COGNITIVE DEVELOPMENT OF PROOF

David Tall, Oleksiy Yevdokimov, Boris Koichu, Walter Whiteley, Margo Kondratieva, Ying-Hao Cheng

1. Introduction

In this chapter our aim is to seek how the growing individuals develop ideas of proof appropriate for their level of maturity at the time, and to see how these ideas develop over the long term, from the young child to the adult user of mathematics and on to the research mathematician. We will focus on the ways in which developing individuals build from real world perceptions and actions to a mental world of sophisticated mathematical knowledge.

This chapter is consonant with the four plenaries presented to the ICMI conference on proof. Longo sees the formalism of modern mathematics growing out of the actions and perceptions of the biological human brain. Grabiner reports significant examples of development in history as mathematical experts build on their experience to develop new mathematical constructs. Borwein observes the changing nature of mathematical thought now that we have computer technology to perform highly complex computations almost immediately and to represent information in dynamic visual ways. Quinn underlines the mathematical concern that proof at the highest level needs to be fundamentally based on the precision of the axiomatic method.

Mathematical proof develops in many different forms both in historical time and in the development of any individual. Various degrees of proof are suggested in school mathematics by terms such as ‘show’, ‘justify’, ‘explain’, ‘prove from first principles’. Rather than begin by debating the difference between them, we will use the word ‘proof’ in its widest sense and analyse the changes in its meaning as the individual matures.

The outline of the chapter is as follows. In Section 2 we consider how the nature of proof is envisioned by professional mathematicians and by novices in mathematics. This is followed, in Section 3, by introducing a three-facet conceptual framework based on perceptions, operations and formal structures that enables us to adequately consider the cognitive journey from the child to the adult, and from the novice to professional mathematician. Section 4 deals with the development of proof from human experience. Section 5 considers the development of proof in the context of Euclidean and non-Euclidean geometry. Section 6 details the increasing sophistication of proofs in arithmetic and algebra from proof using specific calculations, generic arguments, algebraic manipulation and on to algebraic proof based on the rules of arithmetic. Section 7 addresses the development of proof in undergraduates and on to research mathematics, followed by a summary of the whole development.

2. PERCEPTIONS OF PROOF

2.1 What is proof for mathematicians?

Mathematics is a diverse and complex activity, spanning a range of contexts from everyday practical activities, through more sophisticated applications and on to the frontiers of mathematical research. At the highest level of mathematical research, discovery and proof of new theorems may be considered to be the summit of mathematical practice. In the words of three mathematicians:

> Proofs are to mathematics what spelling (or even calligraphy) is to poetry. Mathematical works do consist of proofs, just as poems do consist of characters (Arnold, 2000, p. 403).

> 'Ordinary mathematical proofs'—to be distinguished from formal derivations—are the locus of mathematical knowledge. Their epistemic content goes way beyond what is summarised in the form of theorems (Rav, 1999, p. 5).

> The truth of a mathematical claim rests on the existence of a proof. Stated this way, such a criterion is absolute, abstract, and independent of human awareness. This criterion is conceptually important, but practically useless (Bass, 2009).

We chose these quotations because they suggest not only the importance of proofs in mathematics, but also reveal the debate on the role and nature of proofs within the mathematical community. We learn from the first quotation that proofs are fundamental to the structure of mathematics. The second tells us that the usual ('ordinary') proofs produced by mathematicians have subtleties of meaning that go beyond the application of logic. The third implies that mathematical proof as an absolute argument is conceptually important but may not be what occurs, or even what is achievable, in practice.

A formal proof, in the sense of Hilbert (1928/1967), is a sequence of assertions, the last of which is the theorem that is proved and each of which is either an axiom or the result of applying a rule of inference to previous formulas in the sequence; the rules of inference are so evident that the verification of the proof can be done by means of a mechanical procedure. Such a formal proof can be expressed in first-order set-theoretical language (Rav, 1999). Dawson (2006, p.271) observed that 'formal proofs appear almost exclusively in works on computer science or mathematical logic, primarily as objects to study to which other, informal, arguments are applied.'

An ordinary mathematical proof consists of an argument to convince an audience of peer experts that a certain mathematical claim is true and, ideally, to explain why it is true (cf. Dawson, 2006). Such ordinary proofs can be found in mathematics research journals as well as in school and university-level textbooks. They utilize second or higher-order logic (Shapiro, 1991), but often contain conceptual bridges between parts of the argument rather than explicit logical justification (Rav, 1999). Sometimes a convincing argument for peer experts does not constitute a formal proof, only a justification that a proof can be constructed, given sufficient time, incentive, and resources (Bass, 2009).

From the epistemic point of view, a proof for mathematicians involves thinking about new situations, focusing on significant aspects, using previous knowledge to put new ideas together in new ways, consider relationships, make conjectures, formulate definitions as necessary and to build a valid argument.

In summary, contemporary mathematicians' perspectives on proof are sophisticated yet build on the broad development to 'convince yourself, convince a friend, convince an enemy,' in mathematical thinking at all levels (Mason, Burton & Stacey, 1982). To this should be added Mason's further insight that mathematicians are able to develop an 'internal enemy'– a personally constructed view of mathematics that not only sets out to convince doubters, but shifts to a higher level attempting to make sense of the mathematics itself.

2.2 **What is proof for growing individuals?**

Children or novices do not initially think deductively. The young child begins by interacting with real-world situations, perceiving with the senses including vision, hearing, touch, acting on objects in the world, pointing at them, picking them up, exploring their properties, developing language to describe them.

In parallel with the exploration of objects, the child explores various operations on those objects: sorting, counting, sharing, combining, ordering, adding, subtracting, multiplying, dividing, developing the operations of arithmetic and on to the generalized arithmetic of algebra. This involves observing regularities of the operations, such as addition being independent of order and various other properties that are collected together and named 'the rules of arithmetic'. Properties that were seen as natural occurrences during physical operations with objects are subtly reworded to become rules that must be obeyed. Children may even establish their own rules which are not necessarily correct yet, nevertheless, help them to make initial steps towards deductive thinking. For instance, a four-year old child attempting to persuade her parents that, 'I am older (than another child) because I am taller.' (Yevdokimov, 2010.)

Over the longer term, the convincing power of the emerging rules for the child is often rooted not only in the observation that the rules 'work' in all available situations, but in the external role of authorities such as a parent, a teacher or a textbook (Harel & Sowder, 1998).

It is only much later—usually at college level—that axiomatic formal proof arises in terms of formal definitions and deductions. Unlike earlier forms of proof, the axioms formulated express only the properties required to make the necessary deduction, and are no longer restricted to a particular context, but to *any* situation where the only requirement is that the axioms are satisfied.

This yields a broad categorization of three distinct forms of proof: using figures, diagrams, transformations and deduction in geometry, using established rules of arithmetic in algebra initially encountered in school, and in axiomatically defined structures met only by mathematics majors in university. Our task is to fit the development of proof in general and these three forms of proof in particular into a framework of cognitive growth.

3. THEORETICAL FRAMEWORK

We begin by considering a brief overview of theories of cognitive growth relevant to the development of proof. Then we focus on a single idea that acts as a template for the cognitive development of proof in a range of contexts: the notion of a *crystalline concept*. This will then be used as a foundation for the development of mathematical thinking over time, in which the cognitive development of proof plays a central role.

3.1 Theories of cognitive growth

There are many theories of cognitive growth offering different aspects of development over the longer term. Piaget, the father of cognitive approaches to development, sees the child passing through various stages, from the sensori-motor, through concrete operations, then on to formal operations.

Lakoff (1987) and his co-workers (Lakoff & Johnson, 1999; Lakoff & Núñez, 2000) claim that all human thought is embodied through the sensori-motor functions of the human individual and builds linguistically through metaphors based on human perception and action.

Harel and Sowder (1998, 2005) describe cognitive growth of proof in terms of the learner's development of *proof schemes* – relatively stable cognitive/affective configurations responsible for what constitutes ascertaining and persuading an individual of the truth of a statement at a particular stage of mathematical maturation. A broadly-based empirical study found a whole range of different proof schemes, some categorized as 'external conviction', some 'empirical' and some 'analytical'.

Van Hiele (1986) focuses specifically on the development of Euclidean geometry, proposing a sequence of stages from the recognition of figures, through their description and categorization, the more precise use of definition and construction using ruler and compass and on to the development of a coherent framework of Euclidean deductive proof.

There are also theories of development of symbolism through the encapsulation of processes (such as counting) into concepts (such as number) that reveal a different kind of development in arithmetic and the generalized arithmetic of algebra (Dubinsky & McDonald, 2001; Gray & Tall, 1994; Sfard, 1991).

Many theoretical frameworks speak of multiple representations (or registers) that operate in different ways (e.g., Goldin, 1998; Duval, 1986). The two distinct forms of development through the global visual-spatial modes of operation on the one hand and the sequential symbolic modes of operation on the other can operate in tandem with each supporting the other (Paivio, 1991). Bruner's three modes of communication—enactive, iconic, symbolic—also presume different ways of operating: the sensori-motor basis of enactive and iconic linking to the visual and spatial, and the symbolic forms including not only language but the sub-categories of number and logic (Bruner, 1966).

The hypothesis about distinct cognitive structures for language/symbolism and for visualisation has received empirical support by means of neuroscience.

For instance, figure 1 shows the areas of the brain stimulated when responding to the problem '5×7', which coordinates an overall control in the right hemisphere and the language area in the left recalling a verbal number fact, and the response to the problem 'Is 5×7>25?', which uses the visual areas at the back to compare relative sizes (Dehaene, 1997). This reveals human thinking as a blend of global perceptual processes that enable us to 'see' concepts as a gestalt (Hoffmann, 1998), and sequential operations that we can learn to perform as mathematical procedures.

The brain operates by the passage of information between neurons where connections are excited to a higher level when they are active. Repeated use strengthens the links chemically so that they are more likely to react in future and build up sophisticated knowledge structures. Metzoff, Kuhl, Movellan and Sejnowski (2009) formulate the child's learning in terms of the brain's implicit recognition of statistical patterns:

> Recent findings show that infants and young children possess powerful computational skills that allow them automatically to infer structured models of their environment from

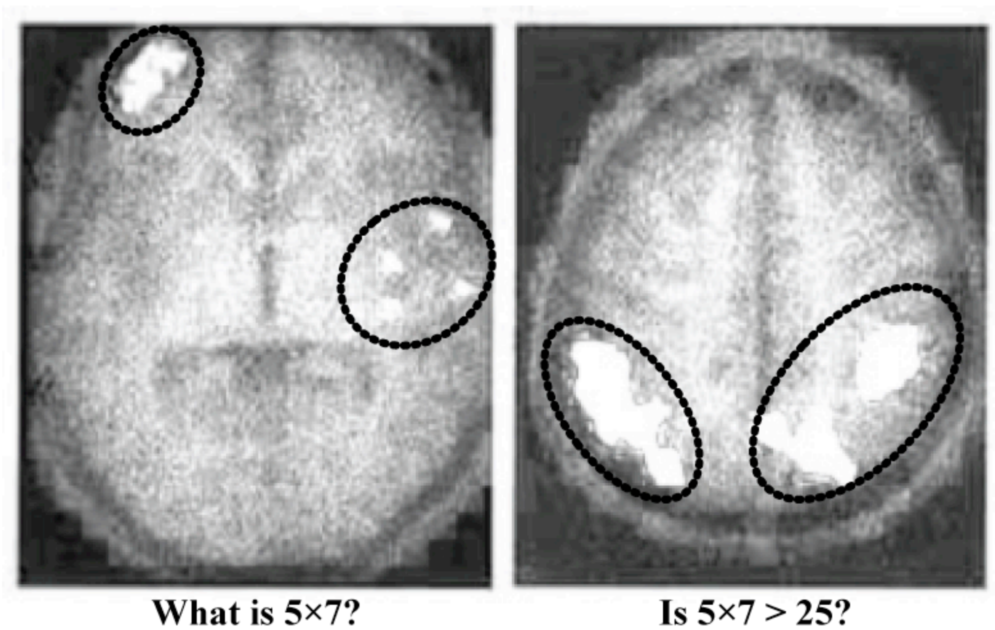


Figure 1: Areas of the brain recalling a fact and performing a comparison

> the statistical patterns they experience. Infants use statistical patterns gleaned from experience to learn about both language and causation. Before they are three, children use frequency distributions to learn which phonetic units distinguish words in their native language, use the transitional probabilities between syllables to segment words, and use covariation to infer cause-effect relationships in the physical world (p. 284).

