Assignment6: Solutions

Problem 1: Find general solution of the system of equations. Sketch few trajectories in the phase space of the system. Is the zero solution stable or unstable?

1. $\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

$$\vec{x} = \vec{\xi} e^{rt}, \qquad \vec{x'} = r e^{rt} \vec{\xi}$$

Find eigenvalues:

$$det(A - rI) = 0, \qquad \text{where} \qquad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$det \begin{bmatrix} 3 - r & -2 \\ 2 & -2 - r \end{bmatrix} = 0$$

$$(3 - r)(-2 - r) - (-4) = 0$$

$$r^2 - r - 2 = 0$$

$$(r - 2)(r + 1) = 0$$

$$r_1 = -1, r_2 = 2$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} 3 + 1 & -2 \\ 2 & -2 + 1 \end{bmatrix} \vec{\xi_1} = 0$$

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} = 0$$

$$4\xi_{11} - 2\xi_{12} = 0$$
$$2\xi_{11} - \xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\xi_{21} - 2\xi_{22} = 0$$
$$2\xi_{21} - 4\xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore, the general solution is:

$$x(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

The origin is a saddle point. It is unstable.

2. $\begin{cases} x_1' = -2x_1 + x_2 \\ x_2' = x_1 - 2x_2 \end{cases}$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\vec{x} = \vec{\xi} e^{rt}, \qquad \vec{x'} = r e^{rt} \vec{\xi}$$

Find eigenvalues:

$$det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$det \begin{bmatrix} -2-r & 1 \\ 1 & -2-r \end{bmatrix} = 0$$

$$(-2-r)(-2-r) - 1 = 0$$

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0$$

$$r_1 = -3, r_2 = -1$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix} \vec{\xi_1} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} = 0$$

$$\xi_{11} + \xi_{12} = 0$$

$$\xi_{11} + \xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$-\xi_{21} + \xi_{22} = 0$$

$$\xi_{21} - \xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, the general solution is:

$$x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

The origin is node. It is stable.

3. $\begin{cases} x_1' = \frac{5}{4}x_1 + \frac{3}{4}x_2 \\ x_2' = \frac{3}{4}x_1 + \frac{5}{4}x_2 \end{cases}$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$$

$$\vec{x} = \vec{\xi} e^{rt}, \qquad \vec{x'} = r e^{rt} \vec{\xi}$$

Find eigenvalues:

$$\det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$\det \begin{bmatrix} \frac{5}{4} - r & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - r \end{bmatrix} = 0$$

$$\left(\frac{5}{4} - r\right)\left(\frac{5}{4} - r\right) - \left(\frac{3}{4}\right)^2 = 0$$

$$r^2 - \frac{5}{2} r + 1 = 0$$

$$r_1 = 2, r_2 = \frac{1}{2}$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} \frac{5}{4} - 2 & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - 2 \end{bmatrix} \vec{\xi_1} = 0$$

$$\begin{bmatrix} \frac{-3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{-3}{4} \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} = 0$$

$$\frac{-3}{4}\xi_{11} + \frac{3}{4}\xi_{12} = 0$$

$$\frac{3}{4}\xi_{11} - \frac{3}{4}\xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} \frac{5}{4} - \frac{1}{2} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} - \frac{1}{2} \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\frac{3}{4}\xi_{21} + \frac{3}{4}\xi_{22} = 0$$

$$\frac{3}{4}\xi_{21} + \frac{3}{4}\xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the general solution is:

$$x(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{1}{2}t}$$

The origin is an unstable node.

4. $\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 8x_1 - 6x_2 \end{cases}$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$$

$$\vec{x} = \vec{\xi} e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$\det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$\det \begin{bmatrix} 4-r & -3 \\ 8 & -6-r \end{bmatrix} = 0$$

$$(4-r)(-6-r) - (-3)(8) = 0$$

$$r^2 + 2r = 0$$

$$r(r+2) = 0$$

$$r_1 = 0, r_2 = -2$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} 4-0 & -3 \\ 8 & -6-0 \end{bmatrix} \vec{\xi_1} = 0$$

$$\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} = 0$$

$$4\xi_{11} - 3\xi_{12} = 0$$
$$8\xi_{11} - 6\xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} 4+2 & -3 \\ 8 & -6+2 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$6\xi_{21} - 3\xi_{22} = 0$$
$$8\xi_{21} - 4\xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore, the general solution is:

$$x(t) = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$$

Problem 2: Find general solution as a real valued function. Sketch the phase portrait of the system.

