arbitrary: things would work just as well if the opposite convention were chosen. The algebraic area of a polygon, broken into a set of triangles oriented according to its boundary orientation, is defined as a sum of algebraic areas of these triangles. Figure 6 (right) shows that algebraic area of $A_1B_1C_1D_1$ is the sum of algebraic areas of $A_1B_1C_1$ and $C_1D_1A_1$ each of which is positive. At the same time, algebraic area of a self-crossing quadrilateral ABCD consist of a negative portion ABE' and a positive area E'CD.

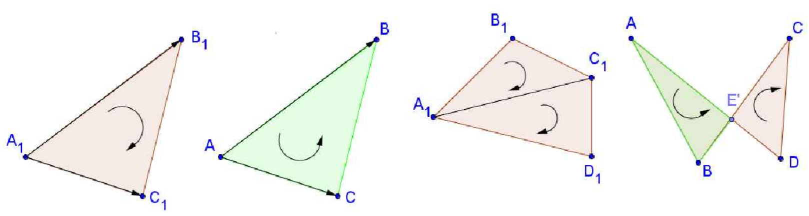

Figure 6. Algebraic area of triangles and quadrilaterals.

With this in hand, we return to the construction of a case-invariant solution of Problem 3. Figure 5 (right) suggests that $[ABCDEF]=[ABF]+[BCEF]+[CDE]$. In both the convex case (figure 5, right) and the non-convex case (figure 7, left) we have $|MN|=|GJ|=|BK|=|BF|/2=|OP|=|HI|=|LE|=|CE|/2$. By Cavalieri's principle areas of two parallelograms $[GMNJ]=[GBKJ]$ or $[LHIE]=[OHIP]$ are equal since both of

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them have the altitude and base of the same lengths. Thus, in both convex and non-convex cases $[ABF] = 2[GBKJ] = 2[GMNJ]$, $[CDE] = 2[LHIE] = 2[OHIP]$. Similarly, since parallelograms MONP and BCEF have the same altitude and bases in ratio 1:2, we obtain $[BCEF] = 2[MOPN]$. Finally, we conclude that $[ABCDEF] = 2([GMNJ] + [MOPN] + [OHIP]) = 2[GHIJ]$. Note that so far all polygons are positively oriented and their algebraic areas are positive.

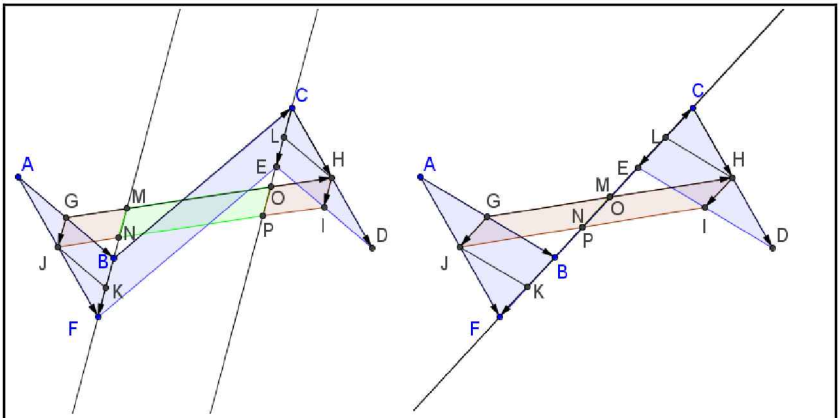

Figure 7. Problem 3: case-invariant geometric solution in non-convex case.

Figure 7 (right) illustrates the case when the hexagon degenerates in such a way that points B,C,E,F, and M,O,P,N become collinear and the area $[BCEF] = [MOPN] = 0$. In this case we have $[ABCDEF] = [ABF] + [CDE] = 2([GMNJ] + [OHIP]) = 2[GHIJ]$.

Now, in case of self-crossing hexagon (Figure 8, left) we have $[ABCDEF] = [ABF] - [BCEF] + [CDE]$. The negative sign results from the fact that BCEF is oriented counterclockwise. One can imagine that rectangle BCEF flips over as we pass to the case of a self-crossing hexagon. Similarly, the parallelogram MOPN is oriented counterclockwise and its area should be subtracted, that is, $[GMNJ] - [MOPN] + [OHIP] = [GHIJ]$. Note that relations

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$[ABF] = 2[GMNJ]$, $[CDE] = 2[OHIP]$, $[BCEF] = 2[MOPN]$ still hold in this case. Thus again, $[ABCDEF] = 2[GHIJ]$.

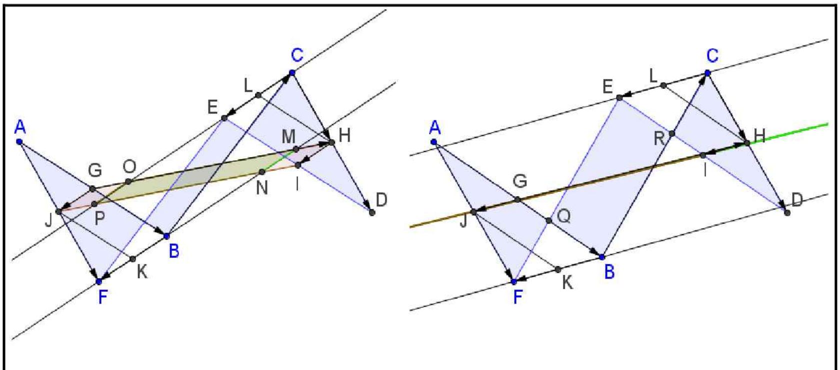

Figure 8. Problem 3: case-invariant geometric solution in self-crossing case.

Another special case of Problem 3 is presented in Figure 8 (right). Here we see that G, H, I, and J are collinear and thus the area [GHIJ]=0. At the same time we see that [ABF]+[CDE]= [BCEF], and BCEF is oriented counterclockwise, so [ABCDEF]=0.

It is important that experimenting with DGE allows us to clarify and confirm the meaning of the relation presented in Problem 3. Precisely speaking, this relation refers to algebraic areas in order to be case-invariant. When it comes to the signs of the contributing area-terms, visual representation of a polygon may be ambiguous if the order of vertices is not specified.

The case-invariant solution allows learning and rethinking other branches of mathematics besides synthetic geometry. It is well known that algebraic area of a parallelogram with vertices $A(0,0)$, $B(v_1, v_2)$, $C(u_1, u_2)$ and $D(v_1 + u_1, v_2 + u_2)$ can be calculated as $\{ABDC\} = v_1 u_2 - v_2 u_1$. This fact can be proved by area decomposition presented in Figures 9. One needs to distinguish geometrical cases but algebraic expression in terms of the components of vectors $\vec{v} = \vec{AB}$ and $\vec{u} = \vec{AC}$ happens to be case-invariant. For example, for the configuration presented in Figure 9.1 we have $[ABDC] = [AJDI] - [ABF] - [DEC] - [ACH] - [BGD] - [FJGB] - [HCEI]$.