The facility for building sophisticated knowledge structure is based on a phenomenal array of neuronal facilities for perception and action that are present in the new-born child and develop rapidly through experience in the early years. Tall (2008) refers to these abilities as 'set-befores' (because they are set before birth as part of our genetic inheritance and develop through usage) as opposed to 'met-befores' that arise as a result of previous experience and may be supportive or problematic when that experience is used in new contexts. He hypothesizes that three major set-befores give rise to three distinct developments of mathematical thinking and proof through:

(a) **recognition** of similarities, differences and patterns through *perception*,
(b) **repetition**: the ability to learn complex sequences of operation through *action*, and
(c) the development of **language** to enable perception and action to be expressed and conceived in increasingly subtle ways.

In the development of mathematical thinking, these three set-befores combine to give the three different ways of constructing mathematical concepts. Language enables the verbal *categorization* of the perception of figures in geometry and other aspects of mathematics. It enables the *encapsulation* of processes as concepts to compress processes that occur in time into manipulable mental objects in arithmetic and algebra. And it enables *definition* of concepts, both in terms of *observed* properties of perception and action in school mathematics and also of the *proposed* set-theoretic properties of axiomatic systems.

As the authors of this chapter reflected together on the total development of mathematical proof and the many cognitive theories available with various links and differences, we came to the view that there is a single broad developmental template that underlies them all, from early perceptions and actions, to the various forms of Euclidean, algebraic and axiomatic proof.

3.2 Crystalline concepts

The foundational idea that underpins our framework can be introduced using a single specific example that starts simply and becomes more general until it provides a template for the total development of mathematical proof, from the first perceptions and actions of the child to the Platonic concepts of Euclidean geometry, algebraic proofs based on the rules of arithmetic, and formal proofs in axiomatic mathematics.

Our example is the notion of an isosceles triangle as seen by a maturing child. At first the child perceives it as a single gestalt. It may have a shape that is broad at the bottom, narrowing to the top with two equal sides, two equal angles and an axis of symmetry about a line down the middle. If the triangle is cut out of paper, it can be folded over this central line to reveal a complete symmetry. The child can learn to *recognise* various isosceles triangles and *describe* some of their properties. However, at this early stage, all these properties occur *simultaneously*, they are not linked together by cause-effect relationship.

In order to be able to make sense of coherent relationships, the child needs to build a growing *knowledge structure* (or *schema*) of experiences involving perceptions and actions that relate to each other. As the child develops, some of the properties described may become privileged and used as a *definition* of a particular concept. For instance, one may describe an isosceles triangle to be 'a triangle with two equal sides.' Now the child may use this criterion to test whether new objects are isosceles triangles. For the child, the 'proof' that a particular triangle is isosceles is that anyone can *see* that it has two equal sides.

Subsequently, the child may be introduced to a more sophisticated knowledge structure, such as physically placing one triangle on top of another and verbalizing this as the principle of congruence. As the child becomes aware of more and more properties of an isosceles triangle and his or her conceptions of relationships develop, it may become possible to see relationships between properties and to use appropriate principles based on constructions and transformations to *deduce* some properties from the others. The idea emerges that it is not necessary to include all known properties in a definition. An isosceles triangle defined only in terms of equal sides, with no mention of the angles, can now be proved by the principle of congruence to have equal angles.

Other, more sophisticated, properties may be deduced using similar techniques. For instance, for an isosceles triangle, the perpendicular bisector of the base can be proved to pass through the vertex, or the bisector of the vertex angle can be proved to meet the base in the midpoint at right angles. Further proofs show that any of these deduced properties mentioned may be used as alternative definitions. At this point there may be several different possible definitions that are now seen to be *equivalent*. It is not that the triangle has a single definition with many consequences, but that it has many equivalent definitions any one of which can be used as a basis for the theory. One of these properties, usually the simplest one to formulate, is then taken as the primary *definition* and then all other properties are deduced from it.

Then something highly subtle happens: the notion of isosceles triangle—which originally was a single gestalt with many simultaneous properties, and was then defined using a single specific definition—now matures into a fully unified concept, with many properties linked together by a network of relationships based on deductions.

We introduce the term *crystalline concept* for such a phenomenon. A crystalline concept may be given a working definition as 'a concept that has an internal structure of constrained relationships that cause it to have necessary properties as part of its context.' A typical crystalline concept is the notion of an idealized Platonic figure in Euclidean geometry. However, as we shall see, crystalline concepts are also natural products of development in other forms of proof such as those using symbol manipulation or axiomatic definition and deduction.

The long-term formation of crystalline concepts matures through the construction of increasingly sophisticated knowledge structures, as follows:

- **perceptual recognition** of phenomena where objects have simultaneous properties,
- **verbal description** of properties, often related to **visual** or **symbolic representations**, to begin to think about specific properties and relationships,
- **definition and deduction,** to define which concepts satisfy the definition and to develop appropriate principles of proof to deduce that one property implies another,
- realising that some properties are **equivalent** so that the concept now has a structure of equivalent properties that are related by deductive proof,

and finally that

- these properties are different ways of expressing an underlying **crystalline concept** whose properties are now connected together by deductive proof.

Crystalline concepts are not isolated from each other. The deductive network of one crystalline concept may intersect with another. For instance, a child may begin to perceive an isosceles triangle as a representative of a broader class of objects—triangles in general—and compare the definitions and properties involved to find new deductive journeys relating concepts that may not always be equivalent. This leads to the distinction between direct and converse deduction of properties and further developments of deductive relationships (Yevdokimov, 2008).

This development represents a broad trend in which successive stages are seen as developing and interrelating one with another, each correlated within the next (Figure 2.) This is represented by the deepening shades of gray as increasingly sophisticated knowledge structures are connected together as each new stage develops and matures. On the right is a single vertical arrow whose shaft becomes more firmly defined as it is traced upwards. Such an arrow will be used subsequently to denote the long-term development from initial recognition of a phenomenon in a given mathematical context through increasing sophistication to deductive knowledge structures. The development is a natural human growth and should not be seen as a rigid growth of discrete levels, rather as a long-term growth in maturity to construct a full range of mathematical thinking from perceptual recognition to deductive reasoning.

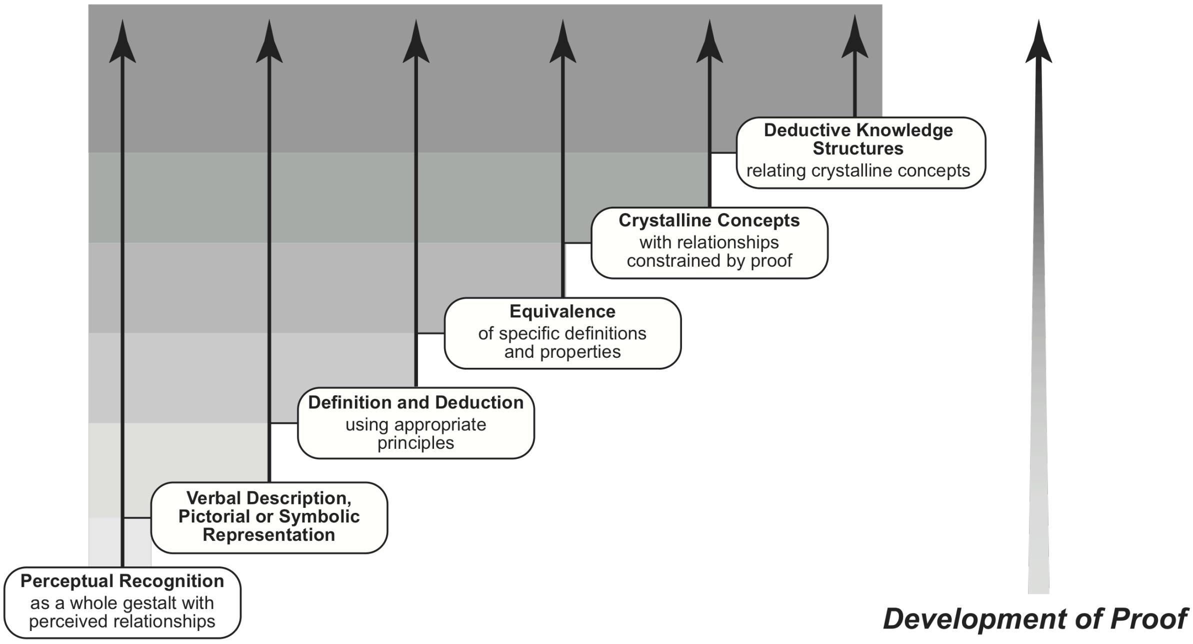


Figure 2: The broad maturation of proof structures

Such a framework is consistent with van Hiele's theory of the teaching and learning of geometry, but it goes further by making explicit the final shift from equivalent figures drawn on the page or in sand to conceiving them as instances of crystalline concepts in the form of perfect platonic objects.

The framework is also consistent with the development of other forms of geometry that arise in new contexts (projective geometry drawing a representation of a three-dimensional scene on a plane, spherical geometry on the surface of a sphere, elliptic and hyperbolic geometries in appropriate contexts, finite geometries, algebraic geometries and so on).

These new frameworks reveal that the 'appropriate principles of deduction' may differ in different forms of geometry. For instance, in spherical geometry, while there is a concept of congruence of spherical triangles, there are no parallel lines, and proof is a combination of embodied experience operating on the surface of a sphere coupled with symbolic computations using trigonometry.

In arithmetic and algebra the symbols have a crystalline structure that the child may begin to realize through experience of counting collections which are then put together or taken away. The sum 8+6 can be computed first by counting, and in this context it may not yet be evident that 8+6 gives the same result as 6+8. As the child builds more sophisticated relationships, it may later be seen as part of a more comprehensive structure in which 8+2 makes 10 and, decomposing 6 into 2 and 4, gives 8+6 is 10+4, which is 14. One might say that 8+6 or 6+8 or 10+4 or various other arithmetic expressions equal to 14 are all *equivalent*, but it is cognitively more efficient to say simply that they are *the same*. Gray and Tall (1994) referred to such symbols as different ways of representing a *procept*, where the symbols can stand dually for a *pro*cess and the con*cept* output by the process. Here the various symbols, 8+6, 10+4, and so on, all represent the same underlying procept that operates as the flexible crystalline concept '14'. This crystalline structure is then used to derive more complex calculations from known facts in a deductive knowledge structure.

Algebra arises as generalized arithmetic, where operations having the same effect, such as 'double the number and add six' or 'add three to the number and double the result', are seen as being equivalent; these equivalences give new ways of seeing the same underlying procept written flexibly in different ways as $2\times x+6$ and $(x+3)\times 2$. These equivalences can be described using the rules that were observed and described as properties in arithmetic, now formulated as rules to *define* the properties in algebra. Finally, a crystalline concept, 'an algebraic expression' arises in which equivalent operations are seen as representing the same underlying operation.

Procepts arise throughout the symbolism of mathematics where symbols such as $4+3$, $\frac{3}{4}$, $2x+6$, dy/dx, $\int \sin x\, dx$, $\sum u_n$ dually represent a process of computation and the result of that process. Such procepts, along with the networks of their deduced properties, form crystalline concepts, which allow the human brain to operate flexibly and efficiently in formulating models, solving problems through symbol manipulation and discovering new properties and connections.

Crystalline concepts also operate at the formal-axiomatic level. Mathematicians construct the successive systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ formally using equivalence relations, such as defining the integers $\mathbb{Z}$ as equivalence classes of ordered pairs (m,n) where $m,n\in\mathbb{N}$ and $(m,n)\sim(p,q)$ if $m+q=p+n$. The whole number m corresponds to the equivalence class $(m+n,n)$. Successive constructions of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ formulate each as being isomorphic to a substructure of the next. Cognitively, however, it is more natural to see the successive number systems contained one within another, with $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, seen as points on the x-axis and $\mathbb{C}$ as points in the plane. This is not a simple cognitive process, however, as such extensions involve changes of meaning that need to be addressed, such as subtraction always giving a smaller result in $\mathbb{N}$ but not in $\mathbb{Z}$, and a (non-zero) square always being positive in $\mathbb{R}$ but not in $\mathbb{C}$.

