ISSN 1064-5624, Doklady Mathematics, 2007, Vol. 76, No. 2, pp. 775–779. © Pleiades Publishing, Ltd., 2007.
Original Russian Text © V.V. Belov, M.F. Kondrat'eva, E.I. Smirnova, 2007, published in Doklady Akademii Nauk, 2007, Vol. 416, No. 2, pp. 177–181.

MATHEMATICAL PHYSICS

Semiclassical Soliton-Type Solutions of the Hartree Equation

V. V. Belov[a], M. F. Kondrat'eva[b], and E. I. Smirnova[a]

Presented by Academician V.P. Maslov October 25, 2006

Received May 18, 2007

DOI: 10.1134/S1064562407050341

INTRODUCTION

Consider the model Hartree equation (self-consistent field equation) in an external field $U(x)$ with a translation-invariant self-action potential $V(x - y)$, x, $y \in \mathbb{R}^n$:

$$-ih\frac{\partial}{\partial t}\Psi + \hat{H}_\kappa\Psi = 0, \quad \|\Psi\|_{\mathbb{L}_2(\mathbb{R}^n_x)} = 1, \tag{1}$$

$$\hat{H}_\kappa = \hat{H}_\kappa[\Psi] = \hat{H}_0 + \kappa\int_{\mathbb{R}^n}|\Psi(y, t)|^2 V(x-y)dy, \tag{2}$$

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + U(x), \quad \hat{p} = -ih\nabla.$$

Here, κ, h are real parameter and $h \in (0, 1]$ is a small real parameter. This unitary-nonlinear equation plays a fundamental role in quantum theory and nonlinear optics [1], in particular, in the theory of a Bose–Einstein condensate [2] and in the description of collective excitations in molecular chains and DNA molecules [3, 4].

Equations of type (1) have been extensively studied. We note only [5, 6], where numerous references can be found. A class of unitary-nonlinear equations containing equations of form (1) but without singularities in their coefficients was studied in [7–9]. The eigenvalue problem for operator (2) with a singular self-action potential was analyzed in [10] by applying a singular version of the WKB method with the help of reference equations.

Given Eq. (1) with smooth potentials $U(x)$ and $V(\tau)$, the goal of this work is to construct localized formal asymptotic solutions as $h \to 0$ and establish the conditions under which these solutions can be treated as semiclassical soliton solutions (semiclassical solitons).

First, the type of localized asymptotic solutions of interest is described in the linear case, i.e., when $\kappa = 0$ in (2). To do this, we consider the evolution of a compressed coherent state in the semiclassical approximation. Specifically, we set

$$\Psi|_{t=0} = \Psi_0(x, h) = N_0\exp\left[\frac{i}{h}(p_0, x - x_0)\right]$$
$$\times \exp\left[\frac{i}{h}(x - x_0, B_0(x - x_0))\right]. \tag{3}$$

Here, (p_0, x_0) is an arbitrary fixed point of the phase space $\mathbb{R}^{2n}$, B_0 is an arbitrary symmetric $n \times n$ matrix with a positive imaginary part ($\operatorname{Im} B_0 > 0$), $N_0 = (\pi h)^{-n/4}(\det \operatorname{Im} B)^{1/4}$, and $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^n$.

It is well known that a major point in the semiclassical approach is that the construction of semiclassical asymptotics for a quantum problem is reduced to the construction of solutions to the equation of motion of the corresponding classical system and to the study of its geometric and topological properties. For "linear" quantum mechanics (with $\kappa = 0$), the corresponding classical system is an a priori Hamiltonian system in $\mathbb{R}^{2n}$ with the Hamiltonian function $H_0 = H_0(p, x)$:

$$\dot{p} = -\nabla_x U(x), \quad \dot{x} = p. \tag{4}$$

The geometric objects generating a semiclassical answer in this Cauchy problem (1), (3), which is simple from the point of view of the general theory [11, 12], are the $[\Lambda^0_t, r^n_t]$-zero-dimensional Lagrangian manifold Λ^0_t with a Maslov complex germ r^n_t, where Λ^0_t is a point on the phase trajectory of system (4) $p = P(x_0, p_0, t) = p_{\text{cl}}(t)$, $x = X(x_0, p_0, t) = x_{\text{cl}}(t)$, which starts at the point (p_0, x_0) at $t = 0$.