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Thus, $[ABDC] = [AJDI] - 2([ABF] + [ACH] + [HCEI])$, and consequently

$$[ABDC] = (u_1 + v_1)(u_2 + v_2) - v_1 v_2 - u_1 u_2 - 2u_1 v_2 = v_1 u_2 - u_1 v_2 .$$

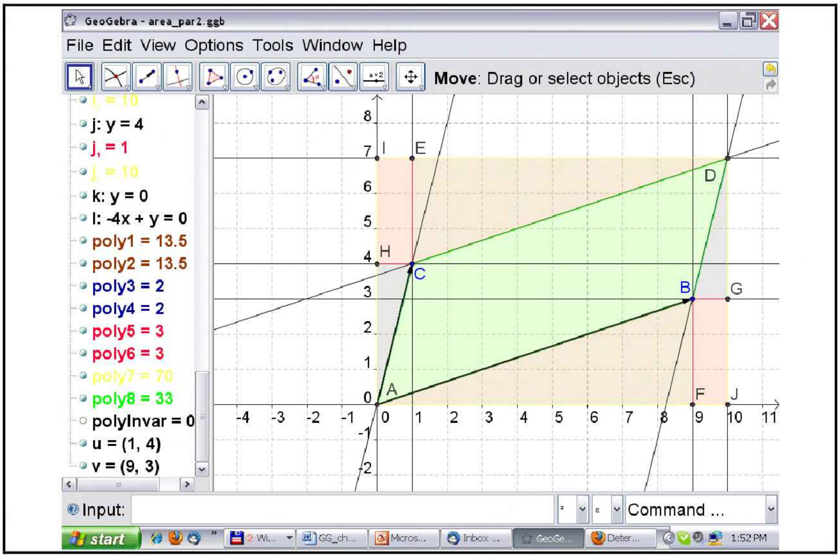

Figure 9.1: Computer screen as one works on the Area of a Parallelogram problem.

However, for the case presented in Figure 9.2 we have $[ABDC] = [AJDI] - [ABF] - [DEC] + [ACH] + [BGD] + [FJGB] + [HCEI]$. In terms of vector components we now obtain for [ABDC]

$$(-|u_1| + v_1)(u_2 + v_2) - v_1 v_2 + |u_1| u_2 + 2|u_1| v_2 = v_1 u_2 + |u_1| v_2 .$$

But since component u_1 is now negative, we have relation $\{ABDC\} = v_1 u_2 - u_1 v_2$ valid in both cases. Note that this expression can be negative. It changes the sign as we interchange vectors $\vec{u}$ and $\vec{v}$, which precisely corresponds to the change of orientation of the parallelogram $\{ABDC\} = -\{ACDB\}$. For this very reason when we talk about geometrical area we must take the absolute value of the corresponding algebraic area.

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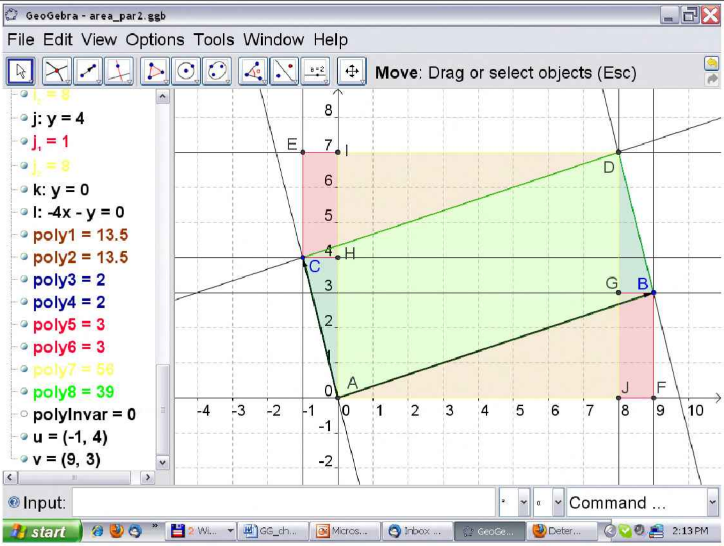

Figure 9.2: Computer screen as one works on the Area of a Parallelogram problem.

The expression for the algebraic area of a parallelogram is known in linear algebra as a determinant of a 2X2 matrix with the first row composed of the two coordinates of point B and second row composed of the two coordinates of point C. The idea that determinants are related to areas of parallelograms and volumes of parallelepipeds was successfully employed by mathematicians since the 19th century, but unfortunately many contemporary students of mathematics are not familiar with this fact. In addition, 2X2-determinants define the vector equal to the cross-product of two 3D vectors. In its turn, the length of cross-product $\vec{v} \times \vec{u}$ represents the geometric area of the parallelogram formed by the two 3D-vectors v and u. To account for area orientation one should also consider the direction of the cross product vector. For example, in our case 3D-vectors $\vec{v} = \vec{AB} = (v_1, v_2, 0)$ and $\vec{u} = \vec{AC} = (u_1, u_2, 0)$ form a parallelogram ABDC. Then the cross product is found as $\vec{v} \times \vec{u} = (0, 0, v_1 u_2 - v_2 u_1)$ and the algebraic area is equal to the third component of the cross-product $\{ABDC\} = v_1 u_2 - v_2 u_1 = (\vec{v} \times \vec{u})_3$.

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Students usually study cross-product in a Vector Calculus course. Problem 3 presents a great opportunity to recall the formula along with its geometrical proof as we presented it here, and elaborate on the notion of algebraic area in the context of self crossing figures. Thus, consideration of a case-invariant solution helps to make mathematical connections between vector calculus, linear algebra and Euclidean geometry. The fact that the problem allows a pure geometrical solution (discussed above) as well as linear-algebraic solution, presented below, makes this problem an interconnected one (Kondratieva, 2011b).

The idea of the case-invariant pure geometric solution is supported by the following linear-algebraic consideration. First of all, the definition of a parallelo-hexagon can be re-written as vector equality: $\vec{AB} = \vec{ED} \equiv \vec{u}$, $\vec{BC} = \vec{FE} \equiv \vec{v}$, $\vec{AF} = \vec{CD} \equiv \vec{w}$ (see figure 5, right). Thus we obtain relations $\vec{BF} = \vec{BA} + \vec{AF} = \vec{w} - \vec{u} = \vec{CD} + \vec{DE} = \vec{CE}$ and $\vec{GJ} = \vec{BF}/2 = \vec{CE}/2 = \vec{HI} \equiv \vec{d}$. Also we have $\vec{GH} = \vec{AB}/2 + \vec{BC} + \vec{CD}/2 = u/2 + v + w/2 = \vec{JI}$. Consequently, we observe that $GHIJ$ is a parallelogram with area $\{GHIJ\} = (\vec{GH} \times \vec{GJ})_3 = ((u/2 + v + w/2) \times (\vec{d}))_3$ which can be rewritten as a sum $\{GHIJ\} = \frac{(\vec{u} \times \vec{d})_3}{2} + (\vec{v} \times \vec{d})_3 + \frac{(\vec{w} \times \vec{d})_3}{2}$. We also have $\{GHIJ\} = \{GMNJ\} + \{MOPN\} + \{OHIP\}$.

