Assignment2 Solutions

Problem 1: For the following autonomous equation answer the questions
(a.) find all equilibrium solutions and determine their types of stability;
(b.) investigate growth/decay and concavity of solutions for different initial conditions;
(c.) plot the solutions in (t,y)-plane;
(d.) give the solution by an analytic formula
1. $y' = (1-y)(y-3)$

Let $f(y) = (1-y)(y-3)$

Equilibrium solutions such that $f(y) = 0$ are:

$$y(t) = 1, \qquad y(t) = 3$$

$$\text{If} \quad y(t) < 1 \quad \text{then} \quad f(y) < 0 \Longrightarrow y' < 0 \Longrightarrow y(t) \downarrow$$

$$\text{If} \quad y(t) > 3 \quad \text{then} f(y) < 0 \Longrightarrow y' < 0 \Longrightarrow y(t) \downarrow$$

$$\text{If} \quad 1 < y(t) < 3 \quad \text{then} f(y) > 0 \Longrightarrow y' > 0 \Longrightarrow y(t) \uparrow$$

From the above information, we can say that $y = 1$ is unstable and $y = 3$ is stable.

Solution of analytic formula:

$$y' = (1-y)(y-3)$$

$$\int \frac{1}{(1-y)(y-3)} dy = \int dx$$

Now must solve LHS by using partial fractions:

$$\frac{A}{y-1} + \frac{B}{y-3}$$

$$\Longrightarrow \frac{Ay - 3A + By - B}{(y-1)(y-3)}$$

$$\Longrightarrow \frac{(A+B)y + B - 3A}{(y-1)(y-3)}$$

Therefore, $A + B = 0$ and $B - 3A = 1$ and you get $A = \frac{1}{2}$ and $B = \frac{-1}{2}$

Now the LHS becomes:

$$\int \frac{1}{2(y-1)} dy - \int \frac{1}{2(y-3)} dy$$
$$\frac{\ln|y-1|}{2} - \frac{\ln|y-3|}{2}$$

So therefore, the implicit formula is:

$$\frac{\ln|y-1|}{2} - \frac{\ln|y-3|}{2} = x + c$$

2. $y' = ry - ky^2$

Let $f(y) = ry - ky^2$, where $f(y) = ry(1 - \frac{k}{r}y)$

Here $k > 0$, $r > 0$ are given constants. Denote $M = \frac{k}{r} > 0$. Then $f(y) = ry(1 - My)$.

Equilibrium solutions such that $f(y) = 0$ are:

$$y(t) = 0, \qquad y(t) = \frac{1}{M}$$

$$\text{If} \quad y(t) < 0 \quad \text{then} \quad f(y) < 0 \Longrightarrow y' < 0 \Longrightarrow y(t) \downarrow$$

$$\text{If} \quad y(t) > \frac{1}{M} \quad \text{then} f(y) < 0 \Longrightarrow y' < 0 \Longrightarrow y(t) \downarrow$$

$$\text{If} \quad 0 < y(t) < \frac{1}{M} \quad \text{then} f(y) > 0 \Longrightarrow y' > 0 \Longrightarrow y(t) \uparrow$$

From the above information, we can say that $y = 0$ is unstable and $y = \frac{1}{M}$ is stable.

Solution of analytic formula:

$$y' = y(1 - My)$$

$$\int \frac{1}{y(1-My)} dy = \int r dx$$

Can use partial fractions here for the LHS as we did in the previous question. So the LHS becomes:

$$\int \frac{1}{y} + \frac{M}{1 - My} dy$$

$$\ln|y| - \ln|1 - My|$$

So therefore, the implicit formula is:

$$\ln|y| - \ln|1 - My| = rx + c$$

3. $y' = (1 - y)(3 - y)^2$

Let $f(y) = (1 - y)(3 - y)^2$

Equilibrium solutions such that $f(y) = 0$ are:

$$y(t) = 1, \qquad y(t) = 3$$

$$\text{If} \quad y(t) < 1 \quad \text{then} \quad f(y) > 0 \Longrightarrow y' > 0 \Longrightarrow y(t) \uparrow$$

$$\text{If} \quad y(t) > 3 \quad \text{then} f(y) > 0 \Longrightarrow y' > 0 \Longrightarrow y(t) \uparrow$$

$$\text{If} \quad 1 < y(t) < 3 \quad \text{then} f(y) < 0 \Longrightarrow y' < 0 \Longrightarrow y(t) \downarrow$$

From the above information, we can say that $y = 1$ is stable and $y = 3$ is semistable.