[a] Moscow State Institute of Electronics and Mathematics (Technical University), Bol'shoi Trekhsvyatitel'skii per. 3/12, Moscow, 109028 Russia
e-mail: belov@miem.edu.ru, katerinasm@bk.ru
[b] Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, A1C 5S7 Canada
e-mail: mkondra@math.mun.ca

The general formulas of the WKB–Maslov complex method give the following leading term of the asymptotics (mod$h^{3/2}$) of problem (1), (3) (at $\kappa = 0$):

$$\Psi(x, t, h) = N\exp\Big[\frac{i}{h}(S_{\text{cl}}(t) + (p_{\text{cl}}(t), x - x_{\text{cl}}(t))\Big]$$

$$\times \exp\Big[\frac{i}{h}(x - x_{\text{cl}}(t), BC^{-1}(t)(x - x_{\text{cl}}(t)))\Big](\det C(t))^{-1/2}. \tag{5}$$

Here, $S_{\text{cl}}(t)$ is the classical action

$$S_{\text{cl}} = \int_0^t\Big(\frac{x_{\text{cl}}^2(\tau)}{2} - U(x_{\text{cl}}(\tau))\Big)d\tau, \tag{6}$$

and the matrices $B(t)$ and $C(t)$ (defining the complex germ $r_t^n \in \mathbb{C}^{2n}_{w,z}$, $r_t^n = \{(w, r), w = BC^{-1}(t)z\}$) are the matrix solutions to the Cauchy problem for the variational system (linearization of system (4) in the neighborhood of Λ_t^0)

$$\begin{pmatrix} \dot{B} \\ \dot{C} \end{pmatrix} = \begin{pmatrix} 0 & -U''_{xx}(x_{\text{cl}}(t)) \\ \mathbb{E}_n & 0 \end{pmatrix}\begin{pmatrix} B \\ C \end{pmatrix}, \tag{7}$$

$$B(0) = B_0, \quad C(0) = \mathbb{E}_n = ((\delta_{ij}))_{n\times n}.$$

Asymptotic solution (5) (Gaussian wave packet) is localized, as $h \to 0$, in the neighborhood of Λ_t^0 on the classical trajectory: $\lim\limits_{h\to 0}\operatorname{supp}\Psi = x_{\text{cl}}(t)$ and $\lim\limits_{h\to 0}\operatorname{supp}\tilde{\Psi} = p_{\text{cl}}(t)$, where $\tilde{\Psi}$ is the Fourier transform of Ψ. Such soliton-type solutions cannot be interpreted as semiclassical solitons, because these wave packets disperse as $t \to \infty$ (at times $t \sim \dfrac{1}{h^{1-\delta}}$, where $0 < \delta < 1$). Except for the case when $U(x)$ is the potential of a quadratic oscillator, the dispersions of coordinates and momenta calculated from the states $\Psi(x, t, \hbar)$ (5) increase as $t \to \infty$ no more slowly than a linear function of t.

It turns out that the focusing effect due to the integral nonlinearity at $\kappa \neq 0$ leads (at least, for convex self-action potentials) to the existence of localized Gaussian-packet asymptotics of Eq. (1) that are similar in form to (5), for which the dispersions of coordinates and momenta are bounded functions of time $t \in [0, +\infty)$. Such asymptotic solutions are naturally interpreted as Gaussian semiclassical solitons.

Obviously, the generalization of the above germ constructions to nonlinear quantum systems is a nontrivial task. Specifically, even the statement of the correspondence problem to classical results as $h \to 0$ is rather problematic, because it is unclear (in contrast to the linear case) what we should mean by the classical equations of motion corresponding to quantum equation (1) with a nonlinear Hamiltonian $\hat{H}_\kappa$ (2).

