AMAT 3260 assignment #5 – solutions

Problem 1 Using integration by parts, find the Laplace transform of the following functions, where a is a real constant :

1. $t\cos(at)$

$$\begin{aligned}
L\left(tcos\left(at\right)\right)\left(s\right) &= \int_0^\infty e^{-st} tcos\left(at\right) dt \\
&= \left[-\frac{1}{s}e^{-st}tcos\left(at\right)\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}cos\left(at\right) dt - \frac{a}{s}\int_0^\infty e^{-st}tsin\left(at\right) dt \\
&= 0 + \frac{1}{s}L\left(cos\left(at\right)\right) \\
&\quad -\frac{a}{s}\left(\left[-\frac{1}{s}e^{-st}tsin\left(at\right)\right]_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}tcos\left(at\right) dt + \frac{1}{s}\int_0^\infty e^{-st}sin\left(at\right) dt\right) \\
&= \frac{1}{s^2+a^2} - 0 - \frac{a^2}{s^2}L\left(tcos\left(at\right)\right)\left(s\right) - \frac{a}{s^2}L\left(sin\left(at\right)\right)\left(s\right) \\
&= \frac{1}{s^2+a^2} - \frac{a^2}{s^2}L\left(tcos\left(at\right)\right)\left(s\right) - \frac{a}{s^2}\frac{a}{s^2+a^2}
\end{aligned}$$

hence, we can solve for $L\left(tcos\left(at\right)\right)\left(s\right)$ *as*

$$\begin{aligned}
L\left(tcos\left(at\right)\right)\left(s\right)\left(1+\frac{a^2}{s^2}\right) &= \frac{1}{s^2+a^2} - \frac{a^2}{s^2\left(s^2+a^2\right)} \\
L\left(tcos\left(at\right)\right)\left(s\right) &= \frac{\frac{s^2-a^2}{\left(s^2+a^2\right)s^2}}{1+\frac{a^2}{s^2}} = \frac{s^2\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2 s^2} = \frac{s^2-a^2}{\left(s^2+a^2\right)^2}.
\end{aligned}$$

2. $t\sinh(at)$, where $\sinh(at) = \dfrac{e^{at}-e^{-at}}{2}$

$$\begin{aligned}
L\left(tsinh\left(at\right)\right)\left(s\right) &= \int_0^\infty e^{-st} t\sinh(at) dt \\
&= \left[-\frac{1}{s}e^{-st}t\sinh(at)\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}sinh\left(at\right) dt + \frac{a}{s}\int_0^\infty e^{-st}tcosh\left(at\right) dt \\
&= -\frac{1}{2s}\left[t\left(e^{(a-s)t} - e^{-(a+s)t}\right)\right]_0^\infty + \frac{1}{s}L\left(sinh\left(at\right)\right)\left(s\right) - \frac{a}{s^2}\left[e^{-st}tcosh\left(at\right)\right]_0^\infty \\
&\quad + \frac{a}{s^2}\int_0^\infty e^{-st}cosh\left(at\right) dt + \frac{a^2}{s^2}\int_0^\infty e^{-st}tsinh\left(at\right) dt \\
&= -\frac{1}{2s}\left[t\left(e^{(a-s)t} - e^{-(a+s)t}\right)\right]_0^\infty + \frac{1}{s}L\left(sinh\left(at\right)\right)\left(s\right) - \frac{a}{2s^2}\left[t\left(e^{(a-s)t} + e^{-(a+s)t}\right)\right]_0^\infty \\
&\quad + \frac{a}{s^2}L\left(cosh\left(at\right)\right)\left(s\right) + \frac{a^2}{s^2}L\left(tsinh\left(at\right)\right)\left(s\right)
\end{aligned}$$

if $s > a \geq 0$*, the square brackets vanish , since then the exponents are negative. Assuming that* $s > a \geq 0$*, we can continue as follows.*