Solution of analytic formula:

$$y' = (1 - y)(3 - y)^2$$

$$\int \frac{1}{(1 - y)(3 - y)^2} dy = \int dx$$

Now must solve LHS by using partial fractions:

$$\frac{A}{1-y}+\frac{B}{3-y}+\frac{C}{3-y}$$

Now the LHS becomes:

$$\frac{-\ln|1-y|}{4}+\frac{\ln|3-y|}{4}+\frac{1}{2(y-3)}$$

So therefore, the implicit formula is:

$$\frac{-\ln|1-y|}{4}+\frac{\ln|3-y|}{4}+\frac{1}{2(y-3)}=x+c$$

4. $y' = \cos y$

Let $f(y) = \cos y$

Equilibrium solutions such that $f(y) = 0$ are:

$$y(t) = \pm\pi/2, \qquad y(t) = \pm 3\pi/2$$

$$\text{If} \quad y \in (-3\pi/2, -\pi/2) \quad \text{then} \quad f(y) < 0 \implies y' < 0 \implies y(t) \downarrow$$

$$\text{If} \quad y \in (-\pi/2, \pi/2) \quad \text{then} f(y) > 0 \implies y' > 0 \implies y(t) \uparrow$$

$$\text{If} \quad y \in (\pi/2, 3\pi/2) \quad \text{then} f(y) < 0 \implies y' < 0 \implies y(t) \downarrow$$

From the above information, we can say that $y = \pi/2$ is stable and $y = 3\pi/2$ is unstable.

Solution of analytic formula:

$$y' = \cos y$$

$$\int \frac{1}{\cos y} dy = \int dx$$

$$\int \sec y dy = \int dx$$

So therefore, the implicit formula is:

$$\ln|\sec y + \tan y| = x + c$$

5. $y' = \cos^2 y$

Let $f(y) = \cos^2 y$

Equilibrium solutions such that $f(y) = 0$ are:

$$y(t) = \pm\pi/2, \qquad y(t) = \pm 3\pi/2$$

$$\text{If} \quad \cos^2 y \geq 0 \quad \text{then} \quad f(y) \geq 0 \Longrightarrow y' \geq 0 \Longrightarrow y(t) \uparrow$$

From the above information, we can say that $y(t)$ is semistable.

Solution of analytic formula:

$$y' = \cos^2 y$$

$$\int \frac{1}{\cos^2 y} dy = \int dx$$

$$\int \sec^2 y dy = \int dx$$

So therefore, the implicit formula is:

$$\tan y = x + c$$

6. $y' = y - y^4$

Let $f(y) = y(1 - y^3)$

Equilibrium solutions such that $f(y) = 0$ are:

$$y(t) = 0, \qquad y(t) = 1$$

$$\text{If} \quad y(t) > 1 \quad \text{then} \quad f(y) < 0 \Longrightarrow y' < 0 \Longrightarrow y(t) \downarrow$$

$$\text{If} \quad y(t) < 0 \quad \text{then} f(y) < 0 \Longrightarrow y' < 0 \Longrightarrow y(t) \downarrow$$

$$\text{If} \quad 0 < y(t) < 1 \quad \text{then} f(y) > 0 \Longrightarrow y' > 0 \Longrightarrow y(t) \uparrow$$

From the above information, we can say that $y = 1$ is stable and $y = 0$ is unstable.

Solution of analytic formula:

$$y' = y(1 - y^3)$$

$$y'y^{-4} = y^{-3} - 1$$

Now let $u = y^{-3}$ and $u' = -3y^{-4}y'$:

$$u' = -3u + 3$$

$$u = ce^{-3x} + 1$$

So therefore, the implicit formula is:

$$y(x) = \frac{1}{(1 + ce^{-3x})^{\frac{1}{3}}}$$

Problem 2: Determine whether or not each of the following equations is exact. If it is exact then find the solution.