1. $\begin{cases} x_1' = -x_1 - 4x_2 \\ x_2' = x_1 - x_2 \end{cases}$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$$

$$\vec{x} = \vec{\xi} e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$\det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$\det \begin{bmatrix} -1-r & -4 \\ 1 & -1-r \end{bmatrix} = 0$$

$$(-1-r)(-1-r)-(-4)(1)=0$$

$$r^2+2r+5=0$$

$$r_1=-1+2i, r_2=-1-2i$$

Find eigenvectors:

$$(A-r_1I)\vec{\xi_1}=0$$

$$\begin{bmatrix} -1-(-1+2i) & -4 \\ 1 & -1-(-1+2i) \end{bmatrix}\vec{\xi_1}=0$$

$$\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix}\begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix}=0$$

$$-2i\xi_{11}-4\xi_{12}=0$$

$$\xi_{11}-2i\xi_{12}=0$$

$$\vec{\xi_1}=\begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$(A-r_2I)=\vec{\xi_2}=0$$

$$\begin{bmatrix} -1-(-1-2i) & -4 \\ 1 & -1-(-1-2i) \end{bmatrix}\begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix}=0$$

$$\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix}\begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix}=0$$

$$2i\xi_{21}-4\xi_{22}=0$$

$$\xi_{21}+2i\xi_{22}=0$$

$$\vec{\xi_2}=\begin{bmatrix} 2i \\ -1 \end{bmatrix}$$

Therefore, the general solution is:

$$x(t)=c_1\begin{bmatrix} 2i \\ 1 \end{bmatrix}e^{(-1+2i)t}+c_2\begin{bmatrix} 2i \\ -1 \end{bmatrix}e^{(-1-2i)t}$$

To obtain real solution:

$$\vec{v_1} = Re\vec{x_1}$$

$$\vec{x_1} = \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{(-1+2i)t}$$

$$= \begin{bmatrix} 2ie^{-t}\cos 2t + (2i)ie^{-t}\sin 2t \\ e^{-t}\cos 2t + ie^{-t}\sin 2t \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{bmatrix} + i \begin{bmatrix} 2e^{-t}\sin 2t \\ -e^{-t}\cos 2t \end{bmatrix}$$

Therefore, the final solution is:

$$\vec{x} = c_1 \begin{bmatrix} 2e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} 2e^{-t}\sin 2t \\ -e^{-t}\cos 2t \end{bmatrix}$$

The origin is a spiral. It is stable.

2. $\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 4x_1 - x_2 \end{cases}$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

$$\vec{x} = \vec{\xi}e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$\det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$\det \begin{bmatrix} 3-r & -2 \\ 4 & -1-r \end{bmatrix} = 0$$

$$(3-r)(-1-r) - (4)(-2) = 0$$

$$r^2 - 2r + 5 = 0$$

$$r_1 = 1 + 2i, r_2 = 1 - 2i$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{bmatrix} \vec{\xi_1} = 0$$

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} = 0$$

$$(2-2i)\xi_{11} - 2\xi_{12} = 0$$

$$4\xi_{11} + (-2-2i)\xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} 3-(1-2i) & -2 \\ 4 & -1-(1-2i) \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$(2+2i)\xi_{21} - 2\xi_{22} = 0$$

$$4\xi_{21} - (-2+2i)\xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

Therefore, the general solution is:

$$x(t) = c_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(1+2i)t} + c_2 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(1-2i)t}$$

To obtain real solution:

$$\vec{v_1} = Re\vec{x_1}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(1+2i)t}$$

$$= \begin{bmatrix} e^t \cos 2t + ie^t \sin 2t \\ (1-i)e^t \cos 2t + (1-i)ie^t \sin 2t \end{bmatrix}$$

$$= \begin{bmatrix} e^t \cos 2t + ie^t \sin 2t \\ e^t \cos 2t - ie^t \cos 2t + e^t i \sin 2t + e^t \sin 2t \end{bmatrix}$$

$$= \begin{bmatrix} e^t \cos 2t \\ e^t \cos 2t + e^t \sin 2t \end{bmatrix} + i \begin{bmatrix} e^t \sin 2t \\ -e^t \cos 2t + e^t \sin 2t \end{bmatrix}$$

Therefore, the final solution is:

$$\vec{x} = c_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{bmatrix}$$

The origin is a spiral. It is unstable.