$$\begin{aligned} L\left(tsinh\left(at\right)\right)\left(s\right) &= \frac{1}{s}L\left(sinh\left(at\right)\right)\left(s\right) + \frac{a}{s^2}L\left(cosh\left(at\right)\right)\left(s\right) + \frac{a^2}{s^2}L\left(tsinh\left(at\right)\right)\left(s\right) \\ &= \frac{1}{s}\frac{a}{s^2-a^2} + \frac{a}{s^2}\frac{s}{s^2-a^2} + \frac{a^2}{s^2}L\left(tsinh\left(at\right)\right)\left(s\right), \end{aligned}$$

from where we can solve for $L\left(tsinh\left(at\right)\right)\left(s\right)$

$$\begin{aligned} L\left(tsinh\left(at\right)\right)\left(s\right)\left(1-\frac{a^2}{s^2}\right) &= \frac{2sa}{s^2\left(s^2-a^2\right)} \\ L\left(tsinh\left(at\right)\right)\left(s\right) &= \frac{\frac{2sa}{s^2\left(s^2-a^2\right)}}{\left(1-\frac{a^2}{s^2}\right)} = \frac{2sa}{\left(s^2-a^2\right)^2}. \end{aligned}$$

3. $t^n e^{at}$, $n = 1, 2, 3, ...$

$$L\left(t^n e^{at}\right)\left(s\right) = \int_0^\infty e^{-st} t^n e^{at} dt = \int_0^\infty t^n e^{-(s-a)t} dt = L\left(t^n\right)\left(s-a\right).$$

It is enough to find the Laplace transform of t^n*.*

$$L\left(t^n\right)\left(u\right) = \int_0^\infty e^{-ut} t^n dt = \left[-\frac{1}{u}e^{-ut}t^n\right]_0^\infty + \frac{1}{u}\int_0^\infty e^{-ut} n t^{n-1} dt = 0 + \frac{n}{u}L\left(t^{-n}\right)\left(u\right),$$

hence

$$L\left(t^n\right)\left(u\right) = \frac{n}{u}\frac{n-1}{u}\frac{n-2}{u}\cdots\frac{1}{u}L\left(t^0\right) = \frac{n!}{u^n}L\left(1\right) = \frac{n!}{u^{n+1}}.$$

Combining these two lines we see that

$$L\left(t^n e^{at}\right)\left(s\right) = \frac{n!}{\left(s-a\right)^{n+1}}.$$

Problem 2 Use the Laplace transform to solve the following initial value problems: Solve the same problem by any other method and compare the results.

1. $y'' - 2y' - 2y = 0; \qquad y\left(0\right) = 1, \quad y'\left(0\right) = 0$
   This equation transforms under the Laplace transform to

$$s^2 L\left(y\right) - y'\left(0\right) - sy\left(0\right) - 2sL\left(y\right) + 2y\left(0\right) - 2L\left(y\right) = 0.$$

   This can be written as

$$\left(s^2 - 2s - 2\right)L\left(y\right) = y'\left(0\right) + sy\left(0\right) - 2y\left(0\right)$$

$$L(y) = \frac{s-2}{s^2-2s-2} = \frac{s-1-1}{(s-1)^2-3} = \frac{s-1}{(s-1)^2-3} - \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2-3}$$

we can see that the right side contains the Laplace transforms of cosh $\left(\sqrt{3}t\right)$ *and sinh* $\left(\sqrt{3}t\right)$ *evaluated at* $(s-1)$*. Hence*

$$y = e^t\left(cosh\left(\sqrt{3}t\right) - \frac{1}{\sqrt{3}}sinh\left(\sqrt{3}t\right)\right).$$