Vector relations allow establishing that $GMNJ$ is a parallelogram, and since $\vec{BK} = \vec{BF}/2 = \vec{GJ} = \vec{MN} = \vec{d}$, we have the following area relations $\{GMNJ\} = \{GBKJ\} = \frac{(\vec{u} \times \vec{d})_3}{2} = \frac{\{ABF\}}{2}$. In the same way, we also derive $\{OHIP\} = \{LHIE\} = \frac{(\vec{w} \times \vec{d})_3}{2} = \frac{\{CDE\}}{2}$. Similarly, we obtain that both MOPN and BCEF are parallelograms. Since $\vec{BF}/2 = \vec{MN}$, we have $\{MOPN\} = \{BCEF\}/2 = (\vec{v} \times \vec{d})_3$. Finally, we conclude that $\{GHIJ\} = \{ABF\}/2 + \{BCEF\}/2 + \{CDE\}/2 = \{ABCDEF\}/2$.

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This point of view gives area relations which remain unchanged across the cases and hence suggests the partition of the hexagon used above in construction of a case-invariant synthetic solution.

Students' Responses

The instructors' aim in this action research project was to familiarize students with a DGE mostly by demonstrating some constructions during the lectures. The assignment included tasks of constructing applets in GG related to theorems and problems assigned. It was of interest to know to what extent the students perceived this approach to teaching the course as being helpful and to observe how the students are going to use dynamic software during their study pertinent to this course.

Thirteen students participated in the study. Most of them had a solid background neither in geometry nor in composing deductive proofs. While few students were familiar with some technology, DGE in general and GeoGebra in particular were new for all of them. At the end of the semester the students were asked to respond to a questionnaire. Most students felt that the combination of theoretical and experimental in DGE approaches helped them to make meaning of the geometrical statements and proofs they studied. A summary of students' responses is given in the Appendix. A few students volunteered to share their thoughts during semi-structured conversations with the instructor after the study was completed. Students' responses can be united under the following themes.

Assistance Required For Starting and Creating Applets

The opinions were divided between those few who were quick in figuring out the new software and the small majority who struggled in the beginning.

"The first time I used GG I thought it was fairly straightforward. When I was unsure of how to create an isosceles triangle I googled how and got a number of results at least one of which was helpful."

"GG is very easy to use. All my difficulties are easily solvable by a quick search online. I found the icons and descriptive hints made it easier to understand what to do."

"At first the program seemed as frustrating as others. But once I started using it more I realized it was not the devil after all."

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The degree of challenge experienced by a student while learning to use GG could be a reflection of their previous experiences with computers but also the fact that some students set higher goals for themselves in terms of the quality of their drawings. "My first experience with GG was confusing. However, after finishing the first assignment I realized that once figuring out the program it is less tedious than drawing by hands. I find it hard to learn software but I like it now that I know it."

Some students at first have not realized that geometrical knowledge is required to draw certain figures and were actually looking for a button that produced desired effect.

"My biggest challenge has been figuring out how to draw certain things that do not have a button. This can be frustrating at times."

Some students' figures, especially at the beginning, were only static and involved concrete measures of angles and lengths which was not a part of the problem description. Gradually students learn how to produce robust constructions that sustain dragging.

"I have trouble producing figures that do not change their configuration when you click on a point and move it. I still have trouble locating the one third point on a line."

Does DGE Call For or Help With Explanations?

Mudaly and De Villiers (2000) observed that secondary school students experience needs for proof and explanations even when they were convinced in the truth of certain statements by experimental evidence from a DGE. In contrast with this observation, many students in my class, that is, at the senior university level were actually happy to accept visual evidence provided by GG.

"GG is a tool to understand better but it does not generate a need for a proof in my mind. It allows producing an accurate drawing and it is never wrong so I would rather accept the fact I observe than start to doubt and seek for a justification. In general I am happy to accept things without proofs."

But again, the opinions were divided which apparently reflected a variation in students' background and attitude for learning.

"I found it convincing to see some properties on the picture. So in these cases it does not call for a proof as such. But it is also interesting

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to know why things are working. So I would not say that empirical evidence completely takes away the necessity of proving for me."

Some students reported that the call for explanation could appear in them only if the figure was in some way unexpected.

"I read geometrical statement and always imagine a figure in my mind. Sometimes GG adds to my mental picture and that is when I possibly ask myself 'why?' "

But even then interacting with the figure was not always helpful to find such explanations:

"When I see the result I sometimes find it difficult to start from BGCs and facts to explain why it is so. GG is helpful to produce nice figures that sometimes made it obvious that BGC or what we are required to prove is true, but I do not find the applets make it easy to produce the solutions [explanations]."

Students agreed that experimenting with applets was not necessarily the best way for them to learn thinking deductively, but nevertheless it was helpful in some other way.

"I took a course in formal logic which helped my development of deductive reasoning. GG does not have the same effect on me, but it allows seeing the details and developing intuition based on experimentation with figures, which I actually like."

What is the Dynamic Feature Used for?

Making connection between symbolic and visual representations and better understanding of the meaning of the statements were the most popular responses.

"I like the interactive aspect of GG and being able to shift and experiment with figures is extremely helpful with visualizing the problems on the assignments."

"It [dynamic feature] is helpful when reading the problems, because it is easy to draw figures, and if you do it right the figures can be changed around, yet preset relations will still hold... The biggest challenge is to create relations which hold."

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Some students used measurement function and relied on the precision offered by the software. For many resolving some visual paradoxes was easy with GG.

"GG makes drawing less time consuming and more precise. It is good for checking your answer and proof by using the measurement feature."

Many thought that working with applets facilitates exactness of their thinking.

"With GG my thinking was more confident and explicit. I used to rely on visual images in a similar fashion when I studied physics."

Students appreciated the ability to embrace different cases of a problem in one applet.

"It has allowed me to really understand the problem and to believe what I am trying to prove. I like that I can manipulate each figure to see different situations without having to redraw a new figure."

While it was not always explicitly emphasized in the lectures, the students seemed to grasp the significance of *construction' case-invariance* and searched for the elements or ideas that remain important in all cases.

"I realized that in order for my figures to have any significance they could not be destructible and had to work in every situation. Now that I know how to do that, the figures are much more 'usable' ".

"I like the ability to generate specific figures that are correct no matter how they are transformed."

Some students associated their understanding of the course with their ability to create the applets and found applets helpful for retention of their knowledge:

"Although I could not get the graphs to be exactly right (sometimes I got them to look right), GG helped with my overall understanding. The better I become with the program hopefully it will help me understand more."

"GG for most part displays images that I perceive in my mind. Other times, it helps my understanding of geometric structures via graphical manipulations. Experimenting with figures during my study helped me to recall some statement on the exam."

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However other students thought of using GG as an independent yet pleasant activity.

"I do not find GG that helpful towards my understanding of the assignments, but it is my favorite part. The challenge is to make a figure that follows multiple rules and being flexible at once, but this is a challenge I always look forward to solving."

Students' Habits in Learning Proofs

Several proofs were discussed during the lectures with varied amounts of details. BGCs were emphasized and applets were used to demonstrate various cases. For some proofs the students were asked to read the book and explore statements with their applets first, and then the proofs were discussed with the whole group if necessary. Many students found this approach helpful.

"I prefer a combination of reading and explanation of teacher. I like both to hear explanations after reading and read after in-class discussions. The point is that you get a couple of slightly different perspectives."

When constructing proofs, some students had more problems with "translating words into applets" than re-drawing applets with given properties posted on-line:

"I like a combination of given explanations and experimentations. I like to create applets: it is challenging but also interesting. I would rather redraw an applet with given relations than create my own applet by transforming word problems onto images on the screen."

Few students preferred to construct their own proofs before or instead of reading the book which was actually encouraged by the instructor.