(a.)$(5x + 7) + (6y + 11)y' = 0$

Here $M(x, y) = 5x + 7$ and $N(x, y) = 6y + 11$.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$0 = 0$$

Therefore, we have an exact equation.

Now we must the function F(x,y):

$$F'_x = M, \qquad F'_y = N$$

$$F(x,y) = \int (5x+7)dx$$

$$= \frac{5x^2}{2} + 7x + C(y)$$

$$F_y'(x,y) = 0 + 0 + C'(y) = N = 6y + 11$$

$$C'(y) = 6y + 11$$

$$C(y) = \frac{6y^2}{2} + 11y$$

$$C(y) = 3y^2 + 11y$$

$$F(x,y) = \frac{5x^2}{2} + 7x + 3y^2 + 11y$$

Therefore the final solution is:

$$\frac{5x^2}{2} + 7x + 3y^2 + 11y = \text{constant}$$

(b.)(5x+7y)+(7x+11y)y'=0

Here $M(x,y) = 5x + 7y$ and $N(x,y) = 7x + 11y$.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$7 = 7$$

Therefore, we have an exact equation.

Now we must the function F(x,y):

$$F_x' = M, \qquad F_y' = N$$

$$F(x,y) = \int (5x+7y)dx$$

$$= \frac{5x^2}{2} + 7yx + C(y)$$

$$F_y'(x,y) = 0 + 7 + C'(y) = N = 7x + 11y$$

$$C'(y) = 7x + 11y$$

$$C(y) = \frac{11y^2}{2}$$

$$F(x,y) = \frac{5x^2}{2} + 7yx + \frac{11y^2}{2}$$

Therefore the final solution is:

$$\frac{5x^2}{2} + 7x + 3y^2 + 11y = \text{constant}$$

(c.) $(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x)y' = 0$

Here $M(x,y) = e^x \sin y - 2y$ and $N(x,y) = e^x \cos y + 2\cos x$.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$e^x \cos y - 2\sin x = e^x \cos y - 2\sin x$$

Therefore, we have an exact equation.

Now we must the function F(x,y):

$$F_x' = M, \qquad F_y' = N$$

$$F(x,y) = \int (e^x \cos y - 2\sin x)dx$$

$$= e^x \sin y + 2y\cos x + C(y)$$

$$F_y'(x,y) = e^x \cos y + 2\cos x + C'(y) = N = e^x \cos y + 2\cos x$$

$$C'(y) = 0$$

$$C(y) = 0$$

$$F(x,y) = e^x \sin y + 2y\cos x$$

Therefore the final solution is:

$$e^x \sin y + 2y \cos x = \text{constant}$$

(d.)$y' = \dfrac{ax+by}{bx+cy}$

Here $M(x,y) = ax + by$ and $N(x,y) = -(bx + cy)$.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$b = -b$$

Therefore, equation is not exact.

(e.)$y' = -\dfrac{ax+by}{bx+cy}$

Here $M(x,y) = ax + by$ and $N(x,y) = bx + cy$.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$b = b$$

Therefore, we have an exact equation.

Now we must the function F(x,y):

$$F'_x = M, \qquad F'_y = N$$

$$F(x,y) = \int (ax + by)dx$$

$$= \frac{ax^2}{2} + byx + C(y)$$

$$F'_y(x,y) = 0 + bx + C'(y) = N = bx + cy$$

$$C'(y) = cy$$

$$C(y) = \frac{cy^2}{2}$$

$$F(x,y)=\frac{ax^2}{2}+bxy+\frac{cy^2}{2}$$

Therefore the final solution is:

$$\frac{ax^2}{2}+bxy+\frac{cy^2}{2}=\text{constant}$$

(f.) $\frac{x}{(x^2+y^2)^{\frac{5}{2}}}dx+\frac{y}{(x^2+y^2)^{\frac{5}{2}}}y=0$

Here $M(x,y)=\frac{x}{(x^2+y^2)^{\frac{5}{2}}}$ and $N(x,y)=\frac{y}{(x^2+y^2)^{\frac{5}{2}}}y$.

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\frac{-5}{2}\left(\frac{2y}{(x^2+y^2)^{\frac{7}{2}}}\right)=\frac{-5}{2}\left(\frac{2xy}{(x^2+y^2)^{\frac{7}{2}}}\right)$$

Therefore, we have an exact equation.