The following section gives an answer to this problem based on the covariant approach developed in [14, 15], which is an extension of the well-known Ehrenfest approach [13] to the derivation of classical equations of motion in the approximation as $h \to 0$ for the (linear) Schrödinger equation. Specifically, let $\Psi(x, t, h)$ be an exact solution (or an h-approximate solution, $h \to 0$) to nonlinear equation (1). By the classical phase trajectory in $\mathbb{R}^{2n}_{p,x}$ of a quantum particle in the state $\Psi = \Psi(x, t, h)$, we mean a vector function $z_\Psi(t, h) = (P(t, h), X(t, h))$ that depends smoothly on h and has components that are the means of the coordinate and momentum operators in the state Ψ: $P(t, h) = \langle\Psi, -ih\nabla_x\Psi\rangle$ and $X(t, h) = \langle\Psi, x\Psi\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{L}_2(\mathbb{R}^n_x)$.

1. EQUATION OF CLASSICAL MECHANICS FOR THE HARTREE EQUATION IN THE CLASS OF SEMICLASSICALLY LOCALIZED STATES

Let $Z(t, h) = (P(t, h), X(t, h))$ be a smooth one-parameter family of phase trajectories in $\mathbb{R}^{2n}_{p,x}$ (with parameter h). Define a class K of functions depending on $x \in \mathbb{R}^n$, $t \geq 0$, and $h \in [0, 1)$, which is called the class of functions semiclassically localized on the trajectory $Z(t, h)$ as $h \to 0$:

$$K = \Big\{\Phi, \Phi = \phi\Big(\frac{x - X(t, \hbar)}{\sqrt{\hbar}}, t\Big)$$

$$\times \exp\Big[\frac{i}{\hbar}(S(t, \hbar) + (P(t, \hbar), x - X(t, \hbar)))\Big]\Big\}, \tag{8}$$

where $S(t, \hbar)$ is a smooth real function of t and h, $S(0, \hbar) = 0$, and $\phi(\xi, t)$ is in the Schwartz space $S(\mathbb{R}^n)$ with respect to ξ.

Lemma 1. *For functions Φ from the class K, the centered moments $\Delta_\alpha(t, h)$ of order $|\alpha|$, $\alpha \in \mathbb{Z}^{2n}_+$ satisfy the estimates*

$$\Delta_\alpha(t, h) = \Delta_\alpha^\Phi(t, h) = \frac{\langle\Phi\{\hat{\Delta z}\}^\alpha\Phi\rangle}{\|\Phi\|^2} = O(h^{|\alpha|/2}),$$

$$h \to 0, \quad |\alpha| \neq 0, \quad \Delta_\alpha(t, h) = 0, \quad |\alpha| = 1,$$

where $\{\hat{\Delta z}\}^\alpha$ is an operator with the Weyl symbol $(\Delta z)^\alpha$, $\Delta z = z - Z(t, h) = (\Delta p; \Delta x)$, $\Delta x = x - X(t, h)$, and $\Delta p = p - P(t, h)$.

Theorem 1. *The classical system* (mod$h^{3/2}$) *associated with the nonlinear operator $\hat{H}_\kappa$ and the class K*

(*which is called the Hamilton–Ehrenfest system*) *has the form*

$$\dot{p} = -\nabla_x U(x) - \kappa \nabla_\tau V(0)$$
$$-\nabla_x\left[\frac{1}{2}\mathrm{Sp}(U''_{xx}(x) + 2\kappa V''_{xx}(x-y))\sigma_{xx}\right]\bigg|_{y=x}, \tag{9}$$
$$\dot{x} = p,$$
$$\dot{\Delta} = JM_\kappa(x)\Delta - \Delta M_\kappa(x)J, \quad (p, x) \in \mathbb{R}^{2n}.$$