3.$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = -5x_1 - x_2 \end{cases}$.

Soltuion:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$$

$$\vec{x} = \vec{\xi} e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$\det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$\det \begin{bmatrix} 1-r & 2 \\ -5 & -1-r \end{bmatrix} = 0$$

$$(1-r)(-1-r) - (-5)(2) = 0$$

$$r^2 + 9 = 0$$

$$r_1 = +3i, r_2 = -3i$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} 1-3i & 2 \\ -5 & -1-3i) \end{bmatrix} \vec{\xi_1} = 0$$

$$(1-3i)\xi_{11} + 2\xi_{12} = 0$$
$$-5\xi_{11} + (-1-3i)\xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 1 \\ \frac{-1+3i}{2} \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} 1-(-3i) & 2 \\ -5 & -1-(-3i) \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} 1+3i & 2 \\ -5 & -1+3i \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$(1+3i)\xi_{21} + 2\xi_{22} = 0$$

$$-5\xi_{21} - (-1+3i)\xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 1 \\ \frac{-1-3i}{2} \end{bmatrix}$$

Therefore, the general solution is:

$$x(t) = c_1 \begin{bmatrix} 1 \\ \frac{-1+3i}{2} \end{bmatrix} e^{(3i)t} + c_2 \begin{bmatrix} 1 \\ \frac{-1-3i}{2} \end{bmatrix} e^{(-3i)t}$$

To obtain real solution:

$$\vec{v_1} = Re\vec{x_1}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ \frac{-1+3i}{2} \end{bmatrix} e^{(3i)t}$$

$$= \begin{bmatrix} \cos 3t + i \sin 3t \\ (\frac{-1+3i}{2} \cos 3t + \frac{-1+3i}{2}) i \sin 3t \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos 3t + 2i \sin 3t \\ -\cos 3t + 3i \cos 3t - i \sin 3t - 3 \sin 3t \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos 3t \\ -\cos 3t - 3\sin 3t \end{bmatrix} + i \begin{bmatrix} 2\sin 3t \\ 3\cos 3t - \sin 3t \end{bmatrix}$$

Therefore, the final solution is:

$$\vec{x} = c_1 \begin{bmatrix} 2\cos 3t \\ -\cos 3t - 3\sin 3t \end{bmatrix} + c_2 \begin{bmatrix} 2\sin 3t \\ 3\cos 3t - \sin 3t \end{bmatrix}$$

The origin is a center.

Problem 3: Find general solution. Sketch few trajectories.

1. $\begin{cases} x_1' = 3x_1 - 4x_2 \\ x_2' = x_1 - x_2 \end{cases}$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$\vec{x} = \vec{\xi}e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$det \begin{bmatrix} 3 - r & -4 \\ 1 & -1 - r \end{bmatrix} = 0$$

$$(3 - r)(-1 - r) - (-4)(1) = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r - 1)(r - 1) = 0$$

$$r_1 = 1, r_2 = 1$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \vec{\xi_1} = 0$$

$$2\xi_{11} - 4\xi_{12} = 0$$
$$\xi_{11} - 2\xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

Therefore, $\vec{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$

To find $\vec{x_2}$ we:

$$(A - rI)\vec{\eta} = \vec{\xi}$$

$$\left[\begin{array}{cc} 3 & -4 \\ 1 & -1 \end{array}\right] - \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \left[\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right] = \vec{\xi}$$

$$\left[\begin{array}{cc} 2 & -4 \\ 1 & -2 \end{array}\right] \left[\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right] = \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$$

$$2\eta_1 - 4\eta_2 = 2$$

$$\eta_1 - 2\eta_2 = 1$$

$$\vec{\eta} = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

$$\vec{x_2} = t \left[\begin{array}{c} 2 \\ 1 \end{array}\right] e^t + \left[\begin{array}{c} 2 \\ 0 \end{array}\right] e^t$$