The various number systems may be conceived as a blend of the visual number line (or the plane) and its formal expression in terms of axioms. For example, the real numbers $\mathbb{R}$ have various equivalent definitions of completeness that the expert recognizes as equivalent ways of defining the same underlying property. The real numbers $\mathbb{R}$ now constitute a crystalline concept whose properties are constrained by the axioms for a complete ordered field.

More generally, any axiomatic system, formulated as a list of specific axioms uses formal proof to develop a network of relationships that gives the axiomatic system the structure of a crystalline concept. While some may be unique (as in the case of a complete ordered field), others, such as the concept of group, have different examples that may be classified by deduction from the axioms.

3.3 A global framework for the development of mathematical thinking

The full cognitive development of formal proof from initial perceptions of objects and actions to axiomatic mathematics can be formulated in terms of three distinct forms of development (Tall, 2004, 2008):

> a development of the *conceptual embodiment* of objects and their properties, with increasing verbal underpinning appropriate for the maturation of Euclidean geometry;
>
> a translation of operations into *proceptual symbolism*, where a symbol such as $2x+3$ represents both a *pro*cess of evaluation, 'double the number and add three', and a manipulable con*cept*, an algebraic expression;
>
> the development of *axiomatic formalism*, in which set-theoretic axioms and definitions are used as a basis of a knowledge structure built up through mathematical proof.

In each form of development, the idea of proof builds through a cycle pictured in figure 2 to give the overall framework in figure 3 as proof develops in the geometric embodiment, algebraic symbolism, and axiomatic formalism.The framework represents the child at the bottom left playing with physical objects,

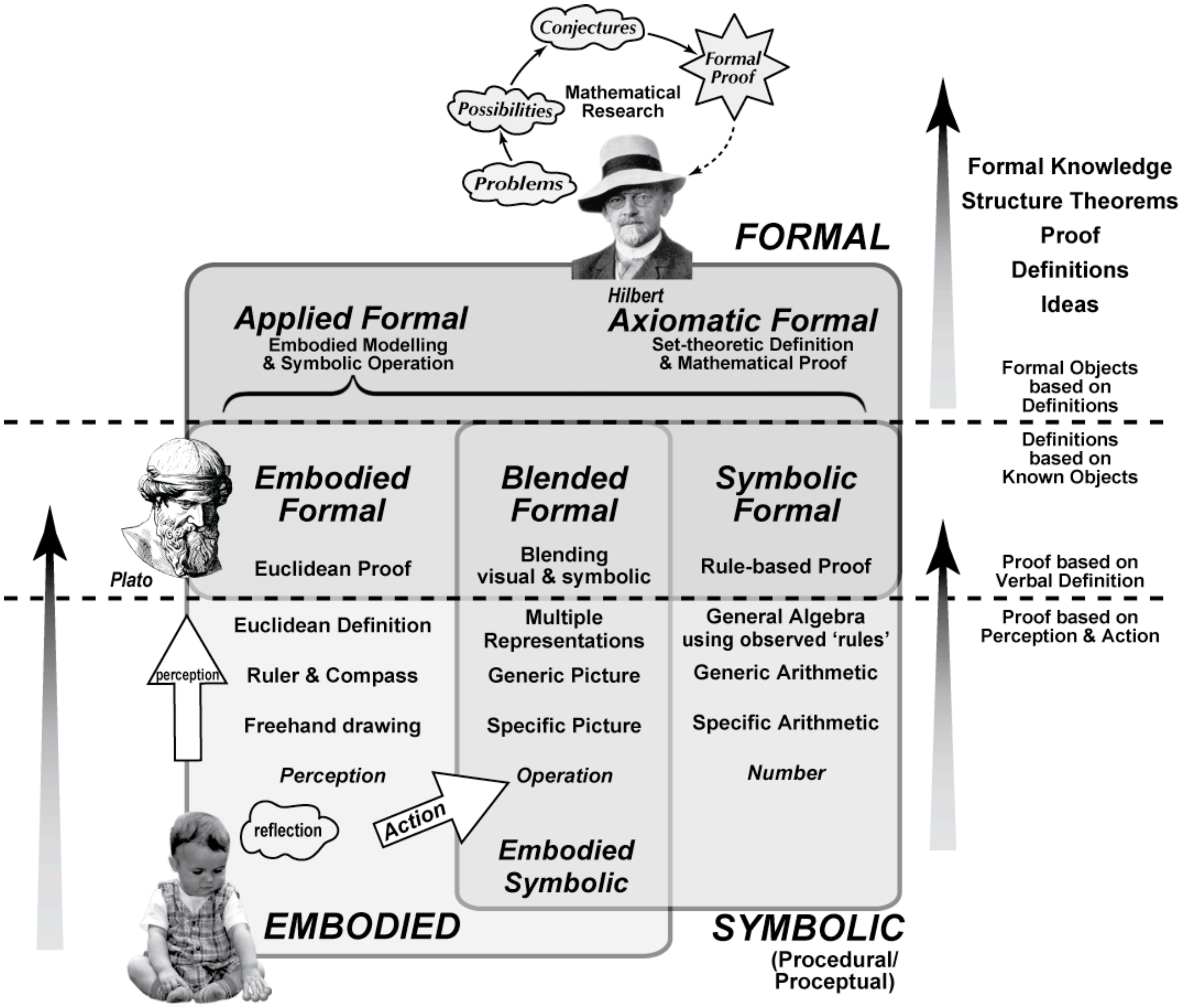


Figure 3: Three strands of conceptual development: embodied, symbolic, formal.

reflecting on their shapes and relationships to build an increasingly sophisticated development of Euclidean geometry through constructions, verbalised definitions and Euclidean proof.

In the centre, as the child reflects on his or her actions on objects, there is a blend of embodiment and symbolism in which properties such as addition are seen visibly to be independent of order (even though the counting procedures and visual representations may be different) and are translated into verbal rules such as the commutative law of addition. Specific pictures may also be seen as generic representations of similar cases, building up generalizations. The culmination of this central overlap includes any proof blending embodiment and symbolism.

On the right-hand side, the operation of counting is symbolised as the concept of number and, in proceptual symbolic terms, specific instances of arithmetic such as $10+7=7+10$ are generalized into an algebraic statement $x+y=y+x$, and such generalities are formulated as rules to be obeyed in algebra.

As the child matures, he or she is introduced to the idea that conceptions based in perception and action can be transformed into proof by verbal definition—a significant cognitive change that leads to Euclidean proof in geometry and rule-based proof in algebra. The figure of Plato here represents a view of crystalline conceptions that are so perfect that they seem to be independent of the finite human brain.

There is a major cognitive change denoted by the horizontal dotted line from inferences based on perception and action to proof based on a definition (of figures in geometry and rules of arithmetic in algebra). For instance, in algebra, the power rule $a^m \times a^n = a^{m+n}$ for whole numbers *m* and *n* can be embodied directly by counting the factors, but when *m* and *n* are taken as fractions or negative numbers, then the idea of counting the factors no longer holds. Now the power rule is used as a definition from which the meanings of $a^{1/2}$ and a^{-1} are deduced. This requires a significant shift of meaning from properties based on perception to properties deduced from rules.

The formal world includes all forms of proof based on an appropriate form of definition and agreed processes of deduction. These include Euclidean proof based on geometric principles such as congruence, algebraic proof based on the rules of arithmetic, and axiomatic formal proof. In applications, applied proof builds on embodiment and symbolism, developing refined strategies of contextual reasoning appropriate to the context.

A major development in formal proof is the shift to the axiomatic method of Hilbert. Now mathematical objects are *defined* as lists of set-theoretic axioms and any other properties must be *deduced* from the axioms and subsequent definitions by formal mathematical proof.

This gives a second major cognitive change, from definitions based on familiar objects or mental entities—as in the thought experiments of Plato—to formal definitions where proofs apply in *any* context where the required axioms hold—as in the formal theory of Hilbert.

At the level of axiomatic formalism, the top right arrow in the figure represents the desired development of the student maturing from a range of familiar ideas to their organization as formal definitions and proof. The student learning the axiomatic method for the first time is faced with a list of axioms from which she or he must make initial deductions, building up a knowledge structure of formally deduced relationships that leads to the proof of successive theorems.

Some theorems, called 'structure theorems' have special qualities that prove that the axiomatic structures have specific embodiments and related embodied operations. For instance the structure theorem that *all finite dimensional vector spaces over F are isomorphic to a coordinate space* F^n reveals that such a vector space can be represented symbolically using coordinates and operations on vectors that can be carried out symbolically using matrices. When $F = \mathbb{R}$, the formalism is then related to embodiments in two and three-dimensional space.

In this way, embodiment and symbolism arise once more, now based on an axiomatic foundation. Mathematicians with highly sophisticated knowledge structures then reflect on new problems, think about possibilities, formulate conjectures and seek formal proofs of new theorems in a continuing cycle of mathematical research and development. At each stage, this may involve embodiment, symbolism and formalism as appropriate for the given stage of the cycle.

4. THE DEVELOPMENT OF PROOF FROM EMBODIMENT

4.1 From embodiment to verbalization

Young children have highly subtle ways of making sense of their observations. For instance, Yevdokimov (2010) observed that young children playing with wooden patterns of different shapes sense symmetry even in quite complicated forms and may build their own conceptions of symmetry without any special emphasis and influence from adults. However, they experience enormous difficulties when attempting to describe a symmetric construction verbally.

The young child may develop ways to recognise and name different shapes in ways that may be quite different from an adult perspective. Two-year old Simon learned to recognise and say the name 'triangle' for a shape he recognised (Tall, 2010.) He learnt it through watching and listening to a television programme in which Mr. Triangle was one of several characters, including Mr. Square, Mr. Rectangle and Mr. Circle where each character had the named shape with a face on it and hands and feet attached.

When Simon saw a triangle, he named it, but then, when playing with some square table mats, he put together a 3 by 2 rectangle, a 2 by 2 square, and then reorganised four squares into an upside down T-shape that he called a triangle (Figure 4.) Although the figure lacks three sides, it is fat at the bottom, thin at the top and symmetric about a vertical axis. It is the nearest word in his vocabulary to describe what he sees, being more like a triangle than any other shape that he can name.

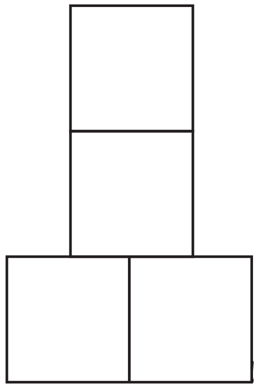


Figure 4: Simon's Triangle

4.2 From embodiment and verbalization to pictorial and symbolic representations

A case of interest is the five-year long study of Maher and Martino (1996), which followed the developing ideas of a single child, called Stephanie. This revealed the emergence of statements that are precursors of quantifiers (such as 'there is an A such that B occurs' or 'for all A, B occurs'.) During her third and fourth grade, Stephanie and her classmates were given variations of the following question, which will be termed the four-cube-tall Tower Problem:

> How many different four-cube-tall towers can be built from red and blue cubes?

When Stephanie and her partner approached this problem for the first time, she started from the search for different four-cube-tall towers using a trial-and-error strategy. Namely, she built a tower, named it (e.g. 'red in the middle' or 'patchwork'), and compared it with all the towers that had been constructed so far to see whether it was new or a duplicate. In several minutes, Stephanie began to spontaneously notice relationships between pairs of towers and put them together. She then called the towers such as the left pair on Figure 5 as 'opposites', and the right pair as 'cousins'. Maher and Martino interpreted this as the beginning of Stephanie's classification of towers into sets by a local criterion. No global organizational criterion emerged at this stage, and Stephanie, who eventually constructed all 16 towers, did not know whether or not she found them all. When asked about it, she explained that she 'continued to build towers until [she] couldn't find any that were different' (p. 204).