Now we must the function F(x,y):

$$F'_x=M, \qquad F'_y=N$$

$$F=\frac{1}{3(x^2+y^2)^{\frac{3}{2}}}$$

Therefore the final solution is:

$$x^2+y^2=\text{constant}$$

Problem 3: Find a value of b for which the following equation is exact and solve it.

$$(ye^{-3xy}+x^2)dx-bxe^{-3xy}dy=0$$

Here $M(x,y)=ye^{-3xy}+x^2$ and $N(x,y)=bxe^{-3xy}$.

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$e^{-3xy}-3xye^{-3xy}=-be^{-3xy}+b3xye^{-3xy}$$

Therefore, in order to be an exact equation $b = -1$.
Now we must the function F(x,y):

$$F'_x = M, \qquad F'_y = N$$

$$F(x,y) = \int (ye^{-3xy} + x^2)dx$$

$$= \frac{e^{-3xy}}{-3} + \frac{x^3}{3} + C(y)$$

$$F'_y(x,y) = xe^{-3xy} + C'(y) = N = -(xe^{-3xy})$$

$$C'(y) = 0$$

$$C(y) = 0$$

$$F(x,y) = x^3 - e^{-3xy}$$

Therefore the final solution is:

$$x^3 - e^{-3xy} = \text{constant}$$

Problem 4: Solve using given integrating factor.

(a.)$ydx + (2x - ye^y)dy = 0, \qquad \mu(x,y) = y$

First check to see if the equation is exact:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = 2$$

Therefore, the equation is not exact.

Now multiply by the integrating factor y:

$$y(ydx + (2x - ye^y)dy) = 0$$

$$y^2dx + 2xy - y^2e^ydy = 0$$

Here $M(x,y) = y^2$ and $N(x,y) = 2xy - y^2e^y$.
Now check to see if the equation is exact again:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$2y = 2y$$

Therefore, the equation is now exact.

Now we must the function F(x,y):

$$F'_x = M, \qquad F'_y = N$$

$$F(x,y) = \int y^2 dx$$

$$= xy^2 + C(y)$$

$$F'_y(x,y) = 2xy + C'(y) = N = 2xy - y^2e^y$$

$$C'(y) = -y^2e^y$$

$$C(y) = y^2e^y - 2(y-1)e^y$$

$$C(y) = (y^2 - 2y + 2)e^y$$

$$F(x,y) = xy^2 - (y^2 - 2y + 2)e^y$$

Therefore the final solution is:

$$xy^2 - (y^2 - 2y + 2)e^y = \text{constant}$$

(b.) $(x+2)\sin y + (x\cos y)y' = 0, \qquad \mu(x,y) = xe^x$

First check to see if the equation is exact:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$(x+2)\cos y = \cos y$$

Therefore, the equation is not exact.

Now multiply by the integrating factor xe^x:

$$xe^x((x+2)\sin y dx + (x\cos y)dy) = 0$$

$$(x^2e^x + 2xe^x)\sin y dx + (x^2e^x\cos y)dy = 0$$

Here $M(x,y) = (x^2e^x + 2xe^x)\sin y$ and $N(x,y) = (x^2e^x\cos y)$.
Now check to see if the equation is exact again:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$(x^2e^x + 2xe^x)\cos y = (x^2e^x + 2xe^x)\cos y$$

Therefore, the equation is now exact.

Now we must find the function F(x,y):

$$F_x' = N, \qquad F_y' = M$$

$$F(x,y) = \int xe^x((x+2)\sin y dx$$

$$= x^2e^x\sin y + C(x)$$

$$F_y'(x,y) = (2x+x^2)e^x\sin y + C'(y) = N = x(x+2)e^x\sin y$$

$$C'(x) = 0$$

$$C(x) = 0$$

$$F(x,y) = x^2e^x\sin y$$

Therefore the final solution is:

$$x^2e^x\sin y = \text{constant}$$

(c.)$(3xy+y^2)+(x^2+xy)y' = 0, \qquad \mu = \frac{1}{xy(2x+y)}$

First check to see if the equation is exact:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$3x + 2y = 2x + y$$

Therefore, the equation is not exact.