2. $y'' - 2y' + 2y = \sin(t)\,; \qquad y(0) = 1, \quad y'(0) = 0$
   This equation transforms under the Laplace transform to

$$s^2L(y) - y'(0) - sy(0) - 2sL(y) + 2y(0) + 2L(y) = \frac{1}{s^2+1}.$$

This can be written as

$$\begin{aligned}
\left(s^2-2s+2\right)L(y) &= y'(0) + sy(0) - 2y(0) + \frac{1}{s^2+1} \\
L(y) &= \frac{s-2+\frac{1}{s^2+1}}{s^2-2s+2} = \frac{\left(s^2+1\right)(s-2)+1}{\left(s^2+1\right)\left((s-1)^2+1\right)} = \\
&= \frac{3}{5}\frac{s-1}{(s-1)^2+1} - \frac{4}{5}\frac{1}{(s-1)^2+1} + \frac{2}{5}\frac{s}{s^2+1} + \frac{1}{5}\frac{1}{s^2+1} \\
&= \frac{1}{5}L(sint)(s) + \frac{2}{5}L(cost)(s) - \frac{4}{5}L(sint)(s-1) + \frac{3}{5}L(cost)(s-1) \\
&= \frac{1}{5}L(sint)(s) + \frac{2}{5}L(cost)(s) + \frac{3}{5}L\left(e^t cost\right)(s) - \frac{4}{5}L\left(e^t sint\right)(s).
\end{aligned}$$

Hence, the solution of the initial value problem is

$$y(t) = \frac{1}{5}sint + \frac{2}{5}cost + \frac{3}{5}e^t cost - \frac{4}{5}e^t sint.$$

3. $y'' + 2y' + y = e^{-t}; \qquad y(0) = 0, \quad y'(0) = 1$
   This equation transforms under the Laplace transform to

$$s^2L(y) - y'(0) - sy(0) + 2sL(y) - 2y(0) + L(y) = \frac{1}{s+1}.$$

This can be written as

$$\begin{aligned}
\left(s^2+2s+1\right)L(y) &= y'(0) + sy(0) + 2y(0) + \frac{1}{s+1} \\
L(y) &= \frac{1+\frac{1}{s+1}}{s^2+2s+1} = \frac{s+2}{(s+1)^3} = \frac{s+1}{(s+1)^3} + \frac{1}{(s+1)^3}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3} \\
&= L(t)(s+1) + \frac{1}{2}L\left(t^2\right)(s+1) \\
&= L\left(e^{-t}t\right) + \frac{1}{2}L\left(e^{-t}t^2\right)
\end{aligned}$$

Hence, the solution of the initial value problem is

$$y(t) = e^{-t}t + \frac{1}{2}e^{-t}t^2.$$

Alternative Solutions for Problem 2:

$1.y'' - 2y' - 2y, \qquad y(0) = 1, \quad y'(0) = 0$

$$\lambda^2 - d\lambda - 2 = 0$$

$$\lambda = 1 \pm \sqrt{3}$$

Therefore the general solution is:

$$y(t) = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}$$

$$y'(t) = (1+\sqrt{3})c_1 e^{(1+\sqrt{3})t} + (1-\sqrt{3})c_2 e^{(1-\sqrt{3})t}$$

Now using the initial values to solve for c_1 and c_2:

$$y(0) = c_1 e^{1+\sqrt{3}(0)} + c_2 e^{1-\sqrt{3}(0)} = 1$$

$$c_1 + c_2 = 1$$

$$c_1 = 1 - c_2$$

$$y'(0) = (1+\sqrt{3})c_1 e^0 + (1-\sqrt{3})c_2 e^0$$

$$(1+\sqrt{3})(1-c_2)+(1-\sqrt{3})c_2=0$$

$$(1-c_2+\sqrt{3}-\sqrt{3}c_2)+c_2-\sqrt{3}c_2=0$$

$$1+\sqrt{3}-\sqrt{3}c_2-\sqrt{3}c_2=0$$

$$1+\sqrt{3}-2\sqrt{3}c_2=0$$

$$c_2=\frac{1+\sqrt{3}}{2\sqrt{3}}$$

$$c_2=\frac{\sqrt{3}+3}{6}$$

$$c_1=1-\left(\frac{\sqrt{3}+3}{6}\right)$$

$$c_1=\frac{3-\sqrt{3}}{6}$$

Therefore, the final solution is:

$$y(t)=\left(\frac{3-\sqrt{3}}{6}\right)e^{(1+\sqrt{3})t}+\left(\frac{\sqrt{3}+3}{6}\right)e^{(1-\sqrt{3})t}=e^t(\cosh(\sqrt{3}t)-\frac{1}{\sqrt{3}}\sinh(\sqrt{3}t))$$

2.