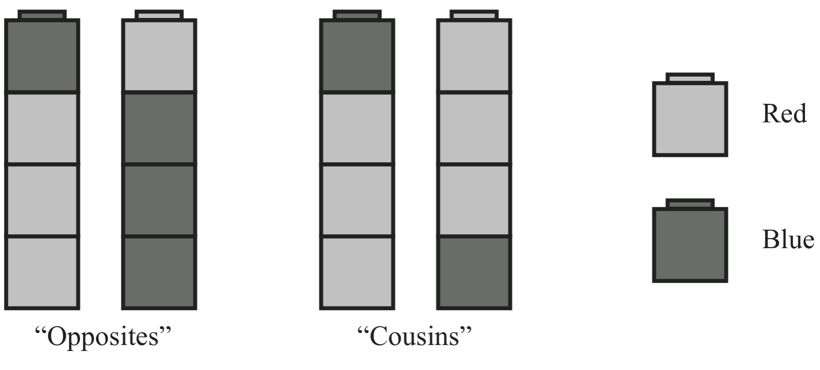


Figure 5: 'Opposites' and 'cousins'

Eighteen months later, when in the fourth grade, Stephanie was presented with the problem concerning the number of five-cube-tall towers that can be built from red and yellow cubes. This time she constructed 28 original combinations organized in the sets consisting of four elements: some tower, its 'opposite', its 'cousin' and 'the opposite of the cousin.' Thus, though a global organizational principle still remained murky for Stephanie, she progressed from the pure trial-and-error strategy to trial-and-error strategy combined with the local classification strategy.

A crucial event occurred when the teacher asked the class about the number of five-cube-tall towers with exactly two red cubes. In response, Stephanie argued that all towers in this category can be accounted by the following organizational criterion: there are towers in which two red cubes are separated by no yellow cube, one yellow cube, two yellow cubes or three yellow cubes (see Figure 6). It is notable that at this stage Stephanie began to draw pictures of the towers she produced rather than building them with plastic cubes.

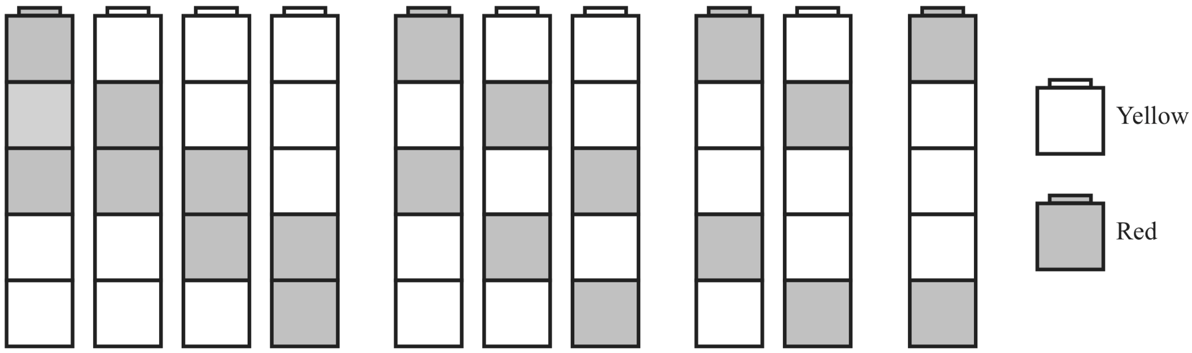


Figure 6: Towers of five with two red and three yellow

In this episode Stephanie for the first time arrived at an existential algebraic statement that required proving. She declared: 'there exist exactly 10 five-cube-tall towers with exactly two red cubes.' Her justification of this statement involved an indirect proof by contradiction as she explained: 'it is not possible to have towers with four or more yellow cubes placed between the two red cubes without violating the requirement that a tower be only five cubes tall,' (p. 205).

One can see here that, though still operating in her embodied mathematical world, Stephanie succeeds in substituting her previous local organizational criterion ('opposites' and 'cousins') with a new, all-inclusive or global, one (the number of yellow cubes separating the red ones). This invention enabled Stephanie to construct—though only for a sub-problem of the five-cube-tall Towers Problem—a mathematically valid argument. An important sign of the development of the idea of proof in Stephanie is that she was able to mentally represent not only what is possible to do with the cubes, but also what is impossible. The spirit of a crystalline concept—a concept that has an internal structure of constrained relationships that cause it to have necessary properties as part of its context—enters here!

Two weeks later, two more cognitive advances emerged when Stephanie shared with the interviewers her thinking about the six-cube-tall Towers Problem that she assigned to herself. The first advance occurred when she introduced a letter-grid notation for representing the towers, instead of drawing them or assembling from the plastic cubes that she used before. This is an important step on her way to the symbolic mathematical world. Interestingly, the authors mentioned that, when in first grade, Stephanie had used a letter-grid notation for solving another combinatorial problem, and then, it seemed, she forgot about it. As Lawler (1980) noted, it may seem that a child regresses in knowledge, whereas he or she may in fact be attempting to insert new knowledge into an existing knowledge structure. (See also Pirie & Kieren, 1994.)

Our point here is that the pictorial and later symbolic representations that she used were indeed rooted in Stephanie's embodied world. The second advance was that Stephanie started to fluently consider different organizational principles for producing the towers in a systematic way. In particular, she introduced a method of holding the position of one color when varying positions for another. Apparently, the second advance is related to the first one: convenient notation makes consideration of various patterns more accessible. These important advances eventually led Stephanie to construct a full proof for the four-cube-tall Tower Problem (by classifying the towers into five categories: towers with no white cube, towers with exactly one cube, etc). We interpret Stephanie's last insight as a verbalisation of the embodied skill to infer cause-effect relationships from statistical patterns: Stephanie found that the answers for three-, four-, and five-cube-tall towers were 8, 16, and 32 respectively, and just expressed her belief that this numerical pattern will work forever. Such a guess is statistically justified for Stephanie, and is appropriate at her stage of development. In the fifth grade she is only just beginning to transfer embodied proof concepts into a symbolic mathematical form.

4.3 From embodiment, verbalization and symbolism to deductions

The experiences just described offer an example of a learner building up increasingly sophisticated knowledge structures by physical experiment with objects, finding ways to formulate similarities and differences, then representing the data observed using drawings and symbols that have an increasingly meaningful personal interpretation. In this section we discuss the general principles underpinning a learner's path towards deductive reasoning from its sensori-motor beginnings, through the visual-spatial development of thought and on to the verbal formulation of proof, particularly in Euclidean geometry.

As the child matures, physical objects, experienced through the senses become associated with pictorial images, and develop into more sophisticated knowledge structures that Fischbein (1993) named *figural concepts*. These are

> mental entities [...] which reflect spatial properties (shape, position, magnitude) and, at the same time, possess conceptual qualities like ideality, abstractness, generality, perfection. (Fischbein, 1993, p.143.)

Figural concepts reflect the human embodiments that underlie our more abstract formal conceptions.

Cognitive science sees the human mental and physical activity underlying all our cognitive acts (Johnson 1987, Lakoff & Johnson, 1999, Lakoff & Núñez, 2000). For example, the schema of 'containment' where one physical object contains another underlies the logical principle of transitivity $A \subset B$, $B \subset C$ implies $A \subset C$ and operates mentally to infer that if A is contained in B and B is contained in C, then A *must* be contained in C. Even Hilbert—on the occasion declaring his famous 23 problems at the International Congress of 1900—noted the underpinning of formalism by visual representation, picturing transitivity as an ordering on a visual line:

> Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea 'between'? (Hilbert, 1900.)

Empirical evidence in support of this can be found throughout the literature. Byrne and Johnson-Laird (1989) studied adults responding to tasks in which they were given verbal evidence of relative positions of objects placed on a table and found that the subjects used visual mental models rather than a pure logical approach to produce their deductions.

The other side of the coin suggests that a mismatch between images (schemata) and formal definitions of mathematical concepts can be a source of difficulty in the study of mathematics and mathematical reasoning. For instance, Tall and Vinner (1981) illustrated how students may interpret real analysis based on their concept imagery of earlier experiences rather than on formal definitions. Hershkowitz and Vinner (1983) similarly revealed how particular attributes of pictures interfere with the general conceptualization process in geometry.

In a similar way, Núñez, Edwards and Mato (1999) suggested that the epsilon-delta approach to continuity is problematic for students because it conflicts with natural embodied conceptions of continuity. The problem, however, is more subtle, because formalism only captures specific explicit properties, such as the 'closeness' of natural continuity formulated in the epsilon-delta definition. Natural continuity also involves other relevant aspects such as as the completeness of the real numbers and the connectedness of the domain.

Sometimes, the practice of generalizing from empirical findings or building argument from intuitively appealing images leads to a possibility that happens to be wrong. As an example of a situation where valid deductive argument is applied to a plausible, but wrong image, we give a '*proof*' that *every triangle is isosceles*. Kondratieva (2009) used this example in order to illustrate the process of learning the art of deduction through the analysis of unexpected or contradictory results.

> Consider an arbitrary triangle *ABC*. Let the point of intersection of the bisector of the angle *B* and the perpendicular bisector of the side *AC* meet in the point *M*. For simplicity, assume that *M* lies inside of the triangle (figure 7).

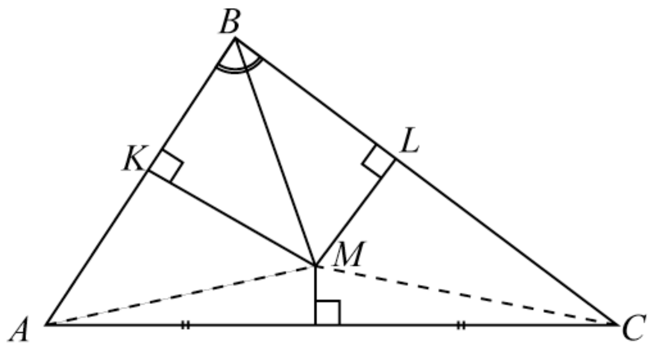


Figure 7: Proof that all triangles are isosceles

> Let points *K* and *L* be the feet of perpendiculars dropped from *M* to the sides adjacent to vertex A. In the triangles *BMK, BML*, the side *BM* is common; the angles *KBM* and *LBM* are equal, as are the right angles *BKM, BLM*. Therefore, triangles *BMK, BLM* are congruent, and *BK* = *BL*. The right-angled triangles *AKM* and *CLM* are also congruent because the legs *MK=ML,* and their hypotenuses are equal *AM* = *CM* by the property of the perpendicular bisector. Thus, *AK* = *CL*. Finally, *AB* = *AK* + *KB* = *CL* + *LB* = *CB*. Hence the triangle *ABC* is isosceles. QED.

Students often perceive the figure to be legitimate and concentrate on looking for mistakes in the argument. The deductions concerning congruency in this proof are supported by reference to appropriate theorems and appear to be correct. It is often a surprise for a student to realise that the actual fallacy arises through drawing *M* inside the triangle when it should be outside, more precisely on the triangle's circumcircle.

Such examples have the potential to prepare the learner for the need to doubt representations and arguments that seem intuitively valid. This can lead to several different ways of emphasizing different aspects of proof.

The first is to realize that *correct reasoning based on a misleading (incorrect) diagram can lead to false conclusions*. Thus, in mathematical derivations we must attend to both the deduction and the assumption(s).

The second is to illustrate the idea of *a proof by contradiction*. For instance, consider the statement: 'The intersection point of an angular bisector and the right bisector of the opposite side in any triangle lies outside the triangle.' Drawing a picture purporting to represent the contrary situation where the intersection lies inside the triangle, as in figure 7, can then be seen as leading to the impossible statement that 'all triangles are isosceles'.

The third possibility is to introduce the concept of *a direct constructive proof* by inviting the student to perform the construction using dynamic software and arguing why the point *M* must lie outside the triangle. (Dynamic software is further discussed in chapter XXX.)