"It is my general habit to start doing homework in the evening and finish up in the morning. Sometimes ideas come to me overnight. So I experiment with GG in the evening and finish my proof on paper next day. I like to create my own proof from scratch before reading the book. I draw figure first to see what the problem or theorem is saying. Then I play with the figure trying to find a solution. I found proofs in the book sometimes long and challenging to understand."

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Aha Moments

Several students reported having an instant insight while trying to create their own proofs. It usually happened after working on a problem and then either having a break, or browsing through lecture notes, talking to a peer or the instructor, or changing the strategy, e.g., working backward.

"I had been working on one question for about 2 hours. I took a break but upon returning to my problem I was still stuck. I went to talk to the instructor and explained my ideas. All she said was 'these angles sum to 180' and it was an instant aha moment. I love these moments, everything just clicks and it is a wonderful feeling."

For very few students the insights happen while using GG applets so that they could relate it to the manipulation of the figure. Using *Trace* function was found to be helpful. The following episode refers to Problem 2 from section 4.2.

"… one problem asked for a locus of points and I could not imagine it in my mind. Then I made an applet and observed the locus which happened to be a circle. But then I almost immediately saw lines intersecting at right angle and this was an explanation. After this insight I had to sit down and write the proof in details."

Most of the students agreed that "with GG I could see certain configuration clearer which sometimes generated an aha-moment in terms of better understanding of what the textbook was saying."

While working with applets some students reported observing "extraneous" facts, that later turned to be useful for other proofs. The following episode refers to Figure 4 (right) originally used in the proof of the Six point circle theorem.

"I accidentally found a proof of the Nine point circle theorem. I looked at picture of an obtuse triangle ACB ($C > 90^o$) and its altitudes and the orthocenter H. But I thought of triangle AHB and C as its orthocenter. First I thought it was my mistake, but then I checked and found that this point of view is also possible and is actually quite useful for the proof of the theorem."

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Conclusions

As it was discussed in the first section, learners should gradually move from observations of geometrical facts to logical explanations of them, towards building local theories and finally global deductive systems of knowledge. However, at the university level there is a demand for a more formal method of instruction. This paper attempts to conceptualize and describe an innovative teaching approach that combines both of these recommendations. This approach tends to employ "mathematical processes of *a posteriori* axiomatization" (De Villiers, 1990, p. 20, italics in the original) when learners start from interesting geometrical facts and analyse their proofs in order to identify what assumptions or other facts they are based on. Thus, freedom of exploration was given to the students within the frames of traditional curriculum shaped by a book and the instruction. The explorations were supported by a DGE that was also used during the in-class discussions. A particular emphasis was made on learning and recognition of basic geometric configuration in both static and dynamic solutions. Case invariance of several solutions was demonstrated as one of the keys to understanding, making connections and knowledge retention.

Based on the instructor's observation and students' responses, which were collected in order to inform the next course delivery, the following conclusions were made.

Guidance. First, students need a bit more guidance regarding how to use DGE and, in particular, how to construct applets with required properties. Students have to be explicitly oriented to the fact that they need to use their geometrical knowledge in order to create 'indestructible by dragging' constructions, and this type of problems should be given more emphasis during the classroom discussions.

Exploration. Second, in order to strengthen the descriptive-pictorial relations, students may be assigned problems of conversion of an observed applet into a word statement before they are asked to build applets related to word problems. Same as with static BGC, students should be asked to protocol the properties they observe when interacting with the applets implementing BGC. Interacting with applets will help the students to notice certain properties and move from concrete to conceptual level. The concepts formulated at first in students' own naïve language should then be converted into formal symbolic state-

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ments acceptable by the mathematics community. The book and lectures should set an example of the latter.

Proofs. Third, reading and analysis of formal proofs from a textbook accompanied by experimentation in a DGE should be continued. Perhaps, more transparency (Hemmi, 2008) regarding the structure and logic of the proofs needs to be introduced. Case-invariance of proofs, when it takes place, should be explicitly emphasised as well as its possible roles in getting general ideas from particular cases and making mathematical connections.

Representation of solutions. Forth, very often synthetic geometry solutions can be supported by ideas of, and restated in the language of linear algebra, analytic geometry or complex analysis. This fact should be discussed in more depth and illustrated in a variety of examples. Figures and problems from synthetic geometry may help to understand better various concepts used in modern mathematics.

In my view, this pedagogical experiment of introduction of a DGE at the university level Euclidean geometry course confirmed many important points regarding the use of technology in teaching that are found in the literature. The freedom of experimentation offered by the use of DGE needs to be very well structured by the instructor in order to help students to conceptualize geometrical knowledge at both intuitive and formal deductive levels. Our next course delivery informed by the observations reported in this article will hopefully shed more light on the subject.

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Appendix: Students survey on the use of GeoGebra in the course

Background

I have a solid background in geometry from my grade (K-12) school	Yes=5	No=8
I did many proofs in my grade school or university courses	Yes=5	No=8
I am learning a lot of new things about geometry in this course	Yes=13	No=0
I am developing my understanding of and ability to prove to a higher degree in this course	Yes=12	No=1
In general I like the use of technology to assist my learning	Yes=7	No=6
I have no problem using GeoGebra as a tool for drawing (static) pictures	Yes=8	No=5
I have no problems in using Geogebra as a tool for creating (dynamic) applets	Yes=8	No=5

Preferences

I like to read proofs given in the book and fill in possible gaps in them	Yes=5	No=8
I like to create my own proofs from scratch	Yes=7	No=6
I like to get some directions in class and complete the same problems at home	Yes=12	No=1
I like to experiment with applets made by others	Yes=4	No=9
I like to illustrate statement and proofs with my own applets made in GeoGebra	Yes=7	No=6

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I like to create my own applets similar to (or better than) ones made by others	Yes=8	No=5
I like to resolve fallacies and geometrical paradoxes	Yes=7	No=6

The use of GeoGebra helps me

To interpret and understand theorems and statements in geometry	Yes=11	No=2
To make a connection of symbolic and visual representations	Yes=12	No=1
To construct mathematical strategies and ideas	Yes=8	No=5
To test my ideas and adopt an action-oriented way of thinking	Yes=8	No=5
To develop reasoning skills and the notion of proof	Yes=6	No=7

I found that creating applets in GeoGebra is

Aesthetically pleasant and rewording	Yes=11	No=2
Time consuming	Yes=12	No=1
Insightful	Yes=12	No=1
Helping to activate my knowledge	Yes=11	No=2

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GeoGebra also

encourages me to make and test conjectures	Yes=11	No=2
encourages to move from naïve to logical thinking	Yes=9	No=4
gives immediate feedback on my actions	Yes=9	No=4
facilitates exactness of my mathematical thinking;	Yes=11	No=2
allows me to search for geometrical relationship that may seem beyond my grasp at that moment	Yes=8	No=5
allows me to try a larger range of possibilities compare to pen and paper approach	Yes=13	No=0

Author Information

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Chapter 8

From Static to Dynamic Representations of Probability Concepts

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Abstract

In this chapter, we explore the nature and the importance of dynamic visualizations for teaching and learning probability. The chapter begins with a discussion of importance of probability as one of the key elements of risk literacy. We identify, in a literature review, the features of dynamic visualizations that make them more suitable for learning mathematics and what differentiates them from static representations. Through the case study of learning conditional probability in the classroom, we describe how students use and transform representations from inert static to kinesthetic/aesthetic representations. We also discuss limitations of static representations and illustrate how those limitations could be resolved using dynamic representations.