$y''-2y'+2y=\sin t, \qquad y(0)=1, \quad y'(0)=0$

$$\lambda^2-2\lambda+2=0$$

$$\lambda=1\pm i$$

Therefore the general solution for the homogeneous equation is:

$$y(t)=c_1e^t\cos t+c_2e^t\sin t$$

Now we must solve for the non-homogeneous equation:

$$y(t) = A\sin t + B\cos t$$

$$y'(t) = A\cos t - B\sin t$$

$$y''(t) = -A\sin t - B\cos t$$

Plugging the above into the non-homogeneous equation:

$$(-A\sin t - B\cos t) - 2(A\cos t - B\sin t) + 2(A\sin t + B\cos t) = \sin t$$

$$-A\sin t + 2B\sin t + 2A\sin t - B\cos t - 2A\cos t + 2B\cos t = \sin t$$

$$(A + 2B)\sin t + (B - 2A)\cos t = \sin t$$

$$A + 2B = 1, \qquad B - 2A = 0$$

Therefore, $A = \frac{1}{5}$ and $B = \frac{2}{5}$, and the particular solution is:

$$y(t) = \frac{1}{5}\sin t + \frac{2}{5}\cos t$$

Therefore, the general equation is:

$$y(t) = c_1 e^t \cos t + c_2 e^t \sin t + \frac{1}{5}\sin t + \frac{2}{5}\cos t$$

$$y'(t) = c_1(e^t\cos t - e^t\sin t) + c_2(e^t\sin t + e^t\cos t) + \frac{1}{5}\cos t - \frac{2}{5}\sin t$$

Now using the initial values to solve for c_1 and c_2:

$$y(0) = c_1 + \frac{2}{5} = 1$$

$$c_1 = 1 - \frac{2}{5}$$

$$c_1 = \frac{3}{5}$$

$$y'(0) = c_1 + c_2 + \frac{1}{5} = 0$$

$$c_2 = -\frac{1}{5} - \frac{3}{5}$$

$$c_2 = -\frac{4}{5}$$

Therefore, the final solution is:

$$y(t) = \frac{3}{5}e^t \cos t - \frac{4}{5}e^t \sin t + \frac{1}{5} \sin t + \frac{2}{5} \cos t$$

3.

$y'' + 2y' + y = e^{-t}, \qquad y(0) = 0, \quad y'(0) = 1$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$\lambda = -1 (\text{repeated root})$$

Therefore the general solution for the homogeneous equation is:

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}$$

Now we must solve for the non-homogeneous equation:

$$y(t) = At^2 e^{-t}$$

$$y'(t) = 2Ate^{-t} - At^2 e^{-t}$$

$$y''(t) = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$$

Plugging the above into the non-homogeneous equation:

$$(2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}) + 2(2Ate^{-t} - At^2e^{-t}) + At^2e^{-t} = e^{-t}$$

$$2Ae^{-t} = e^{-t}$$

$$2A = 1$$

Therefore, $A = \frac{1}{2}$ and the particular solution is:

$$y(t) = \frac{1}{2}t^2e^{-t}$$

Therefore, the general equation is:

$$y(t) = c_1e^{-t} + c_2te^{-t} + \frac{1}{2}t^2e^{-t}$$

$$y'(t) = -c_1e^{-t} + c_2e^{-t} - c_2te^{-t} + te^{-t} - \frac{1}{2}t^2e^{-t}$$

Now using the initial values to solve for c_1 and c_2:

$$y(0) = c_1e^0 + c_2e^0 + \frac{(0)^2e^0}{2} = 0$$

$$c_1 = 0$$

$$y'(0) = -c_1e^0 + c_2e^0 - c_2(0)e^0 + (0)e^0 - \frac{(0)^2e^0}{2} = 1$$

$$-c_1 + c_2 = 1$$

$$c_2 = 1 + c_1$$

$$c_2 = 1$$

Therefore, the final solution is:

$$y(t) = te^{-t} + \frac{t^2 e^{-t}}{2}$$

Problem 3 Find the inverse Laplace transform for the following functions:

1. $\dfrac{2s+2}{s^2+2s+5}$
   We can rewrite the above expression as
   $$2\frac{s+1}{(s+1)^2+4}$$
   what is the Laplace transform of $2\cos(2t)$ *evaluated at* $(s+1)$. *Hence the expression can be written as*
   $$L\left(2e^{-t}\cos(2t)\right)$$
   and thus the inverse Laplace transform is
   $$2e^{-t}\cos(2t).$$

2. $\dfrac{(s-1)\,e^{-2s}}{s^2-4s+3}$
   This expression can be written as
   $$e^{-2s}\frac{s-1}{(s-1)(s-3)} = \frac{e^{-2s}}{s-3}$$
   what is equal to
   $$e^{-2s}L\left(e^{3t}\right)(s) = L\left(h_2(t)\,e^{3(t-2)}\right)(s).$$
   where
   $$h_a(t) = \begin{cases} 0 & \text{for } 0 \le t < a \\ 1 & \text{for } a \le t. \end{cases}$$
   Hence, the inverse Laplace transform of the above expression is
   $$h_2(t)\,e^{3(t-2)}.$$

3. $\dfrac{e^{-2s}}{s^2-4}$

This expression can be written as

$$\begin{aligned}
e^{-2s}\frac{1}{(s-2)(s+2)} &= e^{-2s}\frac{1/4}{s-2} - e^{-2s}\frac{1/4}{s+2} \\
&= e^{-2s}\frac{1}{4}L\left(e^{2t}\right)(s) - e^{-2s}\frac{1}{4}L\left(e^{-2t}\right)(s) \\
&= \frac{1}{4}L\left(h_2(t)\,e^{2(t-2)}\right)(s) - \frac{1}{4}L\left(h_2(t)\,e^{-2(t-2)}\right)(s).
\end{aligned}$$

Hence, the inverse Laplace transform is

$$\frac{1}{4}h_2(t)\left(e^{2(t-2)} - e^{-2(t-2)}\right).$$

4. $\dfrac{e^{-s}+e^{-2s}}{s}$
This expression can be written as

$$\frac{e^{-s}}{s} + \frac{e^{-2s}}{s} = L\left(h_1(t)\right)(s) + L\left(h_2(t)\right)(s).$$

The inverse is

$$h_1(t) + h_2(t).$$

Problem 4 Find the solutions of the given initial value problems:

1. $y'' + y = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}; \qquad y(0) = 1, \quad y'(0) = 0$
The right hand side of the equation can be written as $1 - h_\pi(t)$. *The Laplace transform of the equation is*

$$\begin{aligned}
s^2L(s) - sy(0) - y'(0) + L(y) &= L(1)(s) - L\left(h_\pi(t)\right)(s) \\
\left(s^2+1\right)L(y) - s &= \frac{1}{s} - \frac{e^{-\pi s}}{s} \\
L(y) &= \frac{1-e^{-\pi s}+s^2}{s\left(s^2+1\right)} = \frac{1}{s} - \frac{s}{s^2+1} - e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{s}{s^2+1} \\
&= L(1)(s) - e^{-\pi s}\left(L(1)(s) - L(cost)(s)\right) \\
&= L(1)(s) - L\left(h_\pi(t)\right)(s) + L\left(h_\pi(t)\,cos(t-\pi)\right)(s).
\end{aligned}$$