The need for a more reliable proof leads to the idea of *deductive reasoning,* which may be seen as the ability of an individual to produce new statements in the form of conclusions from given information, particularly in areas where the subject has no prior knowledge other than the information given. The new statements must be produced purely by reasoning with no simultaneous access to hand-on materials and experimentation. However, such methods require the individual to build a knowledge structure that enables the use of logical forms of deduction. As we analyse Euclidean geometry, we find that its deductive methods build on ways of working that are themselves rooted in human embodiment.

5. EUCLIDEAN AND NON-EUCLIDEAN PROOF

5.1 The development of Euclidean Geometry

From a mathematician's viewpoint, the study of Euclidean geometry in school has often been considered as providing a necessary basis for the formal notion of proof and, in particular, the building of a succession of theorems deductively from basic assumptions. From a cognitive viewpoint it is our first example of the long-term development of crystalline concepts. Beginning from personal perceptions and actions, the learner may build personal knowledge structures relating to the properties of space and shape, then develop definitions and deductions to construct the crystalline platonic objects of Euclidean geometry.

Even though the books of Euclid produce a sequence of successively deduced propositions based on specified common notions, definitions and postulates, a closer inspection reveals the use of principles based on human perception and action. For example, the notion of congruence involves the selection of certain minimal properties that enable triangles to be declared to have *all* their properties in common, in terms of requiring only three corresponding sides (SSS), two sides and included angle (SAS), or two angles and corresponding side (AAS). All are based on an embodied principle of superposition of one triangle upon another (perhaps turning it over). This concept of congruence of triangles is not endowed with a specific name in the Books of Euclid. It is used as an established strategy to formulate minimal conditions under which different triangles are equivalent and consequently have all the same properties.

Parallel lines are another special concept in Euclid, defined to be straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction (Euclid, Book I, Definition 23). Using this definition, Euclid establishes various (equivalent) properties of parallel lines, such as alternate angles being equal to one another, corresponding angles being equal, and the sum of the interior angles being equal to two right angles (Proposition 27 et seq.). Once more the concept of parallel lines is a crystalline concept with a range of interlinked properties each of which can be used to furnish a Euclidean proof of the others.

The development of geometry, starting at Euclid Book I, has been declared inappropriate for young children:

> The deductive geometry of Euclid from which a few things have been omitted cannot produce an elementary geometry. In order to be elementary, one will have to start from a world as perceived and already partially globally known by the children. The objective should be to analyze these phenomena and to establish a logical relationship. Only through an approach modified in this way can a geometry evolve that may be called elementary according to psychological principles (van Hiele Geldof, 1984, p. 16).

Some enlightened approaches to geometry have attempted to integrate it with broader ideas of general reasoning skills including the need for clear definitions and proof. For example, Harold Fawcett, the editor of the 1938 NCTM Yearbook on Proof, developed his own high school course that he taught at the University School in Ohio State entitled *The Nature of Proof*. His fundamental idea was to consider any statement, to focus on words and phrases, to ask that they be clearly defined, to distinguish between fact and assumption, and to evaluate the argument, accepting or rejecting its arguments and conclusion, while constantly re-examining the beliefs that guided the actions. Speaking about the course at an NCTM meeting in 2001, Frederick Flener observed:

> Throughout the year, the pupils discussed geometry, creating their own undefined terms, definitions, assumptions and theorems. In all he lists 23 undefined terms, 91 definitions, and 109 assumptions/theorems. The difference between an assumption and a theorem is whether it was proved. Briefly, let me tell you a few of the undefined terms which the pupils understood, but were unable to define. For example, as a class they couldn't come up with 'the union of two rays with a common endpoint' as the definition of an angle, so they left it as undefined. Nor could they define horizontal or vertical, or area or volume. Yet they went on to define terms like dihedral angle, and the measure of a dihedral angle—which I assume involved having rays perpendicular to the common edge (Flener, 2001).

A detailed study of the students who attended this course revealed significant long-term gains in reasoning skills over a lifetime (Flener, 2006). This suggests that making personal sense of geometry and geometric inferences can have long-term benefits in terms of clarity of thinking and reasoning skills.

Modern trends in teaching have led to encouraging young children to build a sense of shape and space by refining ideas through experience, exploring the properties of figures and patterns. They experiment with geometrical objects, seek to recognize the properties of card shapes, sketch the faces of a box when pulled out into a flat figure, fold paper, measure the angles and sides of a triangle, and discuss their ideas with friends or teachers.

They may have experiences in constructing and predicting what occurs in geometric software such as Logo in ways that give figural meanings to geometric ideas. For instance, a 'turtle trip' round a (convex) polygon, may enable the child to sense that 'If the turtle makes a trip back to its starting state without crossing its own path, the total turn is 360 degrees.' According to Papert (1996), this is a *theorem* with several important attributes: first, it is powerful; second, it is surprising; third, it has a proof. It is not a theorem in the formal mathematical sense, of course, but it is a meaningful product of human embodiment, sensing a journey round a circuit and ending up facing the same direction, thus turning through a full turn.

These experiences provide young children with preliminary background on which to develop ideas of proof in a variety of ways. For example, it is possible to experience many ways in which children may attempt to provide a proof for the statement that *the sum of the interior angles of a triangle is 180°*.

In the studies of Lin, Cheng et al. (2003), Healy and Hoyles (1998), and Reiss (2005), eight distinct proofs were collected from students before they had any formal introduction to Euclidean proof as a deductive sequence of propositions.

The first two are pragmatic actions applied to specific cases:

Proof 1: by physical experiment (Figure 8).

> Take a triangle cut out of paper, tear off the corners and place them together to see that they form a straight line. Do this a few times for different triangular shapes to confirm it.

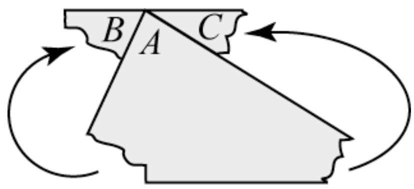


Figure 8

Proof 2: by practical measurement (Figure 9).

> Draw a triangle. Measure the angled to check that the sum is equal to 180°. Repeat the same process on other triangles.

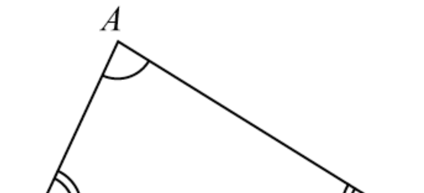


Figure 9

A third proof is a dynamic embodiment, which arises in Logo:

Proof 3: the turtle-trip theorem: imagine walking round a triangle (Figure 10).

Start at any point P and walk all the way round. This turns through 360°. At each vertices, the sum of exterior and interior angles is 180° so the sum of the three exterior and interior angles is 540°. Subtract the total 360° turn to leave the sum of the interior angles as 540°–360° = 180°.

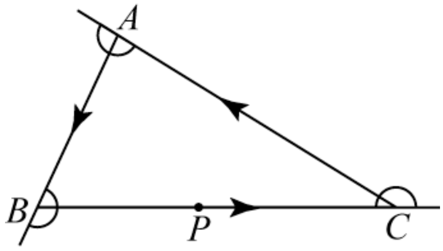


Figure 10

A fourth proof uses a known fact about triangles to infer another fact.

Proof 4: Use the fact that the exterior angle equals the sum of the two interior opposite angles (Figure 11).

Extend the segment CA; the exterior angle $\angle 1$ is equal to the sum of the two interior opposite angles. Because the exterior angle and $\angle BAC$ make a straight line, the sum of all three angles is 180°.

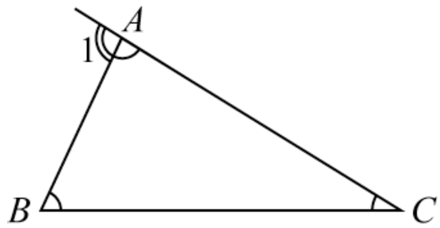


Figure 11

The fifth and sixth proofs introduce additional parallel lines.

Proof 5: A proof using parallel lines (Figure 12).

1. Draw a line parallel to AB through point C.
2. $\angle A = \angle 1$ (alternate angles) and $\angle B = \angle 2$ (corresponding angles)
3. $\angle A + \angle B + \angle C = \angle 1 + \angle 2 + \angle C = 180°$.

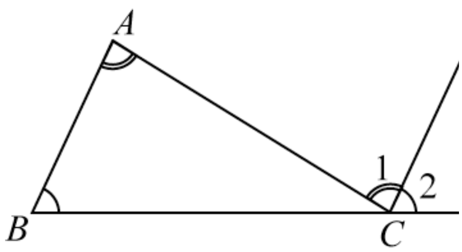


Figure 12

Proof 6: A second proof using parallel lines (Figure 13).

1. Draw a line L parallel to BC through point A. Then $\angle B = \angle 1$ and $\angle C = \angle 2$ (alternate angles).
2. Hence $\angle B + \angle BAC + \angle C = \angle 1 + \angle BAC + \angle 2 = 180°$.

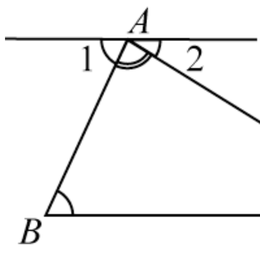


Figure 13

The seventh uses a property of the circle.

Proof 7: Using a property of angles subtended by a chord (Figure 14).

> The angle at the circle is half the angle at the centre, so
> $\angle A = \frac{1}{2}\angle BOC$, $\angle B = \frac{1}{2}\angle AOC$, $\angle C = \frac{1}{2}\angle AOB$.
> Adding these together:
> $\angle A + \angle B + \angle C = \frac{1}{2}360° = 180°$.

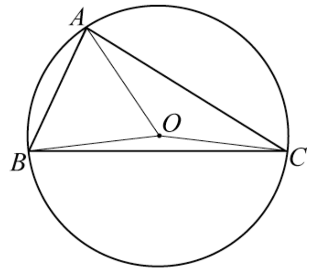


Figure 14

Proof eight appeals to the more general form of the angle sum of an *n*-sided polygon, either as a consequence of the turtle trip theorem, or simply by substitution in the general formula.

Proof 8: An *n*-sided (convex) polygon has angle sum $n \times 180° - 360°$ (Figure 15).

> Put $n = 3$ in the general formula to get the angle sum for a triangle is 180°.

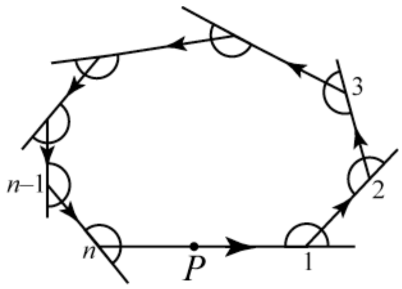


Figure 15

These eight solutions reveal a broad hierarchy. Proofs 1 and 2 are embodied approaches, the first by a physical process of putting the angles together (in a manner that may have been suggested to them earlier), the second by measuring a few examples. Proof 1 contains within it the seeds of more sophisticated proofs 5 and 6 using the Euclidean idea of parallel lines. Proof 3 (and its generalization to a polygon used in proof 8) are dynamic proofs that do not arise in the static formal geometry of Euclid and yet provide a dynamic embodied sense of why the theorem is true. Proof 4, relating to the exterior angle property, nicely links two properties of a triangle and yet, an expert may know that these are given as equivalent results from a single theorem of Euclid (Book 1, proposition 32):

> In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

Though some experts, looking at this proof from a formal viewpoint, may see it relating two equivalent properties in a circular manner, it is a natural connection for a student to make in the early stages of building a knowledge structure of relationships in geometry (Housman & Porter, 2003; Koichu, 2009).

Proofs 5 and 6 are in the spirit of Euclid, constructing a parallel line and using established propositions concerning parallel lines to establish the theorem.

However, here they are more likely to involve an embodied sense of the properties of parallel lines than the specific formal sequence of deductions in Euclid Book I.

Proofs 7 and 8 both use more sophisticated results to prove simple consequences and have a greater sense of a general proof. And yet one must ask oneself, how does one establish the more general proof in the first place? While networks of theorems may have many different paths and possible different starting points, the deeper issues of sound foundations and appropriate sequences of deductions remain.

It has long been known that students have difficulty reproducing Euclidean proof as a sequence of statements where each is justified in an appropriate manner. Senk (1985) showed that only 30% of students in a full-year geometry course reached a 70% mastery on a set of six problems in Euclidean proof.