Here, Δ *is a* $2n \times 2n$ *real block symmetric matrix*,

$$\Delta = \begin{pmatrix} \sigma_{pp} & \sigma_{px} \\ \sigma_{xp} & \sigma_{xx} \end{pmatrix}, \quad \sigma^t_{xp} = \sigma_{px}, \quad \sigma^t_{xx} = \sigma_{xx},$$
$$\sigma^t_{pp} = \sigma_{pp},$$
$$M_\kappa(x) = \begin{pmatrix} \mathbb{E}_n & 0 \\ 0 & U''_{xx}(x) + \kappa V''_{\tau\tau}(0) \end{pmatrix}, \tag{10}$$
$$U''_{xx}(x) = \left(\frac{\partial^2 U(x)}{\partial x_i \partial x_j}\right)_{n\times n},$$

and J is a standard symplectic $2n \times 2n$ *matrix.*

Proof sketch. Since the evolution operator of Eq. (1) is unitary, for solutions $\Psi(x, t, h)$ of this equation, we have the Ehrenfest "theorem" for the quantum averages $\langle \hat{A} \rangle(t, h) = \langle \Psi, \hat{A}\Psi\rangle$, where $\hat{A}$ is a self-adjoint operator in $\mathbb{L}_2(\mathbb{R}^n_x)$:

$$\frac{d}{dt}\langle \hat{A}\rangle(t, h) = \frac{i}{h}\langle[\hat{H}_0, \hat{A}]\rangle$$
$$+\kappa\frac{i}{h}\left\langle \int_{\mathbb{R}^n} dy\, \Psi^*(y, t, h)[V(x-y), \hat{A}]\Psi(y, t, h)\right\rangle. \tag{11}$$

Assume that there exists an asymptotic (modh^N), $N \geq 2$ solution $\Psi(x, t, h) \in K$ to Eq. (1). Using relation (11) for operators from the Heisenberg–Weyl universal enveloping algebra with generators $\hat{I}$, $\hat{\Delta} x_k = \hat{x}_k - X_k(t, h)$, and $\hat{\Delta} p_k = \hat{p}_k - P_k(t, h)$, where $k = 1, 2, \ldots, n$ and $\hat{I}$ is the identity operator, we expand the operator $\hat{H}_\kappa$ in a Taylor series in powers of $\hat{\Delta} x^{\alpha_x}$ and $\hat{\Delta} p^{\alpha_p}$ (where $(\alpha_x, \alpha_p) = \alpha \in \mathbb{Z}^{2n}_+$), take into account the estimates from Lemma 1, and neglect the centered quantum moments of order α ($|\alpha| \geq 3$) to obtain system (9).

Remark. The dynamic variables Δ in system (9), namely, $(\sigma_{pp})_{km} = \langle\Psi, \hat{\Delta} p_k \hat{\Delta} p_m \Psi\rangle$, $(\sigma_{xx})_{km} = \langle\Psi, \hat{\Delta} x_k \hat{\Delta} x_m \Psi\rangle)$, and $(\sigma_{xp})_{km} = \frac{1}{2}\langle\Psi, (\hat{\Delta} x_k \hat{\Delta} p_m + \hat{\Delta} p_m \hat{\Delta} x_k)\Psi\rangle$ for $k, m = 1, 2, \ldots, n$, which are of order h, $h \to 0$, take into account the influence of quadratic quantum fluctuations of coordinates and momenta about their limiting values $X(t, 0)$ and $P(t, 0)$. Thus, the classical equations depend regularly on the small parameter h of the semiclassical approximation in the quantum problem. This fact is of key importance for the construction of localized asymptotics of the Hartree equation in the class of functions K (8).

2. COHERENT STATES OF THE HARTREE EQUATION

For problem (1), (3), let $\Lambda^0_t(\kappa, h)$ denote the projection onto $\mathbb{R}^{2n}_{p,x}$ of the solution $(P(t, h), X(t, h), \Delta(t, h))$ to the Cauchy problem on the interval $[0, T]$, $T > 0$ for Hamilton–Ehrenfest equation (9) with initial data induced by $\Psi_0(x, h)$ (3): $X|_{t=0} = \langle\Psi_0, x\Psi_0\rangle = x_0$, $P|_{t=0} = \langle\Psi_0, -ih\nabla\Psi_0\rangle = p_0$, and $\Delta|_{t=0} = \Delta_{\Psi_0}(h)$, where the $n \times n$ blocks of this matrix obviously have the form $\sigma_{xx}|_{x=0} = \frac{h}{4}\mathbb{E}_{n\times n}$, $\sigma_{pp}|_{x=0} = \frac{h}{4}[B_0 B_0^+ + B_0^* B_0^t]$, and $\sigma_{px}|_{t=0} = \frac{h}{4}[B_0 + B_0^*]$. Here, the symbols +, *, and t on the matrix B_0 denote the Hermite conjugate, complex conjugate, and transpose, respectively. By analogy with linear theory (see [7], Introduction), the Hamilton system is called a variational system with a self-action:

$$\begin{pmatrix} \dot{B} \\ \dot{C} \end{pmatrix} = JM_\kappa(X_0(t))\begin{pmatrix} B \\ C \end{pmatrix}, \tag{12}$$

where $X_0(t)$ is the leading term of the expansion $X(t, h) = X_0(t) + hX_1(t) + O(h^2)$ and the matrix $JM_\kappa(x)$ is defined in (10). Denote by $B = B(t)$ and $C = C(t)$ the matrix solution to this system with the initial data $B|_{t=0} = B_0$ and $C|_{t=0} = \mathbb{E}_n$.

Theorem 2. *Let the potentials* $U(x)$ *and* $V(\tau)$ *be functions of the class* $C^{(3)}(\mathbb{R}^n)$.

Then, for $t \in [0, T]$, *the asymptotic solution* (mod$h^{3/2}$), $h \to 0$ *to problem* (1), (3) *in the* $\mathbb{L}_2(\mathbb{R}^n)$ *norm of the right-hand side is localized in the neighborhood of the point* $\Lambda^0_t(\kappa, h)$ *and has the form*

$$\Psi_\nu(x, t, h)$$
$$= N_\nu \exp\left[\frac{i}{h}(S_\kappa(t, h) + (P(t, h), x - X(t, h)))\right]$$
$$\times \exp\left[\frac{i}{2h}(x - X(t, h), BC^{-1}(t)(x - X(t, h)))\right]\frac{1}{\sqrt{\det C(t)}}. \tag{13}$$

Here, the real phase $S_\kappa(t, h)$ (*the analogue of the action in* (6)) *is*

$$S_\kappa(t,h) = \int_0^t \left(\frac{\dot{X}^2}{2}(\tau,h) - U(X(\tau,h))\right)d\tau - \kappa V(0)t$$
$$-\frac{\kappa}{2}\int_0^t \mathrm{Sp}(V''_{\tau\tau}(0)\sigma_{xx}(\tau,h)d\tau. \qquad (14)$$

To prove the theorem, we expand the equation coefficients in Taylor series in the neighborhood of $\Lambda_t^0(\kappa, h)$, apply Theorem 1, take into account that the last equation in system (9) is the well-known Lax equation in the inverse scattering method with respect to the matrix ΔJ, and then apply Lemma 2 below.

Lemma 2. *Let* $A(t)$ *be the Cauchy matrix of system* (12). *Then the solution to the Cauchy problem for the system* $\dot{\Delta} = JM_\kappa(X_0(t))\Delta - \Delta M_\kappa(X_0(t))J$, $\Delta|_{t=0} = \Delta_{\Psi_0}(h)$ *is given by the formula*

$$\Delta(t,h) = A(t)\Delta_{\Psi_0}A^+(t). \qquad (15)$$

3. SEMICLASSICAL SOLITONS OF THE HARTREE EQUATION WITH NO EXTERNAL FIELD

Assuming that $U(x) = 0$ in (2), we state the problem for Eq. (1) with more general Cauchy data than those in (3). Namely, let $\Psi|_{t=0}$ be the Fock states of a multidimensional oscillator:

$$\Psi|_{t=0} = \Psi_\nu(x,h) = N_\nu\Psi_0(x,h)\phi_\nu\left(\frac{x-x_0}{\sqrt{h}}\right), \quad \nu \in \mathbb{Z}_+^n, \qquad (16)$$