Therefore, the general solution is:

$$x(t) = c_1 \left[\begin{array}{c} 2 \\ 1 \end{array}\right] e^t + c_2 t \left[\begin{array}{c} 2 \\ 1 \end{array}\right] e^t + \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

2.$\left\{\begin{array}{l} x_1' = -\frac{3}{2}x_1 + x_2 \\ x_2' = -\frac{1}{4}x_1 - \frac{1}{2}x_2 \end{array}\right.$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \left[\begin{array}{cc} \frac{-3}{2} & 1 \\ \frac{-1}{4} & \frac{-1}{2} \end{array}\right]$$

$$\vec{x} = \vec{\xi} e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$det(A - rI) = 0, \qquad \text{where} \quad rI = \left[\begin{array}{cc} r & 0 \\ 0 & r \end{array}\right]$$

$$det \left[\begin{array}{cc} \frac{-3}{2} - r & 1 \\ \frac{-1}{4} & \frac{-1}{2} - r \end{array}\right] = 0$$

$$\left(\frac{-3}{2} - r\right)\left(\frac{-1}{2} - r\right) - \left(\frac{-1}{4}\right)(1) = 0$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)(r+1) = 0$$

$$r_1 = -1, r_2 = -1$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi} = 0$$

$$\begin{bmatrix} \frac{-1}{2} & 1 \\ \frac{-1}{4} & \frac{1}{2} \end{bmatrix} \vec{\xi_1} = 0$$

$$\frac{-1}{2}\xi_1 - 4\xi_2 = 0$$

$$\frac{-1}{4}\xi_1 + \frac{1}{2}\xi_2 = 0$$

$$\vec{\xi} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore, $\vec{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$

To find $\vec{x_2}$ we:

$$(A - rI)\vec{\eta} = \vec{\xi}$$

$$\begin{bmatrix} \frac{-1}{2} & 1 \\ \frac{-1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\frac{-1}{2}\eta_1 + \eta_2 = 2$$

$$\frac{-1}{4}\eta_1 + \frac{1}{2}\eta_2 = 1$$

$$\vec{\eta} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\vec{x_2} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t}$$

Therefore, the general solution is:

$$x(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 t \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-t}$$

Problem 4: Solve initial value problem.

1.$\begin{cases} x_1' = 2x_1 + 1.5x_2 \\ x_2' = -1.5x_1 - x_2 \end{cases}$. $\quad x_1(0) = 3, \quad x_2(0) = -2$

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} 2 & 1.5 \\ -1.5 & -1 \end{bmatrix}$$

$$\vec{x} = \vec{\xi}e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$det \begin{bmatrix} 2-r & 1.5 \\ -1.5 & -1-r \end{bmatrix} = 0$$

$$(2-r)(-1-r) - (1.5)(-1.5) = 0$$

$$r^2 - r + 0.25 = 0$$

$$\left(r - \frac{1}{2}\right)\left(r - \frac{1}{2}\right) = 0$$

$$r_1 = \frac{1}{2}, r_2 = \frac{1}{2}$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi} = 0$$

$$\begin{bmatrix} \frac{3}{2} & 1.5 \\ -1.5 & \frac{-3}{2} \end{bmatrix} \vec{\xi_1} = 0$$

$$\frac{3}{2}\xi_1 + \frac{3}{2}\xi_2 = 0$$

$$\frac{-3}{2}\xi_1 - \frac{3}{2}\xi_2 = 0$$

$$\vec{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, $\vec{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{1}{2}t}$

To find $\vec{x_2}$ we:

$$(A - rI)\vec{\eta} = \vec{\xi}$$

$$\begin{bmatrix} \frac{3}{2} & 1.5 \\ -1.5 & \frac{-3}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\frac{3}{2}\eta_1 + \frac{3}{2}\eta_2 = 1$$

$$\frac{-3}{2}\eta_1 - \frac{3}{2}\eta_2 = -1$$

$$\vec{\eta} = \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix}$$

$$\vec{x_2} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{1}{2}t} + \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix} e^{\frac{1}{2}t}$$