Background

This chapter explores the ways in which dynamic visualizations can be used in learning probability. Our first goal is to identify the features of dynamic visualizations that make them appropriate for learning probability. The second goal is to present our research on the use of dynamic area proportional Venn diagrams in order to illustrate the first goal.

Importance of Learning Probability

Gal (2005) describes internal and external reasons for learning probability. Internal reasons are related to the importance of probability within the broader discipline of mathematics. Consistent with the internal rea-

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sons, learning probability is important because it serves as foundations for other mathematical disciplines such as statistics and decision theory. This is because probability is connected to other mathematical concepts such as rational numbers (e.g. theoretical probability is defined as a ratio of favorable outcomes to the total number of outcomes), equations (e.g. many probability problems can be reduced to linear and quadratic equations), and integrals (e.g., cumulative distribution function of a continuous random variable is defined as a definite integral of the probability density function). It follows that solving problems in probability provides opportunity for further mastery of mathematics. External reasons are connected with the fact that probability could be used to explain many natural and social phenomena. Probability models are at the core of many disciplines, theories, and models, including the quantum-theoretic model of the atom, kinetic gas theory, and genetics. For example, in the quantum-theoretic model of the atom, the position of an electron is defined by a probability density function. In addition, many socio-economic issues are approached by using sophisticated probabilistic models. They include interpreting crime rates, determining chance of a new recession, and finding evidence for racial discrimination.

Another external reason for studying probability is that probability is an element of risk literacy, which is gaining momentum in the educational community (Pratt et al., 2011). Decisions that involve the understanding of risk are made in all aspects of life including health (e.g., whether to continue with the course of medication), finances (paying for extra insurance) and politics (preemptive strikes versus political dialogue). These decisions are not only common, but they are also critical for individual and societal health and well-being. Some studies have shown that people are routinely exposed to medical risk information (e.g., prevalence rates of diseases) and that their understanding of this information can have serious implications on their health (Rothman et al., 2008). Despite its importance, most people are unable to adequately interpret and communicate risk (Reyna et al., 2009).

Understanding Conditional Probability

Conditional probability is a mathematical concept describing the likelihood of an event given that another event has occurred. Algebraically, the conditional probability of an event A occurring given that the event B has occurred can be written as

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$$P(A|B) = \frac{P(A \cap B)}{P(B)};$$

where $P(A \cap B)$ is a probability that both events A and B have occurred. In the following discussion, the formula above will be referred to as the conditional probability formula.

The concept plays an important role in understanding of risk because in everyday situations, probability of one event is often contingent on the probability of another. For example, the probability of getting a flu is contingent on many factors (e.g., the strength of one's immune system, whether or not a person has received a vaccine, etc.)

Historically, research on understanding of probability in general and the conditional probability in particular, concentrated on describing people's misconceptions of probability. For example, a very influential work by Kahneman and Tversky provided evidence that individuals tend to make errors in reasoning about conditional probability because of ignoring base rates (Kahneman, Slovic, & Tversky, 1982). In addition, Koehler (1996) describes and provides empirical evidence for inverse fallacy in which the conditional probability of the event A given the event B is taken to be equivalent to the conditional probability of the event B given the event A.

The Nature of Dynamic Visualizations

The use of technology and visualizations in mathematics has been widely discussed (Arcavi, 1999; Duval, 1999; Hitt, 1999; Hoyles, 2008; Kaput & Hegedus, 2000; McDougall, 1999; Moreno-Armella, 1999; Presmeg, 1999; Santos-Trigo, 1999; Thompson, 1999). Extending on this research, recently there has been a focus on dynamic learning environments, such as GeoGebra, which allow users to create mathematical objects and explore them both visually and dynamically.

Moreno-Armella, Hegedus and Kaput (2008) describe learning environment in which students can visualize, construct, and manipulate mathematical concepts. The dynamic learning environments can enable students to act mathematically and to seek relationships between mathematical objects that would not be as intuitive within a static environment of paper and pencil. The authors outline the evolutionary transition from static to dynamic representations by dividing them into five stages: static inert, static kinesthetic/aesthetic, static computational,

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discrete dynamic, and continuous dynamic. While the authors use the term 'inscriptions', we will use the term 'representation', to illustrate the same concept.

In the static inert stage, the representation is inseparable from the media it is presented in. One example would be textbook pages. The main feature of the second stage, static kinesthetic/aesthetic, is erasibility. One example would be writing on the whiteboard and the chalkboard. The writings could be erased over time. The two main features are that the writings are kinesthetic- it is easy to move within the medium (e.g., by adding comments or erasing them on the chalkboard). The second feature is that creation and altering of the writing is an aesthetic process (e.g., using differently coloured markers). The third stage is static computational in which presentations are "artifacts of a computational response to a human's action" (Moreno-Armella, Hegedus, & Kaput, 2008, p. 103). One is working with such representations when using a graphing calculator.

Unlike static stages, in dynamic stages, there is a "co-action between the user and the environment" (p. 103). Discrete dynamic representations can be changed through parametric inputs (i.e., the use of spinners and sliders). For example, a parabola could be moved up or down from the use of a slider that represents a vertical shift. In the continuous dynamic stage, there is an "interaction through space and time" (p. 103) between the user and the representation. For example, there are programs that allow users to explore mathematics through a continuous action of the mouse or in the case of the hand held devices, through the gesture on the touch screen.

Transition from Static to Dynamic Representations: An Example from Probability

In this section, we use a well-known breast cancer problem (Eddy, 1982) to describe how it can be represented by the Moreno-Armella et al.'s (2008) stages; starting from a static representation and culminating with a continuous dynamic representation we use dynamic area proportional Venn diagrams.

Breast cancer problem

A well-known breast cancer problem from Eddy (1982) is an exemplar used for assessing individuals' understanding of conditional probability

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(Gigerenzer, 2002). Recently, Eliezer Yudkowsky, an artificial intelligence theorist and a blogger, has re-introduced the breast cancer problem on line formulizing it as follows:

> 1% of women at age forty who participate in routine screening have breast cancer. 80% of women with breast cancer will get positive mammograms. 9.6% of women without breast cancer will also get positive mammograms. A woman in this age group had a positive mammogram in a routine screening. What is the probability that she actually has breast cancer? (Yudkowsky, 2003)

The problem as stated by Yudkowsky was also used in our research and the solution to the problem is given in the Appendix.

Static representations

The problem, when presented on paper, is an inert static representation. The move from the inert static inscription to the kinetic/aesthetic static inscription can be done in many ways. One of the ways or representing the problem is to draw a Venn diagram representing the population with and without breast cancer as presented in the Figure 1(a). The second way is to draw a tree diagram representing the situation which will be described later in the chapter. There is also a symbolic way to represent the problem by introducing the Bayes' formula and matching various probabilities with the context of the problem as it is done in the Appendix.

Computational stage: Area proportional Venn diagrams

There are many ways to illustrate the computational stage. We suggest that the use of area proportional Venn diagrams, generated by a computer program, can help explain and solve the problem. Venn diagrams were first used to describe qualitative relationships between sets. Researchers have only begun to investigate geometric properties of Venn diagrams (see Ruskey & Weston, 2005).