where $\Psi_0(x, h)$ is defined in (3) and ϕ_ν is a multidimensional Hermite polynomial of multi-index ν, which is represented in terms of the generalized creation operators $\phi_\nu\left(\frac{x-x_0}{\sqrt{h}}\right) = \hat{\Lambda}^{+\nu}(0)1$, where $\hat{\Lambda}^{+\nu}(0)1 = [\hat{\Lambda}_1^+(0)]^{\nu_1}\dots[\hat{\Lambda}_n^+(0)]^{\nu_n}1$ and $\hat{\Lambda}_j^+(0) = \frac{1}{\sqrt{h}}\left[-ih\frac{\partial}{\partial x_j} + 2i(\mathrm{Im}w_j, x-x_0)\right]$ (see, e.g., [12]). Here, w_j $(j = 1, 2, \dots, n)$ are the column vectors of B_0. In the case $U(x) = 0$, the system of equations for Δ in (9) has constant coefficients. Therefore, the Hamilton–Ehrenfest system, as well as the variational system in self-action (12) with the initial data induced by initial function (16), can easily be integrated. The initial conditions for system (9) have the form

$$X|_{t=0} = x_0, \quad P|_{t=0} = p_0, \quad \sigma_{xx}|_{t=0} = \frac{h}{4}D_\nu,$$
$$D_\nu = ((2\nu_j+1)\delta_{ij})_{n\times n},$$
$$\sigma_{pp}|_{t=0} = \frac{h}{4}[B_0D_\nu B_0^+ + B_0^*D_\nu B_0^t], \qquad (17)$$
$$\sigma_{px}|_{t=0} = \frac{h}{4}[B_0D_\nu + D_\nu B_0^t].$$

Assume that system (12) is stable; i.e., the self-action potential satisfies the condition

$$\kappa V''_{\tau\tau}(0) > 0. \qquad (18)$$

Denote by ω_j^2 $(j = 1, 2, \dots, n)$ the eigenvalues of the matrix $\kappa V''_{\tau\tau}(0)$.

Lemma 3. *Let condition* (18) *be satisfied, and let the frequencies be not resonant; i.e.,*

$$\omega_l \neq \omega_m, \quad l, m = 1, 2, \dots, n, \quad l \neq m. \qquad (19)$$

Then the solution to Cauchy problem (9), (17) *is given by the formulas*

$$X(t,h) = x_0 + p_0t + \frac{(a_0+ha_1)t^2}{2}$$
$$+ h\sum_\pm \sum_{l,m=1,l\neq m}^n (d_{l,m}^\pm\cos((\omega_l\pm\omega_m)t) + k_{l,m}^\pm\sin((\omega_l\pm\omega_m)t))$$
$$+ h\sum_{l=1}^n (e_l\cos((2\omega_l)t) + s_l\sin(2(\omega_l)t)),$$
$$P(t,h) = \dot{X}(t,h), \quad a_0 = -\kappa\nabla_\tau V(0),$$
$$\sigma_{xx}(t,h) = h\Gamma_0 + \sum_\pm \sum_{l,m=1,l\neq m}^n (\cos((\omega_l\pm\omega_m)t)A_{l,m}^\pm - \sin((\omega_l\pm\omega_m)t)B_{l,m}^\pm)$$
$$+ \sum_{l=1}^n A_{l,l}^+\cos(2\omega_lt) - B_{l,l}^+\sin(2\omega_lt),$$

where Γ_0, $A_{l,m}^\pm$, *and* $B_{l,m}^\pm$ *are* $n\times n$ *constant real matrices and* a_1, $d_{l,m}^\pm$, $k_{l,m}$, s_l, *and* e_l $(l, m = 1, 2, \dots, n)$ *are real vectors from* $\mathbb{R}^n$, *whose explicit form can easily be derived using formulas* (15) (*for* σ_{xx}) *with the Cauchy matrix*

$$A = \begin{pmatrix} \cos[t\sqrt{R}] & R^{-1/2}\sin[t\sqrt{R}] \\ -\sqrt{R}\sin[t\sqrt{R}] & \cos[t\sqrt{R}] \end{pmatrix},$$
$$R = \kappa V''_{\tau\tau}(0)$$

and the first pair of equations in system (9) (*for* $X(t, h)$, $P(t, h)$).

Lemma 3 and Theorem 2 imply the following result.

Theorem 3. *Let conditions* (18) *and* (19) *be satisfied, and let* $V(\tau) \in C^3(\mathbb{R}^n)$.