Hence, the solution of the initial value problem is

$$y(t) = 1 - h_\pi(t) - h_\pi(t)\,cos(t).$$

2. $y'' + y = \begin{cases} \sin(t) & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$; $\quad y(0) = 0, \quad y'(0) = 1$

The right hand side of the equation is $sin(t)(1 - h_\pi(t))$*. The Laplace transform of the initial value problem is*

$$\begin{aligned} s^2 L(y) - sy(0) - y'(0) + L(y) &= L(sint)(s) - L(h_\pi(t)\, sin(t))(s) \\ \left(s^2+1\right) L(y) - 1 &= \frac{1}{s^2+1} - L(h_\pi(t)(-sin(t-\pi)))(s) \\ &= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} \\ L(y) &= \frac{1}{(s^2+1)^2} + e^{-\pi s}\frac{1}{(s^2+1)^2} + \frac{1}{s^2+1} \end{aligned}$$

The inverse Laplace transform of $(s^2+1)^{-2}$ *can be found by using the convolution. Another possible way to solve this problem is to split it into two initial value problems, one, for* $0 \le t < \pi$ *,and one for* $\pi \le t$ *with appropriate initial conditions. Let us consider first initial value problem for* $0 \le t < \pi$.

$$y_1'' + y_1 = sin(t), \qquad y_1(0) = 0, \quad y_1'(0) = 1.$$

Using method of variation of coefficients, we can see that the solution is

$$y_1(t) = \frac{3}{2} sin(t) - \frac{1}{2} t cos(t).$$

In order to determine the initial conditions for the second initial value problem, we have to find, from the above equation, values $y(\pi)$ *and* $y'(\pi)$*. Computing these, we can write the second initial value problem as*

$$y_2'' + y_2 = 0, \qquad y_2(\pi) = \frac{\pi}{2}, \quad y_2'(\pi) = -1.$$

The solution of this is

$$y_2(t) = sin(t) - \frac{\pi}{2} cos(t).$$

If we combine these two functions together, we get the solution of the original initial value problem:

$$\begin{aligned} y(t) &= (1 - h_\pi(t))\, y_1(t) + h_\pi(t)\, y_2(t) \\ &= (1 - h_\pi(t)) \left(\frac{3}{2} sin(t) - \frac{1}{2} t cos(t)\right) + h_\pi(t)\left(sin(t) - \frac{\pi}{2} cos(t)\right). \end{aligned}$$

Note that using the convolution we would get the same answer.

3. $y'' + 4y = \begin{cases} \sin(t) & 0 \le t < \pi \\ \sin(t) + \sin(t-\pi) & t \ge \pi \end{cases}$; $\quad y(0) = 0, \quad y'(0) = 0$

The right side of the equation can be written as $sin(t) + h_\pi(t)\, sin(t-\pi)$*. The Laplace transform of the* *initial value problem is*

$$\begin{aligned}
s^2L(y) - sy(0) - y'(0) + 4L(y) &= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} \\
\left(s^2+4\right)L(y) &= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} \\
L(y) &= \frac{1}{(s^2+4)(s^2+1)} + e^{-\pi s}\frac{1}{(s^2+4)(s^2+1)} \\
&= -\frac{1/3}{s^2+4} + \frac{1/3}{s^2+1} + e^{-\pi s}\left(-\frac{1/3}{s^2+4} + \frac{1/3}{s^2+1}\right) \\
&= -\frac{1}{3\cdot 2}\frac{2}{s^2+4} + \frac{1}{3}\frac{1}{s^2+1} + e^{-\pi s}\left(-\frac{1}{3\cdot 2}\frac{2}{s^2+4} + \frac{1}{3}\frac{1}{s^2+1}\right) \\
&= -\frac{1}{6}L(sin(2t))(s) + \frac{1}{3}L(sin(t))(s) + \\
&\quad + e^{-\pi s}\left(-\frac{1}{6}L(sin(2t))(s) + \frac{1}{3}L(sin(t))(s)\right) \\
&= L\left(-\frac{1}{6}sin(2t) + \frac{1}{3}sin(t)\right)(s) + \\
&\quad L\left(-\frac{1}{6}h_\pi(t)\,sin(2(t-\pi)) + \frac{1}{3}h_\pi(t)\,sin(t-\pi)\right)(s).
\end{aligned}$$