For example, the breast cancer problem (Eddy, 1982) could be represented by the original Venn diagrams and area proportional Venn diagrams in following ways (Figure 1):

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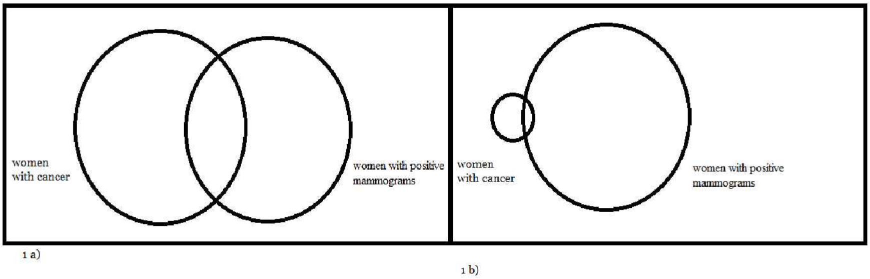

Figure 1. Original and area proportional Venn diagrams.

The key feature of the area proportional Venn diagram is that the probability of each event matches the area of the region within the diagram. For example, in Figure 1(b), the area of the circle representing the proportion of women with breast cancer (1%) is equal to 1% of the area of the diagram.

The area proportional Venn diagram presented in the Figure 1(a) is still a static inert representation. To make it static computational, we can use a software to draw the diagrams given the specific probabilities (for a detailed description of the construction of area proportional Venn diagrams, see Radakovic & McDougall, 2011).

Dynamic representations: Dynamic area proportional Venn diagrams

By the use of sliders and using, what Martinovic and Karadag (2011) describe as a dynamic and interactive mathematics learning environment (DIMLE) such as GeoGebra, we could create a discrete dynamic representation such as the one represented in Figure 2.

The applet consists of two circles representing the set of all women with cancer and the set of all women tested positive. There are three variables that can be manipulated by the users. They are the base rate (probability that a woman over forty has cancer), the conditional probability of testing positive given that one has breast cancer, and the conditional probability of testing positive given that one does not have breast cancer. These variables can be manipulated with the sliders on the right from the Venn diagrams.

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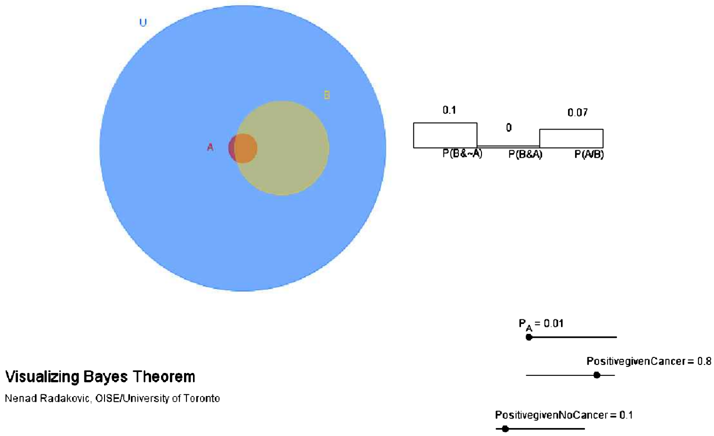

Figure 2. A discrete dynamic representation using area proportional Venn diagrams.

There are two ways in which the conditional probability of a female having breast cancer given that she tested positive is visualized by the applet. First, the conditional probability is displayed in the bar graph. Second, the user can estimate the probability by estimating the size of the intersection in relation to the size of the set corresponding to the women that tested positive. For example, in Figure 1 the base rate, given by the area of the circle A, is 0.01. The conditional probability of P(A|B) is represented by the ratio of the intersection of the circle A and circle B. The value of the conditional probability is given in the bar graph (P(B|A)=0.07).

To illustrate the dynamic nature of the applet, one can increase the base rate (probability of a person having cancer) by increasing the area of the circle A. This is done using the slider. The screen shot of the applet when P(A)=0.1 is given in the Figure 3.

One can see that the conditional probability, which is the ratio of the area of the intersection to the area of the circle A is now greater than in the Figure 2. Furthermore, the probability P(A|B) is given on the bar graph as 0.47.

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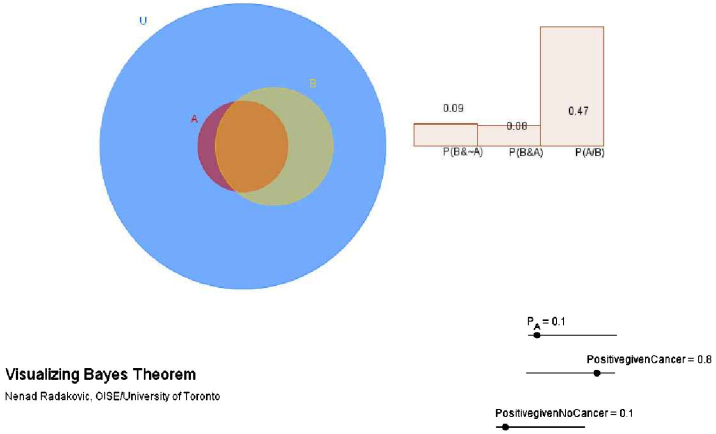

Figure 3. A discrete dynamic representation using area proportional Venn diagrams with the base rate of 0.1.

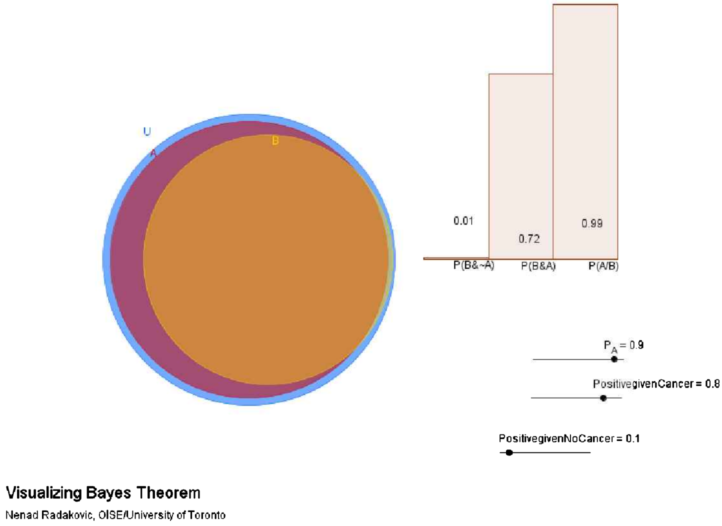

Figure 4. A discrete dynamic representation using area proportional Venn diagrams with the base rate of 0.9.

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Finally, we can increase the base rate to 0.9, as in Figure 4, to obtain the conditional probability of 0.99.

In summary, the dynamic visualization could serve two instructional purposes. First, it can help students understand, possibly deduce, the conditional probability formula as well as observe the relationship between the base rate (the probability of a woman having cancer) and the conditional probability (the probability of a woman having cancer given that she has tested positive).

Case Study: From Static to Dynamic Representations

In order to explore the function of dynamic representations in the probability instruction, we present a case study of solving the conditional probability problem in a grade 11 mathematics classroom.