Then the asymptotic semiclassical soliton solution to problem (1), (16) *has the form*

$$\Psi_\nu(x, t, h)$$

$$= N_\nu \exp\left[\frac{i}{h}(S_\kappa(t, h) + (P(t, h), x - X(t, h)))\right]$$

$$\times \exp\left[\frac{i}{2h}(x - X(t, h), \dot{C}C^{-1}(t)(x - X(t, h)))\right]$$

$$\times \frac{1}{\sqrt{C(t)}}\hat{\Lambda}^{+\nu}(t)1, \qquad (20)$$

$$S_\kappa(t, h) = \int_0^t \frac{\dot{X}^2}{2}(\tau, h)d\tau - \kappa V(0)t$$

$$-\frac{\kappa}{2}\int_0^t \mathrm{Sp}(V''_{\tau\tau}(0)\sigma_{xx}(\tau, h))d\tau.$$

Here, the functions $X(t, h)$ *and* $\sigma_{xx}(t)$ *are defined in Lemma* 3; $C(t)$ *satisfies the system* $\ddot{C} + \kappa V''(0)C = 0$, $C(0) = \mathbb{E}_n$, $\dot{C}(0) = B_0$, $\hat{\Lambda}^{+\nu}(t) = [\hat{\Lambda}_1^+(t)]^{\nu_1} \dots [\hat{\Lambda}_n^+(t)]^{\nu_n}$, $\hat{\Lambda}_j^+(t) = \frac{1}{\sqrt{h}}((z_j^*(t), \hat{p}) - (\dot{z}_j^*(t) - \dot{C}C^{-1}(t)z_j^*(t), (x - X(t, 0))))$; *and* $z_j(t)$, $j = 1, 2, \dots, n$ *are the columns of the matrix* $C(t)$.

CONCLUSIONS

In nonlinear quantum mechanics with the model Hamiltonian $\hat{H}_\kappa$ given by (2), we have derived explicit formulas (20) for Gaussian wave packets that do not disperse in the semiclassical approximation, at least, in the case of translation-invariant nonresonant convex potentials. The construction of such wave packets in the case of an arbitrary external electromagnetic field requires an additional study. In the case of homogeneous fields, the formulas for nondispersive solitons will be given elsewhere.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project nos. 05-01-000968 and INI 05-01-2202. M.F. Kondrat'eva acknowledges the support of the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

1. Y. Lai and H. A. Haus, Phys. Rev. A **40**, 854–866 (1989).
2. L. P. Pitaevskii, Usp. Fiz. Nauk **168**, 641–653 (1988).
3. A. S. Davydov, *Solitons in Molecular Systems* (Naukova Dumka, Kiev, 1984; Kluwer, Dordrecht, 1991).
4. V. D. Lakhno and N. S. Fialko, *Computers and Supercomputers in Biology* (Izhevsk, Moscow) [in Russian].
5. E. H. Lieb and B. Simon, Commun. Math. Phys. **53**, 185 (1977).
6. J. M. Chadam and R. T. Glassey, J. Math. Phys. **16**, 1122 (1975).
7. V. P. Maslov, *Complex Markov Chains and Feynman Continual Integral* (Nauka, Moscow, 1975) [in Russian].
8. V. P. Maslov, Sovrem. Probl. Mat. **11**, 153 (1978).
9. M. V. Karasev and V. P. Maslov, Sovrem. Probl. Mat. **13**, 145 (1979).
10. M. V. Karasev and A. V. Pereskokov, Izv. Akad. Nauk, Ser. Mat. **65**, 33–72 (2001).
11. V. P. Maslov, *Operator Methods* (Nauka, Moscow, 1973) [in Russian].
12. V. P. Maslov, *The Complex WKB Method for Nonlinear Equations* (Nauka, Moscow, 1977; Birkhäuser, Basel, 1994).
13. P. Ehrenfest, Ztschr. Ph. **45**, 455–457 (1927).
14. V. V. Belov and A. Yu. Trifonov, Ann. Phys. (New York) **246** (2), 231–280 (1996).
15. V. V. Belov, A. Yu. Trifonov, and A. V. Shapovalov, Teor. Mat. Fiz. **130**, 460 (2002).