Hence, the solution of the initial value problem is

$$y(t) = -\frac{1}{6}sin(2t) + \frac{1}{3}sin(t) - \frac{1}{6}h_\pi(t)\,sin(2(t-\pi)) + \frac{1}{3}h_\pi(t)\,sin(t-\pi).$$

4. $y'' + 2y' + 2y = \delta(t-2\pi);\qquad y(0)=1,\quad y'(0)=0$
The Laplace transform of this initial value problem is

$$\begin{aligned}
s^2L(y) - sy(0) - y'(0) + 2sL(y) - 2y(0) + 2L(y) &= e^{-2\pi s} \\
\left(s^2+2s+2\right)L(y) &= e^{-2\pi s} + s + 2
\end{aligned}$$

This means that

$$\begin{aligned}
L(y) &= \frac{e^{-2\pi s}}{s^2+2s+2} + \frac{s+2}{s^2+2s+2} \\
&= \frac{e^{-2\pi s}}{(s+1)^2+1} + \frac{s+1+1}{(s+1)^2+1} \\
&= e^{-2\pi s}L(sin(t))(s+1) + L(cos(t))(s+1) + L(sin(t))(s+1) \\
&= e^{-2\pi s}L\left(e^{-t}sin(t)\right)(s) + L\left(e^{-t}cos(t)\right)(s) + L\left(e^{-t}sin(t)\right)(s) \\
&= L\left(h_{2\pi}(t)\,e^{-(t-2\pi)}sin(t-2\pi) + e^{-t}cos(t) + e^{-t}sin(t)\right).
\end{aligned}$$

The solution of the initial value problem is

$$y(t) = h_{2\pi}(t)\, e^{-(t-2\pi)} sin(t-2\pi) + e^{-t} cos(t) + e^{-t} sin(t).$$

5. $y'' + 2y' + 3y = \sin(t) + \delta(t-\pi)$; $\quad y(0) = 0, \quad y'(0) = 0$

The Laplace transform of this initial value problem is

$$\begin{aligned} s^2 L(y) - y'(0) - sy(0) + 2sL(y) - 2y(0) + 3L(y) &= \frac{1}{s^2+1} + e^{-\pi s} \\ \left(s^2 + 2s + 3\right) L(y) &= \frac{1}{s^2+1} + e^{-\pi s} \\ L(y) &= \frac{1}{(s^2+2s+3)(s^2+1)} + e^{-\pi s}\frac{1}{s^2+2s+3} \end{aligned}$$

$$\begin{aligned} L(y) &= \frac{1}{4}\frac{s+1}{(s+1)^2+2} - \frac{1}{4}\frac{s-1}{s^2+1} + \frac{1}{\sqrt{2}} e^{-\pi s}\frac{\sqrt{2}}{(s+1)^2+2} \\ &= \frac{1}{4} L\left(cos\left(\sqrt{2}t\right)\right)(s+1) - \frac{1}{4} L(cos(t))(s) + \frac{1}{4} L(sin(t))(s) + \frac{1}{\sqrt{2}} e^{-\pi s} L\left(sin\left(\sqrt{2}t\right)\right)(s+1) \\ &= \frac{1}{4} L\left(e^{-t} cos\left(\sqrt{2}t\right) - cos(t) + sin(t)\right)(s) + \frac{1}{\sqrt{2}} e^{-\pi s} L\left(e^{-t} sin\left(\sqrt{2}t\right)\right)(s) \\ &= \frac{1}{4} L\left(e^{-t} cos\left(\sqrt{2}t\right) - cos(t) + sin(t)\right)(s) + \frac{1}{\sqrt{2}} L\left(h_\pi(t)\, e^{\pi - t} sin\left(\sqrt{2}(t-\pi)\right)\right)(s). \end{aligned}$$