Research Design

The present study uses design experiment approach to investigate the use of various representations in the classroom. The design experiment (design research) is an iterative process consisting of assessment and instructional intervention (Brown, 1992). Through iterative steps, the assessment and the intervention inform each other. The goal of the process is to improve the instruction as well as to gain insight into students' learning processes (Cobb & Gravemeijer, 2008). Specific phases of the design experiment depend on the unique contextual features of a research. According to Cobb and Gravemeijer, the phases of the design experiment are (a) the preparation for the experiment, (b) experiment to support learning, and (c) conducting retrospective analyses. On the other hand, Middleton et al. (2008) present the design experiment consisting of seven phases. These phases are construction of grounded models, development of artefact, feasibility study, prototyping and trials, field study, definitive tests, and dissemination and impact (p. 33).

We found Cobb and Gravemeijer's (2008) phases suitable for our context. However, we also thought that feasibility study, which includes negotiations with teachers and school administrators on the specific features of research, was important. This is why we also added the feasibility study stage from Middleton et al.'s (2008) description of the design experiment. In other words, we have arrived at four stages appropriate for the context of the research. These stages are (1) preparing for

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the experiment, (2) feasibility study, (3) experimenting to support learning, and (4) conducting retrospective analyses of the data.

The phases of the design research are given in Figure 5. The design experiment research we are conducting consists of two cycles (phases) of the design experiment. The first cycle took place in a Grade 11 mathematics classroom at an all-boy private secondary school in Ontario during the probability and statistics unit. There were 23 participants in the study, all of them male. This chapter only focuses on the first, third, and fourth stages of the cycle since the description of the feasibility study is not relevant for the purpose of this article. After completing the first cycle, the second design experiment will be conducted in a different educational setting, i.e., a co-educational independent school in Ontario in a Grade 11 classroom during the probability and statistics unit.

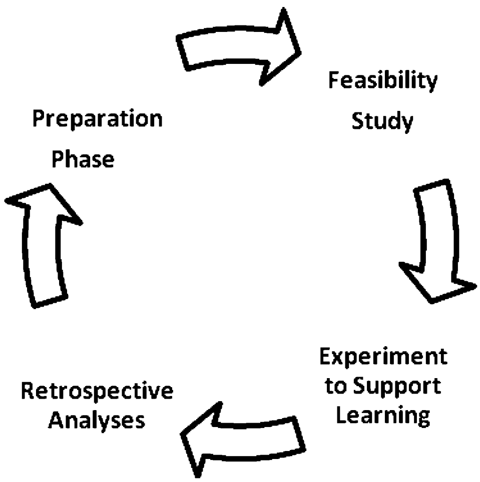

Figure 5. Design experiment cycle.
Adapted from Cobb and Gravemeijer (2008) and Middleton et al. (2008).

Preparation Stage: Instructional End Points

According to Cobb and Gravemeijer (2008), the preparation stage consists of five sub-stages: preparing, specifying instructional endpoints, documenting starting points, formulating conjectured instructional theory, and locating experiment in a broader context. In the case of the breast cancer problem, the instructional end point is to have students correctly solve the problem. The purpose of using this problem was to

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enable students to differentiate between the conditional probability of the event A given the event B and the probability of B given the event A. Furthermore, the students should be able to understand that the conditional probability of the event depends on the base rate (e.g., the greater the prevalence of the disease, the greater the probability of the true positives).

Cobb and Gravemeijer (2008) propose that, after clarifying instructional goals, teachers should "specify an envisioned or a hypothetical learning trajectory" (p. 70). In our study, the starting point was to introduce static representations using area proportional Venn diagrams and tree diagrams and see how the students could make sense of the problem. More specifically, we decided to use tree diagrams as a primary representation and area proportional diagrams to additionally illustrate the problem. Prior to the research, we developed the area-proportional Venn diagrams applet in GeoGebra.

The Experiment to Support Learning

The purpose of the experiment to support learning is to improve and test the envisioned trajectory from the preparation stage (Cobb & Gravemeijer, 2008). In this phase, the classroom data collection effectively begins. The experiment to support learning is a phase in which documenting students' shifts in reasoning is crucial. For this purpose we used three types of data: classroom observations, interviews, and assessments (written as well as oral). We also audio taped meetings with the teacher. After each session, there was debriefing with both students and the teacher to discuss the outcomes of the sessions. As a part of the initial assessment, students were presented with the breast cancer problem. Only two students out of 20 who participated in the initial assessment were able to give the correct answer to the question. After the initial assessment, followed by individual interview with students and a debriefing with the teacher, it became apparent that students found the context of the breast cancer problem as well as the issue of false positive results alien. We decided to change the context while still addressing the same underlying concept. The new problem was restated as follows:

> In May of 2009, Security Vendor Symantec released a report that 90.4 % of ALL email is spam. Let's assume that the number of spam messages is high, say 80% (this is consistent with the number of spam messages I am getting). Suppose, further

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> that 85% of spam messages are correctly identified as spam (end up in spam folder). Also, 5% of messages that are not spam also end up in spam folder.

There were five parts to this question:

> a) What is the probability that a message (any message) will end up in a spam folder;
>
> b) What is the probability that the actual spam message will end up in the spam folder;
>
> c) What is the probability that a message that is in spam folder is actually spam;
>
> d) What is the probability that a message is not spam; and
>
> e) How would the answer change if only 10% of all messages were spam.

From Inert to the Kinesthetic/Aesthetic Representations

The students, who were divided into groups of four or five, were instructed to picture the problem using tree diagrams, which were introduced in previous lectures. In this part, we use video data to report on one of the groups of four students as they proceed to solve the problem. The data were analyzed with a specific focus on the way they created and transformed representations and used representations to support their problem solving and to communicate with the others. We also paid close attention to the gestures the students were using in their arguments.

The Group

The group consisted of four Grade 11 students, Ray, Samir, David, and Blair. Based on the grades and the teacher's perception, Ray could be labeled a weak mathematics learner, Samir as medium, whereas David and Blair could be considered strong students. Samir was designated by the teacher to be the recorder, which means that he was responsible for drawing the diagram and recording answers to all parts of the question given above. The answers were recorded on the construction paper.

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Solving the problem: Transforming the representations

Throughout the activity students were transforming the representations from inert static to kinesthetic aesthetic representations by adding the features that enabled them to move within the representation and solve the problem. The process of transforming representations started with the handout containing the text of the problem and all of the sub-questions. As it is, the problem was represented by the static inert representation. Students started exploring the problem by drawing a tree diagram (i.e., using the kinesthetic/aesthetic representation). David instructed Samir how to draw the tree diagram by gesturing with his fingers how it should branch out. Based on this explanation, Samir proceeded to draw the diagram, mapping out the complete sample space for the situation described in the problem. The nodes students labeled as S and S' represent spam messages and messages that are not spam respectively, whereas F and F' represent messages that end up and do not end up in the spam folder, respectively. As the time progressed, students drew all the branches and used them to label various probabilities. The final product is given in Figure 6.