Therefore, the general solution is:

$$x(t) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{1}{2}t} + c_2 t\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{1}{2}t} + c_2\begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix} e^{\frac{1}{2}t}$$

Now to solve the initial value problem:

$$x_1(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t} + \frac{2}{3}c_2 e^{\frac{1}{2}t}$$

$$x_2(t) = -c_1 e^{\frac{1}{2}t} - c_2 t e^{\frac{1}{2}t} + (0)e^{\frac{1}{2}t}$$

$$x_1(0) = c_1 + 0 + \frac{2}{3}c_2 = 3$$

$$c_1 = 3 - \frac{2}{3}c_2$$

$$c_2 = \frac{3}{2}$$

$$x_2(0) = -c_1 - (0) = -2$$

$$c_1 = 2$$

Therefore, final solution is:

$$\vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{\frac{1}{2}t} + \frac{3t}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{1}{2}t}$$

2. $\begin{cases} x_1' = x_1 - 5x_2 \\ x_2' = x_1 - 3x_2 \end{cases}$. $\quad x_1(0) = 1, \quad x_2(0) = 1$

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$$

$$\vec{x} = \vec{\xi} e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$\det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$\det \begin{bmatrix} 1-r & -5 \\ 1 & -3-r \end{bmatrix} = 0$$

$$(1-r)(-3-r) - (-5)(1) = 0$$

$$r^2 + 2r + 2 = 0$$

$$r_1 = -1+i, r_2 = -1-i$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{bmatrix} \vec{\xi_1} = 0$$

$$(2-i)\xi_{11} - 5\xi_{12} = 0$$
$$\xi_{11} + (-2-i)\xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 1 \\ \frac{1}{2+i} \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} 1-(-1-i) & -5 \\ 1 & -3-(-1-i) \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$(2+i)\xi_{21} - 5\xi_{22} = 0$$
$$\xi_{21} + (-2+i)\xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 1 \\ \frac{1}{2-i} \end{bmatrix}$$

Therefore, the general solution is:

$$x(t) = c_1 \begin{bmatrix} 1 \\ \frac{1}{2+i} \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ \frac{1}{2-i} \end{bmatrix} e^{(-1-i)t}$$

To obtain real solution:

$$\vec{v_1} = Re\vec{x_1}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ \frac{1}{2+i} \end{bmatrix} e^{(-1+i)t}$$

$$= \begin{bmatrix} e^{-t}\cos t + e^{-t} i \sin t \\ (\frac{1}{2+i} e^{-t}\cos t + \frac{1}{2+i}) i e^{-t}\sin t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}\cos t + e^{-t} i \sin t \\ \frac{2}{5}e^{-t}\cos t - \frac{i}{5}e^{-t}\cos t + \frac{2i}{5}e^{-t}\sin t + \frac{1}{5}e^{-t}\sin t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}\cos t \\ \frac{2}{5}e^{-t}\cos t + \frac{1}{5}e^{-t}\sin t \end{bmatrix} + i \begin{bmatrix} e^{-t}\sin t \\ -\frac{1}{5}e^{-t}\cos t + \frac{2}{5}e^{-t}\sin t \end{bmatrix}$$

Therefore, the general solution is:

$$\vec{x} = c_1 \begin{bmatrix} e^{-t}\cos t \\ \frac{2}{5}e^{-t}\cos t + \frac{1}{5}e^{-t}\sin t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\sin t \\ -\frac{1}{5}e^{-t}\cos t + \frac{2}{5}e^{-t}\sin t \end{bmatrix}$$