Now multiply by the integrating factor $\dfrac{1}{xy(2x+y)}$:

$$\frac{3x+y}{x(2x+y)} + \frac{x+y}{y(2x+y)}y' = 0$$

Here $M(x,y) = \dfrac{3x+y}{x(2x+y)}$ and $N(x,y) = \frac{x+y}{y(2x+y)}$.

Now check to see if the equation is exact again:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{x(2x+y) - x(3x+y)}{x^2(2x+y)^2} = \frac{y(2x+y) - 2y(x+y)}{y^2(2x+y)^2}$$

$$\frac{2x^2 + xy - 3x^2 - xy}{x^2(2x+y)^2} = \frac{2yx + y^2 - 2yx - 2y^2}{y^2(2x+y)^2}$$

$$-\frac{1}{2x+y} = -\frac{1}{2x+y}$$

Therefore, the equation is now exact.

Now we must find the function F(x,y):

$$F'_x = N, \qquad F'_y = M$$

$$F(x,y) = \int \frac{3}{2x+y}dx + \int \frac{y}{x(2x+y)}dx$$

$$= \frac{3}{2}\ln|2x+y| + \int \frac{1}{x} + \frac{-2}{2x+y}dx$$

$$= \frac{3}{2}\ln|2x+y| + \ln|x| - \ln|2x+y| + C(y)$$

$$F'_y(x,y) = \frac{3}{2}\frac{1}{2x+y} - \frac{1}{2x+y} + C'(y)$$

$$= \frac{1}{2}\frac{1}{2x+y} + C'(y) = N = \frac{x}{y(2x+y)} + \frac{1}{2x+y}$$

$$= \frac{1}{2}\frac{1}{2x+y} + C'(y) = N = \frac{1}{2}\left(\frac{1}{y} + \frac{-1}{2x+y}\right) + \frac{1}{2x+y}$$

$$= \frac{1}{2}\frac{1}{2x+y} + C'(y) = N = \frac{1}{2y} + \frac{1}{2}\left(\frac{1}{2x+y}\right)$$

$$C'(y) = \frac{1}{2y}$$

$$C(y) = \frac{1}{2}\ln|y|$$

Therefore the final solution is:

$$\frac{1}{2}\ln|2x+y| + \ln|x| + \frac{1}{2}\ln|y| = \text{constant}$$

Problem 5: Find an integrating factor and solve.

(a.)$1 + \left(\frac{x}{y} - \sin y\right) y' = 0$

First check to see if equation is exact or not:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$0 = \frac{1}{y}$$

Therefore, equation is not exact.

Now we must find the integrating factor which makes the equation exact:

$$\frac{M'_y - N'_x}{N} = \frac{0 - \frac{1}{y}}{\frac{x}{y} - \sin y}$$

Above we have a function of both x and y, this is no good.

$$-\frac{M'_y - N'_x}{M} = -\left(\frac{0 - \frac{1}{y}}{1}\right) = \frac{1}{y}$$

Therefore, the integrating factor is:

$$\mu = e^{\ln y} = y$$

$$y(1 + \left(\frac{x}{y} - \sin y\right)) = 0$$

$$y + (x - y \sin y)y' = 0$$

Now check to see if the equation is exact again:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = 1$$

Therefore, the equation is exact.

Now we must find the function F(x,y):

$F'_x = M$ and $F'_y = N$

$$F(x, y) = xy + C(y)$$

$$F'_y = x + C'(y) = N = x - y \sin y$$

$$C(y) = -\int y \sin yy$$

$$C(y) = -\sin y + y \cos y$$

Therefore, the final solution is:

$$xy - \sin y + y \cos y = \text{constant}$$

(b.)$ydx + (3x - e^y)dy = 0$

First check to see if equation is exact or not:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = 3$$

Therefore, equation is not exact.

Now we must find the integrating factor which makes the equation exact:

$$\frac{M'_y - N'_x}{M} = -\frac{2}{y}$$

Therefore, the integrating factor is:

$$\mu = e^{2\ln y} = y^2$$

$$y^2(ydx + (3x - e^y)dy = 0$$

$$y^3dx + (3xy^2 - y^2e^y)dy = 0$$

Now check to see if the equation is exact again:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$3y^2 = 3y^2$$

Therefore, the equation is exact.