Given the perceived difficulties in Euclidean geometry, the NCTM Standards (2000) suggested that there should be decreased attention to the overall idea of geometry as an axiomatic system and increased attention on short sequences of theorems. These can in themselves relate to Papert's notion that a theorem should be powerful, surprising, and have a proof. Two examples include the theorem that two parallelograms on the same base and between the same parallels have the same area, even though they may not look the same, and the theorem that the angles subtended by a chord in a circle are all equal (Figure 16).

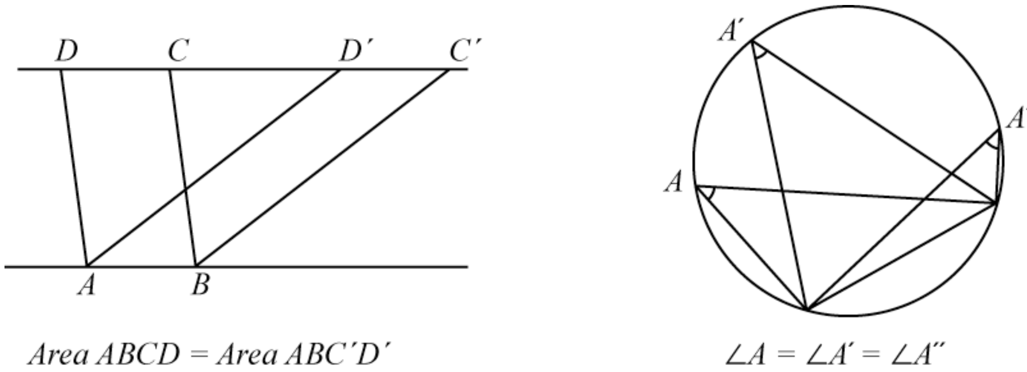


Figure 16: Interesting theorems

Here, short sequences of construction are possible. For instance, in the area of parallelograms, one may prove that the triangles *ADD'* and *BCC'* are congruent and that each parallelogram is given by taking one of these triangles from the whole polygon *ABC'D'*. The equalities of the angles subtended by a chord is established by constructing the angle at the centre of the circle and proving that it is twice the angle at the circle.

5.2 The Beginnings of Spherical and Non-Euclidean Geometries

The teaching of non-Euclidean geometries is not central in the current curriculum, but practical experience in such geometries is becoming part of the development of college mathematics in the USA.

> College-intending students also should gain an appreciation of Euclidean geometry as one of many axiomatic systems. This goal may be achieved by directing students to investigate properties of other geometries to see how the basic axioms and definitions lead to quite different – and often contrary – results. (NCTM Standards, 1989, p.160.)

Although spherical geometry goes back to the time of Euclid, it does not share the axiomatic tradition developed in plane geometry. Spherical geometry may be approached as a combination of embodiment and trigonometric measurement. One may begin with the physical experience of operating on the surface of a sphere (say an orange, or a tennis ball with elastic bands to represent great circles). This produces surprising results quite different from the axiomatic geometry of Euclid. When exploring spherical triangles whose sides are great circles, the learner may find a new geometry that shares properties predicting congruence (SAS, SSS, AAS), but is fundamentally different from plane Euclidean geometry, in that parallel lines do not occur and the angles of a triangle always add up to *more* than 180°.

This can be proved by a combination of embodiment and trigonometry. Figure 17 shows a spherical triangle *ABC* produced by cutting the surface with three great circles. It has a corresponding triangle *A'B'C'* where the great circles meet on the opposite side of the sphere having exactly the same shape and area.

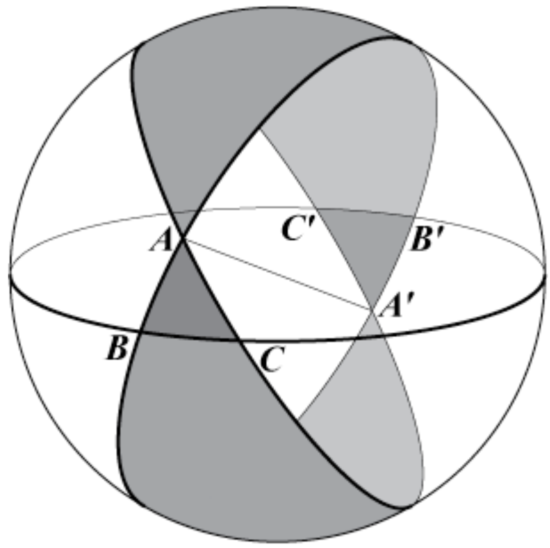


Figure 17: Area of a spherical triangle

The total area of the shaded parts of the surface between the great circles through *AB* and *AC* can be seen by rotation about the diameter *AA'* to be α / π of the total area, where α denotes the size of the angle *A* measured in radians. This area is $4\pi r^2 \times \alpha / \pi = 4\alpha r^2$. The same happens with the slices through *B* and through *C* with area $4\beta r^2$ and $4\gamma r^2$. These three areas cover the whole surface area of the sphere and all three overlap over the triangles *ABC* and *A'B'C'*. Adding all three together, allowing for the double overlap gives the surface area of the sphere as

$$4\pi r^2 = 4\alpha r^2 + 4\beta r^2 + 4\gamma r^2 - 4\Delta$$

where Δ is the area of the spherical triangle *ABC*. This gives the area Δ as $(\alpha + \beta + \gamma - \pi) r^2$ and the sum of the angles as

$$\alpha + \beta + \gamma = \pi + \frac{\Delta}{r^2}.$$

The experience challenges learners to rethink the 'deductive arguments' given above that the sum of the angles of a triangle is 180° when the sum of the angles of a spherical triangle is always more by a quantity proportional to its area.

In long-term implementations with in-service teachers and university students, learners report that they could not have developed concepts and the arguments without access to materials to handle and dynamic geometry sketches to explore the tasks. In short, this provides evidence for the continuing role for embodied experience in the cognitive development of proof in adults. Examples of such an approach arise in the book *Experiencing Geometry* (Henderson & Tamina, 2001) and the work of Lénárt (2003), encouraging comparison of concepts and reasoning in spherical geometry and plane geometry, through practical activities such as handling spheres or folding paper. These explorations give an emphasis to transformations and symmetry, matching the 'modern' definition of geometry of Klein (1879) and offer a setting for increasingly sophisticated reflections on two distinct embodied geometries. The similarities and contrasts between the two structures provoke a reflective reworking of unexamined concepts such as 'straight lines' and 'angles' and also of principles related to 'congruence of triangles' and the properties of 'parallel lines'.

Other geometries may be studied, perhaps as axiomatic systems, but more often as a combination of embodiment and symbolism. For instance, projective geometry of the plane can be studied using embodied drawing or symbolic manipulation of homogeneous coordinates. Non-Euclidean geometries include the Poincaré model of hyperbolic geometry in the upper half plane where 'points' are of the form (x, y) for $y > 0$ and 'lines' are semi-circles with centre on the x-axis; two 'lines' are said to be 'parallel' if they do not meet. In this new context, students must reflect on new meanings to make deductions dependent upon the definitions in the new context (Neta et al., 2009).

6. SYMBOLIC PROOF IN ARITHMETIC AND ALGEBRA

As the mathematics becomes more sophisticated, increasingly subtle forms of proof develop in arithmetic and algebra. They build from demonstrations or calculations for single examples, to considering a specific example to represent a generic proof that applies to all similar cases, and then to general proofs expressed algebraically. Induction proofs can operate at two distinct levels, one the potentially infinite process of proving a specific case and repeating a general step as often as is required, the second involving a three stage proof that compresses the potentially infinite repetition of steps to a single use of the induction axiom in a more formal setting.

6.1 The increasing sophistication of proof in arithmetic and algebra

An example of such successively sophisticated forms of proof is the Gauss Little Theorem that is reputed to have been produced by the schoolboy Gauss when his teacher requested the class to add up all the whole numbers from 1 to 100.

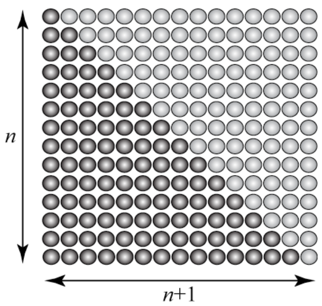
<table>
<tr>
<td>Approach 1:
specific arithmetical calculation
(only works for even numbers):
$\begin{array}{l} 1+2+3+\ldots+50 \\ \underline{100+99+98+\ldots+51} \\ 101+101+101+\ldots+101=101\cdot 50=5050 \end{array}$</td>
<td>Approach 2:
generic arithmetical argument (n =100)
(works for all numbers even and odd):
$\begin{array}{l} 1+2+3+\ldots+100 \\ \underline{100+99+98+\ldots+1} \\ 101+101+101+\ldots+101=101\cdot 100=2\cdot 5050 \end{array}$</td>
</tr>
<tr>
<td>Approach 3: generic pictorial argument

The sum 1 +…+ n is half of $n(n+1)$</td>
<td>Approach 4: algebraic proof (for any n)
$\begin{array}{l} 1+2+3+\ldots+n \\ \underline{n+(n-1)+(n-2)+\ldots+1} \\ (n+1)+(n+1)+(n+1)+\ldots+(n+1) \end{array}$
$=(n+1)\cdot n$
$=2\cdot\frac{(n+1)\cdot n}{2}$</td>
</tr>
<tr>
<td colspan="2">Approach 5: a potentially infinite proof by induction (for any n so far)
$1+\ldots+n=\frac{1}{2}n(n+1)$ is true for $n=1$, because $1=\frac{1}{2}\times 1\times(1+1)$.
If it is true for $n=k$, use the formula $1+\ldots+k=\frac{1}{2}k(k+1)$ and add $k+1$ to both sides to deduce the truth for $n=k+1$,
and then repeat this general step as often as is required:
It is true for $n=1$ hence it is true for $n=2$, hence for $n=3$, hence for $n=4$, …
and so on for any specific whole number ... (ad infinitum).</td>
</tr>
<tr>
<td colspan="2">Approach 6: a finite proof by the Peano axioms (for all n)
The Peano axioms define natural numbers as a set satisfying the following conditions:
• There is a natural number 1.
• Every natural number a has a natural number successor, denoted by s(a).
• Distinct natural numbers have distinct successors: if $a \neq b$, then $s(a) \neq s(b)$.
• 1 is not the successor to any natural number.
• If A is a sub-set of natural numbers containing 1 and if the successor of any number in A is also in A, then A contains all the natural numbers.
Let us now prove the formula.
Let A be a subset of natural numbers such that for any $n\in A$ $1+\ldots+n=\frac{1}{2}n(n+1)$.
1. Prove, by substitution in the formula, that $1\in A$.
2. If $k\in A$ then use the formula for k as above to show that $k+1\in A$.
3. Observe that A satisfies the Peano axioms, and thus, is the whole of $\mathbb{N}$.</td>
</tr>
</table>

Figure 18: Proofs of the formula for the sum of the first *n* whole numbers

These successive proofs are not all as successful in giving meaning to students. Rodd (2000) found that both the generic pictorial and algebraic proofs made more sense to the students because they gave a meaningful explanation as to *why* the proof is true, while the formal proof by induction is more obscure because it seems to use the result of the proof (assuming P(n) for a specific n to prove P(n+1)) during the proof itself. In addition, the finite proof by induction (Proof 6) may cause further problems because, although it has only three steps, the set defined by the Peano axioms must itself be infinite.

6.2 Proof by contradiction and the development of aesthetic criteria

On one hand, it is well established that proving by contradiction is problematic for many students (e.g. Tall, 1979; Leron, 1985; Epp, 1998; Reid & Dobbin, 1998; Antonioni, 2001). On the other hand, Freudenthal (1973, p. 629) notes that indirect proof arises spontaneously in young children in statements such as 'Peter is at home since otherwise the door would not be locked'. Such a phenomenon occurred with Stephanie (see section 4.2) where she observed that it is not possible to have a tower of five cubes to have four or more yellow cubes placed between two red cubes.