Now to solve the initial value problem:

$$x_1(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

$$x_2(t) = \frac{2}{5}c_1 e^{-t}\cos t + \frac{1}{5}c_1 t e^{-t}\sin t + \frac{1}{5}c_2 e^{-t}\cos t + \frac{2}{5}c_2 e^{-t}\sin t$$

$$x_1(0) = c_1 e^0 \cos 0 + c_2 e^0 \sin 0 = 1$$

$$c_1 = 1$$

$$x_2(0) = \frac{2}{5}c_1 e^0 \cos 0 + \frac{1}{5}c_1 e^0 \sin 0 + \frac{1}{5}e^0 \cos 0 + \frac{2}{5}c_2 e^0 \sin 0 = 1$$

$$\frac{2}{5}(1) + \frac{1}{5}c_2 = 1$$

$$c_2 = -3$$

Therefore, final solution is:

$$\vec{x} = e^{-t}\begin{bmatrix} \cos t \\ \frac{2}{5}\cos t + \frac{1}{5}\sin t \end{bmatrix} - 3e^{-t}\begin{bmatrix} \sin t \\ -\frac{1}{5}\cos t + \frac{2}{5}\sin t \end{bmatrix}$$

$$\vec{x} = e^{-t}\begin{bmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{bmatrix}$$

3.$\begin{cases} x_1' = -2x_1 + x_2 \\ x_2' = -5x_1 + 4x_2 \end{cases}$. $x_1(0) = 1, \quad x_2(0) = 3$

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}$$

$$\vec{x} = \vec{\xi}e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$det\begin{bmatrix} -2-r & 1 \\ -5 & 4-r \end{bmatrix} = 0$$

$$(-2-r)(4-r) - (-5)(1) = 0$$

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$r_1 = 3, r_2 = -1$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} -2-3 & 1 \\ -5 & 4-3 \end{bmatrix}\vec{\xi_1} = 0$$

$$\begin{bmatrix} -5 & 1 \\ -5 & 0 \end{bmatrix}\begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} = 0$$

$$-5\xi_{11} + \xi_{12} = 0$$
$$-5\xi_{11} + \xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} -2+1 & 1 \\ -5 & 4+1 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$-\xi_{21} + \xi_{22} = 0$$

$$-5\xi_{21} + 5\xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, the general solution is:

$$x(t) = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Now to solve the initial value problem:

$$x_1(t) = c_1 e^{3t} + c_2 e^{-t}$$

$$x_2(t) = 5c_1 e^{3t} + c_2 e^{-t}$$

$$x_1(0) = c_1 e^0 + c_2 e^0 = 1$$

$$c_1 = 1 - c_2$$

$$c_1 = \frac{1}{2}$$

$$x_2(0) = 5c_1 e^0 + c_2 e^0 = 3$$

$$c_2 = 3 - 5c_1$$

$$c_2 = \frac{1}{2}$$

Therefore, final solution is:

$$\vec{x} = \frac{1}{2}e^{3t}\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \frac{1}{2}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 5: Find general solution of non-homogeneous linear system.

1. $\begin{cases} x_1' = 2x_1 - x_2 + e^t \\ x_2' = 3x_1 - 2x_2 + t \end{cases}$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

$$\vec{x} = \vec{\xi}e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$det\begin{bmatrix} 2-r & -1 \\ 3 & -2-r \end{bmatrix} = 0$$

$$(2-r)(-2-r) - (-1)(3) = 0$$

$$r^2 - 1 = 0$$

$$(r-1)(r+1) = 0$$

$$r_1 = -1, r_2 = 1$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} -2-(-1) & -1 \\ 3 & -2-(-1) \end{bmatrix}\vec{\xi_1} = 0$$

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$$

$$3\xi_{11} - \xi_{12} = 0$$

$$3\xi_{11} - \xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$\xi_{21} - \xi_{22} = 0$$

$$3\xi_{21} - 3\xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = (\vec{a}t + \vec{b})e^t + \vec{c}t + \vec{d}$$

substitute into equation

$$\vec{x'}(t) = (\vec{a} + \vec{a}t + \vec{b})e^t + \vec{c}$$

$$A\vec{x} + \vec{g} = (A\vec{a}t + \vec{b})e^t + A\vec{c} + A\vec{d} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\vec{a} = \vec{a}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$2a_1 - a_2 = a_1$$

$$3a_1 - 2a_2 = a_2$$

$$\vec{a} = \begin{bmatrix} k \\ k \end{bmatrix},$$

where k is an arbitrary number to be determined in the next step.