Now we must find the function F(x,y):

$F'_x = M$ and $F'_y = N$

$$F(x,y) = xy^3 + C(y)$$

$$F'_y = 3xy^2 + C'(y) = N = 3xy^2 - y^2e^y$$

$$C(y) = -(y^2 - 2y + 2)e^y$$

Therefore, the final solution is:

$$y^3x - (y^2 - 2y + 2)e^y = \text{constant}$$

(c.)$xdy + (2y - e^x)dx = 0$

First check to see if equation is exact or not:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$2 = 1$$

Therefore, equation is not exact.

Now we must find the integrating factor which makes the equation exact:

$$\frac{M'_y - N'_x}{N} = \frac{1}{x}$$

Therefore, the integrating factor is:

$$\mu = e^{lnx} = x$$

$$x(xdy + (2y - e^x)dx = 0)$$

$$x^2dy + (2yx - xe^x)dx = 0$$

Now check to see if the equation is exact again:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$2x = 2x$$

Therefore, the equation is exact.

Now we must find the function F(x,y):

$F'_x = M$ and $F'_y = N$

$$F(x, y) = x^2y + C(x)$$

$$F'_x = 2xy + C'(x) = N = 2xy - xe^x)$$

$$C'(x) = -xe^x$$

$$C(x) = -(x - 1)e^x$$

Therefore, the final solution is:

$$x^2y - (x-1)e^x = \text{constant}$$

Problem 6: Determine an interval in which the solution of the given initial value problem is certain to exist. Do not solve the equation.

(a.) $(t+3)y' + (\ln t)y = 2t$

$$y' + \frac{\ln t}{t+3}y = \frac{2t}{t+3}, \qquad t > 0 \text{and} t \neq -3$$

$$y(1) = 2$$

Therefore the interval in which the solution is certain to exist is $(0, \infty)$.

(b.) $y' + (\cot t)y = \cos t$

$$y' + (\cot t)y = \cos t, \qquad t \neq 0, \pi, \cdots$$

$$y(\frac{\pi}{2}) = 0$$

Therefore the interval in which the solution is certain to exist is $(0, \pi)$

(c.) $\frac{y'}{t} + ty = 0$

$$y' = -t^2y$$

$$y(-1) = 1$$

Here there are no problems of discontinuity so therefore the interval in which the solution is certain to exist is $(-\infty, \infty)$

Problem 7: (a.)Find the escape velocity for a rocket launched straight upward from a point $x_o = 0.1R$ above the surface of the earth, where $R = 4000$ miles is the radius of the earth. Neglect air resistance. Find the initial altitude from which the rocked must be launched in order to reduce the escape velocity to 80% of its value on the earth surface.

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c$$

at initial time $x = 0.1R$ and $v = v_0$

$$\frac{v_0^2}{2} = \frac{gR^2}{(R+0.1R)^2} + c$$

$$c = \frac{v_0^2}{2} - \frac{g}{1.1}R$$

$$v(x) = \pm\sqrt{\frac{2gR^2}{R+x} + v_0^2 - \frac{2g}{1.1}R}$$

$$v(x) = 0 \Longrightarrow x = x_max$$

$$0 = \frac{2gR^2}{R + x_max} + v_0^2 - \frac{2g}{1.1}R$$

Escape: $x_max \to \infty$

$$v_0 = \sqrt{\frac{2gR}{1.1}}$$

$$= 6.6 miles/sec$$

Escape velocity for a rocket started at $x_0 = 0.1R$ is approximately 6.6 miles/sec or 10.6 km/sec.

If rocket starts at $x_0 = \epsilon R$ we obtain:

$$v_0 = \sqrt{\frac{2gR}{1+\epsilon}}$$

To start from the surface $v_0 = \sqrt{2gR}$

Solve:

$$\sqrt{\frac{2gR}{1+\epsilon}}$$

$$= 0.8\sqrt{2gR}$$

$$\epsilon = 0.56$$

To reduce escape velocity to 80 percent at $\sqrt{2gR}$ we need to start at $x_0 = 0.56R$