The full notion of proof by contradiction (to prove P, assume P is false and deduce that this leads to a contradiction) is an altogether more problematic mode of proving as the prover must simultaneously hold a falsehood to be true and attempt to argue why it is false while in a state of stress. Leron (1985) suggests that it is preferable to reorganise a contradiction proof so that it initially involves a positive construction and the contradiction is postponed to the end. For instance, in order to prove there is an infinite number of primes, start by proving positively that given any finite number of primes one can always construct another one, then—and only then—deduce that, if there is only a finite number of primes, then this would lead to a contradiction.

A similar technique is to use a generic proof. For instance, rather than prove that √2 is irrational by contradiction, first prove that if one squares a rational number where denominator and numerator are factorised into different primes, then its square has an even number of each prime factor in the numerator or denominator. Then, and only then, deduce that √2 cannot be a rational because its square is 2, which only has an odd number of occurrences of the prime 2. (At a formal level, we note that this proof is dependent on the uniqueness of factorization into primes, but this is not a concern that occurs to students on first acquaintance with the proof.)

Tall (1979) presented a choice of two proofs: the contradiction proof that √2 is irrational and the generic proof that √(5/8) is irrational because 5/8 contains an odd number of 5s and also an odd number of 2s in its prime factorization. The generic proof has explanatory power that generalizes easily while the contradiction proof is both problematic and not easily generalized. Students significantly preferred the generic proof over the contradiction proof and even among students who were already familiar with the contradiction proof, their preference for the generic proof increased over a period of days.

The process of turning all contradiction proofs to more direct proofs with a later contradiction, however, may not be profitable in the long-term as proofs by contradiction are central to mathematical analysis. This suggests that direct proofs will be helpful in the early stages, but there is a need to shift to the use of contradiction proofs to enable the student to build more powerful knowledge structures that are sufficiently robust for more advanced mathematical analysis.

Dreyfus and Eisenberg (1986) found that mathematicians comparing different proofs of the irrationality of $\sqrt{2}$ ranked the proofs by contradiction that lack the need for prerequisite knowledge as elegant and appropriate for their teaching. They also found that college students had not yet developed the sense of aesthetics of a proof and proposed that such a sense should be encouraged.

Koichu and Berman (2005) found that gifted high school students, who were asked to prove the Steiner-Lehmus theorem ('If the bisectors of two angles of a triangle are equal, then it is an isosceles triangle'), could fluently operate in the mode of proving by contradiction. In addition, they manifested the developed aesthetic sense, by their incentive to find the most parsimonious proof. For instance, they realized that it was possible to build a concise proof that used the minimum of prerequisite knowledge. However, in the pressure of a contest, they found that the brute force of using well-rehearsed procedures could prove to be more efficient if less aesthetic.

The development of more sophisticated insight into proof reveals the fact that the use of contradiction requires the propositions involved to be either true or false with no alternative. More general forms of logic are possible, for instance allowing an extra alternative that a theorem might be undecideable (it could be true, but the truth cannot be established in a finite number of steps), or there may be different gradations between absolute truth and absolute falsehood. This might occur in multi-valent logic allowing in-between possibilities between 0 (false) and 1 (true). It also occurs when a conjecture is formulated, which may be considered 'almost certain', 'highly probable', 'fairly likely' or some other level of possibility prior to the establishment of a formal proof.

7. AXIOMATIC FORMAL PROOF

Formal proof, as introduced earlier in section 2.1, refers either to a precise logical form as specified by Hilbert or to forms of proof used by mathematicians to communicate to each other in conversation and in journal articles. We begin by considering the undergraduate development in formal proof as part of the long-term cognitive growth of proof concepts from child to adult. The nature of undergraduate development will be further studied in chapter XXX.

7.1 Student development of formal proof

Initial student encounters with formal proof can occur in a number of ways. At one end of the spectrum is the Moore method in which students are given basic definitions and theorems and encouraged to seek proofs for themselves. R. L. Moore reckoned that 'That student is taught the best who is told the least' (Parker, 2005). He produced a rich legacy of graduates in mathematics who advanced the frontiers of research, producing a phenomenon consistent with the long-term success of Fawcett's open-ended approach to geometry in school.

When students are introduced to formal proof in a university pure mathematics course, the objective is for them to make sense of formal definitions in a way that can be used for deduction of theorems. Pinto (1998) found that there was a spectrum of approaches in an analysis course, which she classified in two main groups: those who *gave* meaning to the definition from their current concept imagery, and those who *extracted* meaning from the definition by learning to reproduce it fluently and studying its use in proofs presented in class. She found that both approaches could lead to the building of successful formal knowledge structures but that either could break down as a result of conflict between previous experience and the theory, or through the sheer complexity of the definitions that could involve three or more nested quantifiers.

These observations are consistent with the work of Duffin and Simpson (1993, 1995) who distinguished between 'natural' approaches to describe a new experience that fits a learner's current mental structures, 'alien' approaches when the learner finds no connection with any of his or her internal structures, and 'conflicting' when the learner realizes the experience is inconsistent with them.

Pinto and Tall (1999) used the terms 'natural' to describe the process of extracting meaning and 'formal' for the process of giving meaning by working with the formal deductions. To the categories 'natural' and 'formal', Weber (2004) added the notion of 'procedural' learning for those students who simply attempted to cope with formal definitions and proof by learning them by rote.

Alcock and Weber (2004) later classified the responses of students into 'semantic' and 'syntactic', using terms from linguistics that essentially refer to the *meaning* of language (its semantic content) and the *structure* of the language (its syntax). They described a syntactic approach as one in which 'the prover works from a literal reading of the involved definitions' and a semantic approach 'in which the prover also makes use of his or her intuitive understanding of the concepts.' These are broadly consistent (though not identical) with the extracting and giving of meaning and the related categories of 'formal' and 'natural'.

This reveals the complexity that students face in attempting to make sense of formal mathematical theory. Overall, it may be seen as a journey making sense of axiomatic systems by natural or formal means, beginning with specific axioms and seeking to construct a more flexible knowledge structure. For instance, one may begin with the definition of completeness in terms, say, of an increasing sequence bounded above having a limit, and then deduce other properties such as the convergence of Cauchy sequences or the existence of a least upper bound for a non-empty subset bounded above. All of these properties may then be seen to be equivalent ways of defining completeness, with the ultimate goal being the conception of a complete ordered field as a crystalline concept having a tight coherent structure with the various forms of completeness being used flexibly as appropriate in a given situation.

In this way, expertise develops as hypothesized in figure 3, from the students' previous growing experience, shifting from description to formal definition, constructing links through formal proof, establishing the equivalence of various definitions, and then building increasingly flexible links to construct crystalline concepts that can be mentally manipulated in flexible ways.

Evidence for this further development was given by Weber (2001) who investigated how four undergraduates and four research students responded to formal problems, such as the question of whether there is an isomorphism between the group of integers $\mathbb{Z}$ under addition and the group $\mathbb{Q}$ of rational numbers under addition. None of the four undergraduates could provide a formal response but all four graduates were able to do so. While the undergraduates attempted to deal with the proof in terms of definition to find a bijection that preserved the operation, the graduates used their more flexible knowledge structures to consider whether an isomorphism was even possible.

Some undergraduates focused on their memory that $\mathbb{Z}$ and $\mathbb{Q}$ have the same cardinal number and so already had a bijective correspondence between them. They had part of the idea—a bijection—but not a bijection that respected the operation. Meanwhile the graduates used their wider knowledge structures, including a wider repertoire of proving strategies, to suggest possible ways of thinking about the problem. For instance, one immediately declared that $\mathbb{Q}$ and $\mathbb{Z}$ could not be isomorphic, first by speculating that $\mathbb{Q}$ is dense but $\mathbb{Z}$ is not, then that '$\mathbb{Z}$ is cyclic, but $\mathbb{Q}$ is not' (meaning that the element 1 generates the whole of $\mathbb{Z}$ under addition but no element in $\mathbb{Q}$ does so).

7.2 Structure Theorems and new forms of Embodiment and Symbolism in Research Mathematics

In the process of building formal knowledge structures, certain central theorems play an essential role. These are *structure* theorems. They state that an axiomatic structure has certain essential properties which can often involve enriched visual and symbolic modes of operation that are now based not on naïve intuition but on formal proof. Typical examples are:

> An equivalence relation on *S* corresponds to a partition of *S*.
>
> A finite group is isomorphic to a subgroup of a group of permutations.
>
> A finite dimensional vector space over a field *F* is isomorphic to F^n.
>
> A field contains a subfield isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}_n$.
>
> An ordered field contains a subfield isomorphic to $\mathbb{Q}$.
>
> An ordered field extension *K* of $\mathbb{R}$ contains finite and infinite elements and any finite element is of the form $c + \varepsilon$ where $c \in \mathbb{R}$ and $\varepsilon \in K$ is infinitesimal.

Such theorems enable research mathematicians to develop personal knowledge structures that operate in highly flexible ways. Some build formally, others naturally, as observed by the algebraist Saunders MacLane (1994) speaking of a conversation with the geometer Michael Atiyah:

> For MacLane it meant getting and understanding the needed definitions, working with them to see what could be calculated and what might be true, to finally come up with new 'structure' theorems. For Atiyah, it meant thinking hard about a somewhat vague and uncertain situation, trying to guess what might be found out, and only then finally reaching definitions and the definitive theorems and proofs.

This division between those 'preoccupied with logic' and those 'guided by intuition' was noted long ago by Poincaré (1913), citing Hermite as a logical thinker who 'never evoked a sensuous image' in mathematical conversation and Riemann as an intuitive thinker who 'calls geometry to his aid' (p. 212).

Such 'intuition' used by mathematicians relates to deeply embedded subtle linkages in their own personal knowledge structures that suggest likely relationships before they may be amenable to formal proof. Formal proof is the final stage of research in which the argument is refined and given in a deductive manner based on precise definitions and appropriate mathematical deductions. Making sense of mathematics requires more:

> Only professional mathematicians learn anything from proofs. Other people learn from explanations. A great deal can be accomplished with arguments that fall short of proofs (Boas, 1981.)
>
> If mathematics were formally true but in no way enlightening this mathematics would be a curious game played by weird people (Rota, 1997).

Burton (2002) interviewed seventy research mathematicians (with equal numbers of males and females) to study a range of aspects under headings such as thinking styles, socio-cultural relatedness, aesthetics, intuition and connectivities. She initially hypothesized that she would find evidence of the two styles of thinking formulated above (which she described as visual and analytic) and that mathematicians would move flexibly between the two. However, her analysis of the data showed a more complex situation that she organised into three categories—visual (thinking in pictures, often dynamic), analytic (thinking symbolically, formalistically) and conceptual (thinking in ideas, classifying). The majority of those interviewed (42/70) embraced two styles, a small number (3/70) used all three and the rest referred only to one (15 visual, 3 analytic and 7 conceptual). Rather than a simple dualism, this suggests a range of thinking styles used in various combinations.

This finer analysis remains consistent with the broad framework of embodiment, symbolism and formalism, while revealing subtle distinctions. The research cycle of development builds on the experiences of the researchers and involves exploration in addition to formal proof. The visual category is concerned with embodiment in terms of 'thinking in pictures, often dynamic'. The analytic category specifically describes 'thinking symbolically, formalistically'. The conceptual category refers to 'thinking in ideas', which occurs at an exploratory phase of a cycle of research, while 'classifying' relates to classifying structures that satisfy an appropriate definition. These definitions may vary in their origins, as verbally defined embodiments, symbolic concepts related to rules of operation, or formally defined concepts that may then be proved to have a given structure. For instance simple groups are defined formally but their classification may be performed using structural properties such as generators and relations rather than formal proof as a sequence of quantified statements.

Applied mathematicians develop contextually based inference in their area of expertise, often based on embodiment of situations, translated into some kind of symbolism (such as systems of algebraic or differential equations) to manipulate and predict a solution.

Research mathematicians have a variety of ways of thinking creatively to develop new theorems. Byers (2007) observed that true creativity arises out of paradoxes, ambiguities and conflicts that occur when ideas from different contexts come into contact. This encourages mathematicians to attempt to make sense of the problem, thinking about possibilities, suggesting possible hypotheses, making new definitions, and seeking mathematical proofs.