Solve the second equation.

$$\vec{b} + \vec{a} = A\vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{a} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\vec{b} - \vec{b}$$

$$\begin{bmatrix} k \\ k \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (A - I)\vec{b}$$

$$\begin{bmatrix} k-1 \\ k \end{bmatrix} = \begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} \vec{b}$$

$$\begin{bmatrix} k-1 \\ k \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \vec{b}$$

$$b_1 - b_2 = k - 1$$

$$3b_1 - 3b_2 = k$$

Here k is any number. To have the system consistent we require $k = 3(k-1)$. Thus $k = 3/2$. Then

$$b_1 = b_2 + \frac{1}{2}$$

$$\vec{b} = \begin{bmatrix} b_2 + \frac{1}{2} \\ b_2 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Here b_2 is any number, so taking it to be 0 we have

$$\vec{b} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \quad \vec{a} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

To solve for $\vec{c}$:

$$0 = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$0 = 2c_1 - c_2 + 0$$

$$3c_1 - 2c_2 + 1$$

$$\vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Now to solve for $\vec{d}$:

$$\vec{c} = A\vec{d}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$1 = 2d_1 - d_2$$
$$2 = 3d_1 - 2d_2$$

$$\vec{d} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Therefore, the general solution is:

$$\vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + te^t \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} + e^t \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

2. $\begin{cases} x_1' = x_1 + \sqrt{3}x_2 + e^t \\ x_2' = \sqrt{3}x_1 - x_2 + \sqrt{3}e^{-t} \end{cases}$.

Solution:

$$\vec{x} = A\vec{x}, \qquad A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

$$\vec{x} = \vec{\xi}e^{rt}, \qquad \vec{x'} = re^{rt}\vec{\xi}$$

Find eigenvalues:

$$det(A - rI) = 0, \qquad \text{where} \quad rI = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

$$det \begin{bmatrix} 1 - r & \sqrt{3} \\ \sqrt{3} & -1 - r \end{bmatrix} = 0$$

$$(1 - r)(-1 - r) - (-\sqrt{3})(\sqrt{3}) = 0$$

$$r^2 - 4 = 0$$

$$(r - 2)(r + 2) = 0$$

$$r_1 = 2, r_2 = -2$$

Find eigenvectors:

$$(A - r_1 I)\vec{\xi_1} = 0$$

$$\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{bmatrix} \vec{\xi_1} = 0$$

$$-\xi_{11} + \sqrt{3}\xi_{12} = 0$$
$$\sqrt{3}\xi_{11} - 3\xi_{12} = 0$$

$$\vec{\xi_1} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

$$(A - r_2 I) = \vec{\xi_2} = 0$$

$$\begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} = 0$$

$$3\xi_{21} + \sqrt{3}\xi_{22} = 0$$
$$\sqrt{3}\xi_{21} + \xi_{22} = 0$$

$$\vec{\xi_2} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$

$$\vec{x}(t) = \vec{a}e^t + \vec{b}e^{-t}$$

substitute into equation

$$\vec{x'}(t) = \vec{a}e^t - \vec{b})e^{-t}$$

$$A\vec{x} + \vec{g} = \vec{x'}$$

$$A\vec{a}e^t + Ae^{-t}\vec{b} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-t} \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} = \vec{a}e^t - \vec{b}e^{-t}$$

$$A\vec{a} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{a}$$

$$\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$a_1 + \sqrt{3}a_2 - a_1 = -1$$
$$\sqrt{3}a_1 - a_2 - a_2 = 0$$

$$\vec{a} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A\vec{b} + \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} = -\vec{b}$$

$$\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} = - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$b_1 + \sqrt{3} b_2 + b_1 = 0$$
$$\sqrt{3} b_1 - b_2 + b_2 = -\sqrt{3}$$

$$\vec{b} = \begin{bmatrix} -1 \\ \frac{2}{\sqrt{3}} \end{bmatrix}$$

Therefore, the general solution is:

$$\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} + e^{t} \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} + e^{-t} \begin{bmatrix} -1 \\ \frac{2}{\sqrt{3}} \end{bmatrix}$$