Hence, the solution of the problem is

$$y(t) = \frac{1}{4}\left(e^{-t} cos\left(\sqrt{2}t\right) - cos(t) + sin(t)\right) + \frac{1}{\sqrt{2}} h_\pi(t)\, e^{\pi - t} sin\left(\sqrt{2}(t-\pi)\right).$$

6. $y^{(\mathrm{iv})} - y = \delta(t-1)$; $\quad y(0) = y'(0) = y''(0) = y'''(0) = 0$

The Laplace transform of the initial value problem is

$$s^4 L(y) - y'''(0) - sy''(0) - s^2 y'(0) - s^3 y(0) - L(y) = e^{-s}$$

$$\begin{aligned} \left(s^4 - 1\right) L(y) &= e^{-s} \\ L(y) &= e^{-s}\frac{1}{(s^2+1)(s+1)(s-1)} = e^{-s}\left(-\frac{1/2}{s^2+1} - \frac{1/4}{s+1} + \frac{1/4}{s-1}\right) \\ &= e^{-s}\left(-\frac{1}{2} L(sin(t))(s) - \frac{1}{4} L\left(e^{-t}\right)(s) + \frac{1}{4}\left(e^{t}\right)(s)\right) \\ &= L\left(-\frac{1}{2}\left(h_1(t)\, sin(t-1)\right) - \frac{1}{4} h_1(t)\, e^{-t+1} + \frac{1}{4} h_1(t)\, e^{t-1}\right)(s). \end{aligned}$$

The solution of the initial value problem is

$$y(t) = -\frac{1}{2}\left(h_1(t)\, sin\,(t-1)\right) - \frac{1}{4}h_1(t)\, e^{-t+1} + \frac{1}{4}h_1(t)\, e^{t-1}.$$

Problem 5 Setup and solve the equation of motion for an undamped oscillator with $m = k = 1$ and the external force given by set of impulses $f(t) = \delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi)$, where at time $t = 0$ the oscillator was at rest.
The equation of motion for such an oscillator is

$$y'' + y = \delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi).$$

The initial conditions are

$$y(0) = y'(0) = 0.$$

The Laplace transform of this problem gives us

$$\begin{aligned}
s^2 L(y) - y'(0) - sy(0) + L(y) &= e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} \\
\left(s^2+1\right) L(y) &= e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} \\
L(y) &= \frac{1}{s^2+1}\left(e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s}\right) \\
&= \left(e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s}\right) L(sin\,(t))(s) \\
&= L\left(h_\pi(t)\, sin\,(t-\pi) + h_{2\pi}(t)\, sin\,(t-2\pi) + h_{3\pi}(t)\, sin\,(t-3\pi)\right)(s).
\end{aligned}$$

Hence, the motion of the oscillator can be described by the following expression.

$$\begin{aligned}
y(t) &= h_\pi(t)\, sin\,(t-\pi) + h_{2\pi}(t)\, sin\,(t-2\pi) + h_{3\pi}(t)\, sin\,(t-3\pi) \\
&= -h_\pi(t)\, sin\,(t) + h_{2\pi}(t)\, sin\,(t) - h_{3\pi}(t)\, sin\,(t).
\end{aligned}$$

The oscillator is not moving for time between $t = 0$ and $t = \pi$ and then again for time between $t = 2$ and $t = 3\pi$. Between $t = \pi$ and $t = 2\pi$ and for time greater than $t = 3\pi$ the oscillator follows sinusoidal motion.