In creating new mathematical theorems, mathematicians will work not only according to their personal preferences, but also with respect to the particular mathematical context. The possible combinations of embodiment, symbolism and formalism lead to a variety of techniques in which a particular activity may lean towards one or more modes of operation appropriate to the situation.

Mathematical proof at the highest level is an essential part of the story of development, with differently oriented mathematicians having different ways of thought but sharing common standards as to the need for proof to establish a desired result. However, a mathematical proof is not the end of the story, it is the full stop at the end of a paragraph, a place to pause and celebrate the proof of a new theorem which, in turn, becomes the launching pad for a new cycle of research and development.

8. SUMMARY

In considering the development of proof from the child to the adult and the mathematical researcher, we have embraced a wide range of viewpoints that relate directly to the ideas of proof and proving. Overall, however, we see the cognitive development beginning with the perceptions, actions and reflections of the child, developing out of the sensori-motor foundation of human thought, as the child observes regularities, builds mental concepts and makes links between them.

Different developments of proof occur in the visuo-spatial development of geometry and the symbolic development of arithmetic and algebra. In geometry, perceptions are described and knowledge structures are developed that construct relationships, develop definitions of figures, deduce equivalent properties and build up a coherent structure of Euclidean geometry. In arithmetic, general properties of operations are observed, described, then later formulated as rules that should be obeyed in algebra, enabling a form of proof based on algebraic manipulation subject to the rules of arithmetic.

The general population builds mainly on the physical, spatial and symbolic aspects of mathematics. Pure mathematicians in addition develop formally defined set-theoretic entities and formal proof. This leads to structure theorems that give the entities a rich combination of embodiment and symbolism, now supported by formal deduction. Applied mathematicians, develop contextual ways of using structures to model embodied situations as symbolic representations, solved by symbolic operations.

In each case the strand of development begins from human perception and action, through experience of objects and properties, that are described, with meanings that are refined and defined, then built into more sophisticated thinkable concepts that have a rich knowledge structure. These are connected together through relationships constructed by appropriate forms of deduction and proof to lead to sophisticated crystalline concepts whose properties are constrained by the given context. In Euclidean geometry the crystalline concepts are platonic figures that represent the essentials underlying the physical examples, in arithmetic and algebra they are flexible procepts that enable fluent calculation and manipulation of symbols, and in formal mathematics they are the total entities arising through deduction from axioms and definitions, with their essential structure revealed by structure theorems.

Proof involves a lifetime of cognitive development of the individual that is shared within societies and is further developed in sophistication by successive generations of mathematicians. Mathematical proof is designed to furnish theorems that can be used in a given context as both the culmination of a process of seeking certainty and explanation and also as a foundation for future developments in mathematical research.

References

Alcock, L., & Weber K. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, 56, 209-234.

Antonini, S. (2001). Negation in mathematics: obstacles emerging from an exploratory study. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, Vol.2, (pp. 49–56). Utrecht, The Netherlands.

Bass, H. (2009). How do you know that you know? Making believe in mathematics. Distinguished University Professor Lecture given at the University of Michigan on March 25, 2009.

Boas, R. P. (1981). Can We Make Mathematics Intelligible? *American Mathematical Monthly*, 88 (10), 727-73.

Bruner, J. S. (1966). *Towards a theory of instruction*. Cambridge, MA: Harvard University Press.

Burton, L. (2002). Recognising Commonalities and Reconciling Differences in Mathematics Education. *Educational Studies in Mathematics*, 50(2) 157-175.

Byers, W. (2007). *How Mathematicians Think*. Princeton University Press: Princeton.

Byrne, R. M. J., & Johnson-Laird, P. N. (1989). Spatial reasoning. *Journal of Memory and Language*, 28, 564-575.

Dawson, J. (2006). Why do mathematicians re-prove theorems? *Philosophia Mathematica*, 14 (3), 269-286.

Dehaene, S. (1997). *The Number Sense: How the mind creates mathematics.* New York: Oxford University Press.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thoughts. *For the Learning of Mathematics*, 6(1), 2-10.

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study.* Series: New ICMI Study Series, Vol.7 (pp. 273-280). Dordrechet: Kluwer.

Duffin, J. M., & Simpson, A. P. (1993). Natural, conflicting and alien. *Journal of Mathematical Behavior*, 12(4), 313-328.

Duffin, J. M., & Simpson, A. P. (1995). A theory, a story, its analysis, and some implications. *Journal of Mathematical Behavior*, 14, 237-250.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103-131.

Epp, S. (1998). A unified framework for proof and disproof. *The Mathematics Teacher*, 91(8) 708-713.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics,* 24, 139-162.

Flener, F. (2001). A geometry course that changed their lives: The Guinea pigs after 60 years. Paper presented at the Annual Conference of The National Council of Teachers of Mathematics April 6, 2001: Orlando, Florida. Retrieved from the internet on January 26, 2010 from http://www.maa.org/editorial/knot/NatureOfProof.html.

Flener, F. (2006). *The Guinea pigs after 60 years*. Xlibris Corporation.

Freudenthal, H. (1973). *Mathematics as an educational task*. Reidel Publishing Company: Dordrecht, Holland.

Goldin, G. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior*, 17(2), 137-165.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *The Journal for Research in Mathematics Education*, 26(2), 115–141.

Harel, G., & Sowder, L. (1998). Student's proof schemes: results from exploratory studies. In A. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education III*, (pp. 234–283). Providence, RI.

Harel, G., & Sowder, L. (2005). Advanced mathematical thinking at any age: Its nature and its development, *Mathematical Thinking and Learning*, 7, 27-50.

Healy, L., & Hoyles, C. (1998). Justifying and proving in school mathematics. Summary of the results from a survey of the proof conceptions of students in the UK. Research Report. Mathematical Sciences, Institute of Education, University of London, 601-613.

Henderson, D. W., & Taimina, D. (2005). *Experiencing geometry: Euclidean and Non-Euclidean with history,* (3rd edition). Prentice Hall, New Jersey.

Hershkowitz, R. & Vinner, S. (1983). The role of critical and non-critical attributes in the concept image of geometrical concepts. In R. Hershkowitz (Ed.), *Proceedings of the 7th international conference of the International Group for the Psychology of Mathematics Education*, 223-228. Rehovot, Israel.

Hoffmann, D. (1998). *Visual Intelligence: how we change what we see*. W.W. Norton, 1998.

Hilbert, D. (1900). The problems of mathematics. *The Second International Congress of Mathematics*. Retrieved from the internet, January 31, 2010 from http://aleph0.clarku.edu/~djoyce/hilbert/problems.html

Hilbert, D. (1928/1967). The foundations of mathematics. In J. Van Heijenoort (Ed.), From Frege to Gödel (p. 475). Cambridge, MA: Harvard University Press.

Housman, D., & Porter, M. (2003). Proof schemes and learning strategies of above-average mathematics students. *Educational Studies in Mathematics*, 53(2), 139-158.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason.* Chicago: Chicago University Press.

Klein, F. (1872). *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Erlangen: Verlag von Andreas Deichert. Available in English translation as a pdf at http://math.ucr.edu/home/baez/erlangen/erlangen_tex.pdf.

Koichu, B. (2009). What can pre-service teachers learn from interviewing high school students on proof and proving? In F.-L. Lin, F.-J. Hsieh, G. Hanna & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education,* Vol. 2, (pp. 9-15), Taipei, Taiwan: National Taiwan Normal University.

Koichu, B., & Berman, A. (2005). When do gifted high school students use geometry to solve geometry problems? *The Journal of Secondary Gifted Education*, 16(4), 168-179.

Kondratieva, M. (2009). Geometrical sophisms and understanding of mathematical proofs. In F.-L. Lin, F.-J. Hsieh, G. Hanna & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education*, Vol. 2, (pp. 3-8). Taipei, Taiwan: National Taiwan Normal University.

Lakoff, G. (1987). *Women, fire, and dangerous things: What categories reveal about the mind*. Chicago: Chicago University Press.

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh*. New York: Basic Books.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Lawler, R. W. (1980). The progressive construction of mind. *Cognitive Science*, 5, 1-34.

Lenart, I. (2003). *Non-Euclidean adventures on the Lenart sphere*. Emeryville, CA: Key Curriculum Press.

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics*, 16 (3), 321-325.

Lin, F. L., Cheng, Y. H. et al (2003). The competence of geometric argument in Taiwan adolescents. *Proceedings of the International Conference on Science and Mathematics Learning,* (pp. 16-18)*,* National Taiwan Normal University.

MacLane, S. (1994). Responses to theoretical mathematics, *Bulletin (new series) of the American Mathematical Society*, 30, 2, 190-191.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education*, 27(2), 194-214.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking Mathematically*. Addison Wesley.

Meltzoff, A. N., Kuhl, P. K., Movellan, J., & Sejnowski, T. J. (2009). Foundations for a new science of learning. *Science*, 325, 284-288.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Neto, T., Breda, A., Costa, N., & Godino, J. D. (2009). Resorting to Non-Euclidean plane geometries to develop deductive reasoning: An onto-semiotic approach. In F.-L. Lin, F.-J. Hsieh, G. Hanna & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education*, Vol. 2, 106-111. Taipei, Taiwan: National Taiwan Normal University.

Núñez, R., Edwards, L. D., & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in Mathematics Education. *Educational Studies in Mathematics,* 39, 45–65.

Paivio, A. (1991). Dual coding theory: Retrospect and current status. *Canadian Journal of Psychology*, 45, 255–287.

Papert, S. (1996). An exploration in the space of mathematics educations. *International Journal of Computers for Mathematical Learning*, Vol. 1 (1), 95-123.

Parker, J, (2005). *R. L. Moore: Mathematician and Teacher*. Mathematical Association of America, p. vii.

Pinto, M. M. F. (1998). *Students' understanding of real analysis*. PhD, University of Warwick.

Pinto, M. M. F., & Tall, D. O. (1999). Student constructions of formal theory: giving and extracting meaning. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, (pp. 65-73). Haifa, Israel.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how we can represent it? *Educational Studies in Mathematics*, 26, 2-3, 165-190.

Poincaré, H. (1913/1982). *The Foundations of Science,* (translated by Halsted G.B.) The Science Press, Reprinted: University Press of America.

Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica*, 7(1), 5-41.

Reid, D., & Dobbin, J. (1998). Why is proof by contradiction difficult? In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, (pp. 41-48). *Stellenbosch*, South Africa.

Rodd, M. M. (2000). 'On mathematical warrants'. *Mathematical Thinking and Learning*, 2, 3, 221-244.

Rota, G. C. (1997). *Indiscrete Thoughts.* Boston: Birkhauser.

Senk, S. L. (1985). How well do students write geometry proofs? *The Mathematics Teacher,* 78(6), 448-456.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics,* 22, 1-36.

Shapiro, S. (1991). *Foundations without foundationalism: A case for second-order logic*. Oxford: Clarendon Press.

Tall, D. O. (1979). Cognitive aspects of proof, with special reference to the irrationality of √2. *Proceedings of the 3rd Conference of the International Group for the Psychology of Mathematics Education*, (pp. 203-205).Warwick, UK.

Tall, D. O. (2004). Thinking through three worlds of mathematics. In M. J. Hoines and A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol.4, (pp. 281-288). Bergen, Norway.

Tall, D. O. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal,* 20 (2)*,* 5-24.

Tall, D. O. (2010, in preparation). *How humans learn to think mathematically*.

Tall, D. O., & Vinner S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12 (2), 151-169.

Thompson, D. R. (1996). Learning and teaching indirect proof. *The Mathematics Teacher*, 89 (6), 474-82.

Van Hiele, P.M. (1986). *Structure and Insight*. New York. Academic Press.

Van Hiele, P.M., & Geldof, D. (1984). The didactics of geometry in the lowest class of secondary school. In D. Fuys, D. Geddes & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele*. Brooklyn NY: Brooklyn College.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48 (1), 101-119.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, 23, 115–133.

Yevdokimov, O. (2008). Making generalisations in geometry: Students' views on the process. A case study. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepulova (Eds.), *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, (pp. 441-448). Morelia, Mexico.

Yevdokimov, O. (2010, in preparation). *Mathematical concepts in Early Childhood.*