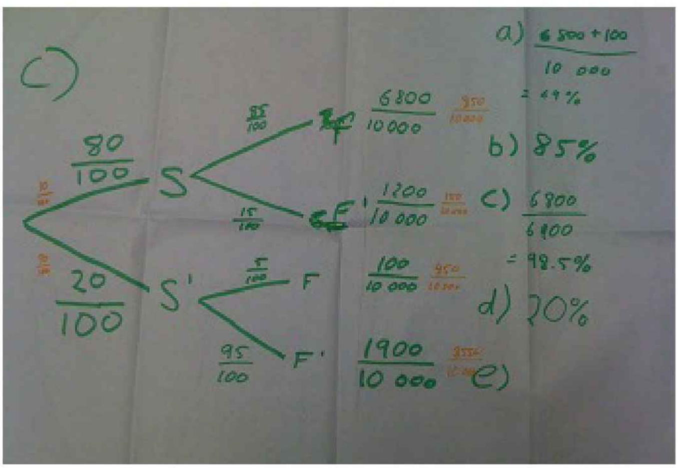

Figure 6. Student-generated tree diagram.

Kinesthetic and aesthetic features were used by the students to clarify the problem as well as to present their mathematical arguments. For example, when listening to Blair's explanation of the solution to part (b), Samir quietly pointed at various parts of the diagram in order to make sense of the problem. Students also moved within the representa-

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tion to argue their points of view. For example, answering the part (a) of the question, David pointed with his pen at the parts of the diagram that relate to this situation and stated that the probability was 68%. Blair countered David's argument by pointing out that "they" were starting at S, pointing at S (spam). Blair's hand then slides along the second branch that is labeled F (spam folder). Blair then explains "given that you have spam message, there is 85% chance it will end up as spam." This argument was convincing enough for the group members and they moved to the next problem.

The students also used the aesthetic features of the representation. For example, in order to solve the last part of the problem, Samir wrote the new probabilities in orange color to contrast them with the old ones that were in green.

Limits of Static Kinesthetic/Aesthetic representations.

The last part of the problem, namely, finding how the answer would change the probability of the message being spam was 80%, tested the limits of tree diagram as a tool for solving this problem. The fact that the probability depends on the base rate was not obvious to students since they responded to the question by calling the teacher and asking whether they had to re-calculate the answer. Seeing that they did not know how to estimate the probability, the teacher and one of the researchers suggested that students re-calculate the probabilities and write them next to the "old" probabilities. The teacher tried to get the members of the group estimate the new probability based on comparing by how much the base rate changed. However, the students did not seem to understand the teacher's line of reasoning.

Toward Dynamic Representations

The objective of the part "e" was for students to grasp the relationship between the base rates and the conditional probability. As explained above, the static features of the tree diagram did not allow for an intuitive way to describe the relationship. One way to approach the problem would be to create a computer or a calculator program that would return the values for the conditional probability, given the base rates. This would introduce the static computational representation into the activity.

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Our approach was to introduce students to the GeoGebra applet scribed above consisting of the area proportional Venn diagrams in order to offer the alternative representation of the breast cancer problem. The representation contains dynamic features that could assist students in exploring the relationship between the base rate and the conditional probability. The applet was projected on an Interactive Smart Board and manipulated by the teacher. The teacher was effectively modeling the manipulation with the applet. The students were able to see how the size of the set A (base rate) influences the conditional probability.

In order to gather more evidence that dynamic representations were beneficial for solving the problem, one of the authors conducted two separate one-on-one interviews with the students in the class. The students were presented the breast cancer problem (since they had already seen the spam problem and since they were at that point more familiar with the structure of the problem). They were then shown the applet and through Socratic dialogue encouraged to come up with a way to calculate the conditional probability of getting cancer, given positive test results. Similar to the group activity described above, the students did not work with the applet. Instead, the researcher used the applet to explain how to find the probabilities needed to solve the problem. Furthermore, by using the slider to change the base rate, the researcher demonstrated how the change in base rate changes the conditional probability.

Final Assessment

Towards the end of the project, each student was given a test that contained the following problem which is equivalent to the breast cancer problem as well as the spam problem.

> About 5% of hard discs have a computer virus. A company makes a computer software that detects 95% of infected programs. However, it also falsely identifies 10% of non-infected hard disks as infected. What is the probability that the computer program will identify any hard disk as infected?

Out of 21 students who took the test, 11 (52%) students solved this problem correctly. There were two students who solved the pre-test question correctly and one of them took the post-test. That means that 10 students who solved the pre-test question incorrectly solved the

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post-test question correctly. More specifically, most of the students were able to calculate the overall probability of being identified as defective, as well as the probability of being defective and being identified as defective. What they failed to do is put the two pieces of information together and divide the latter by the former thus calculating the conditional probability.

Conclusion

In this chapter, we illustrated the transformation from the inert static representations to discrete dynamic representations on an example from teaching probability in a secondary school mathematics classroom. The study contributes to the field by giving examples outside of geometry and algebra, originally presented in Moreno-Armella, Hegedus, and Kaput's (2008) article. As it can be seen from the results presented, the dynamic representations above were not used extensively, rather only as the secondary resource to the static representations. We described students' use of static representations, as well as their limitations, thus indirectly identifying features of representations necessary for identifying base rates. More specifically, the episode in which Samir writes the new probabilities next to the old one in order to induce the relationship between the base rate and the conditional probability shows the need for "co-action between the user and the environment" (Moreno-Armella, Hegedus, & Kaput, 2008, p. 103). This "co-action" could be provided by the dynamic visualization in which the user instead of recalculating the values could simply increase the size of the set. This would enable the user to directly observe the cause and effect of changing the conditional probability by changing the base rate.

However, as the results show, there is only limited evidence that students improved their ability to solve the breast cancer type problems while using dynamic visualizations. After all, although 52% of the students solved the post-correctly compared to the 10% who solved the pre-test incorrectly, 48% did not solve the question correctly. This could be because dynamic visualizations were only used for a brief period of time and they were not used directly by students, thus not allowing the students to participate themselves in the "co-action" with the technology as described in the above mentioned article.

In the second design cycle, we intend to enable each student to use the applet and freely explore the relationship between various parts of the area proportional Venn diagrams. In addition, the researcher used the

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GeoGebra applet as a dynamic representation (i.e., to manipulate objects (Venn diagrams) through sliders). For the second phase of the design experiment, we intend to create an applet that will allow students to change the base rate by clicking directly on the circles rather than doing it indirectly via sliders or inputting the values. This would create more of a continuous dynamic representation in which there is a direct interaction between the user and the machine.

Although there needs to be more empirical evidence of the usefulness of dynamic visualizations, the case study illustrates the range of representations of conditional probability used in the classroom and what each one of them brings to the instruction. It also sheds more light on the nature of dynamic area proportional Venn diagrams and their possible pedagogical role.

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Appendix:
Solution to the breast cancer problem by Eddy (1982)

To solve the following problem,

1% of women at age forty who participate in routine screening have breast cancer. 80% of women with breast cancer will get positive mammograms. 9.6% of women without breast cancer will also get positive mammograms. A woman in this age group had a positive mammogram in a routine screening. What is the probability that she actually has breast cancer?

Let A represent the event of having breast cancer

B the event of testing positive

We are given: P(A)= 0.01

$$P(B|A) = 0.8$$

$$P(B|A') = 0.096$$

We can calculate P(B) as follows:

$$P(B) = P(B|A)P(A) + P(B|A')P(A')$$

$$P(B) = 0.103$$

We then substitute this value into the conditional probability formula:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{(0.8)(0.01)}{1.094} = \frac{0.008}{1.094} = 0.078$$

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