AMAT 3260 Solutions for assignment #1

Problem 1 Determine the order of the given differential equations and state whether the equation is linear or nonlinear.

1. $t^2 \frac{d^2y}{dt^2} - sin(t)\frac{dy}{dt} + y - e^{2t} = 0$
   second order linear equation
2. $2\left(\frac{dy}{dt}\right)^2 + 5y = 1$
   first order nonlinear equation
3. $\frac{d^2y}{dt^2} + 2y\frac{dy}{dt} + y = 0$
   second order nonlinear equation

Problem 2 Verify that $y = e^{-t^3}\int_0^t e^{s^3} ds + 3e^{-t^3}$ is a solution of the differential equation

$$y' + 3t^2 y = 1.$$

We can substitute the function $y = e^{-t^3}\int_0^t e^{s^3} ds + 3e^{-t^3}$ *and its derivative*

$$\begin{aligned} y' &= -3t^2 e^{-t^3}\int_0^t e^{s^3} ds + e^{-t^3}\left(e^{t^3}\right) - 9t^2 e^{-t^3} \\ &= -3t^2 e^{-t^3}\int_0^t e^{s^3} ds + 1 - 9t^2 e^{-t^3} \end{aligned}$$

to the above equation, obtaining

$$\begin{aligned} &-3t^2 e^{-t^3}\int_0^t e^{s^3} ds + 1 - 9t^2 e^{-t^3} + 3t^2\left(e^{-t^3}\int_0^t e^{s^3} ds + 3e^{-t^3}\right) \\ &\qquad = -3t^2 e^{-t^3}\int_0^t e^{s^3} ds + 3t^2 e^{-t^3}\int_0^t e^{s^3} ds - 9t^2 e^{-t^3} + 3t^2 3e^{-t^3} + 1 \end{aligned}$$

what is equal to 1 *and hence the function* y *is a solution of the given equation.*

Problem 3 Solve the following initial value problems

1. $ty' + (t+1)\,y = e^{2t}$ and $y(1) = 1$
   We can devide the equation by t *to write it in the standard form of a linear ODE:*

$$y' + \frac{t+1}{t}y = \frac{e^{2t}}{t}$$

The solution, either using integrating factors or variation of coefficients, is given by

$$y(t) = \frac{1}{e^{\int_0^t \frac{s+1}{s}ds}} \left(\int \left(e^{\int_0^t \frac{s+1}{s}ds} \right) \frac{e^{2t}}{t} dt \right),$$

what gives

$$y(t) = \frac{e^{2t}}{3t} + C\frac{e^{-t}}{t}.$$

To satisfy the initial value, we have to choose the constant C *to be such that*

$$\frac{e^2}{3} + C\frac{e^{-1}}{1} = 1$$

or

$$C = e - \frac{e^3}{3}.$$

2. $y' + \frac{1}{4}y = 3 + 2cos(2t)$ and $y(0) = 0$

   Similarly as in the previous problem, the solution is given by

$$\begin{aligned} y(t) &= \frac{1}{e^{\int_0^t \frac{1}{4}ds}} \left(\int \left(e^{\int_0^t \frac{1}{4}ds} \right) (3 + 2cos(2t))\, dt \right) \\ &= e^{-t/4} \left(\int e^{t/4} 3dt + \int e^{t/4} 2cos(2t)\, dt \right) \end{aligned}$$

The only difficult part is the second integral that can be solved by integrating by parts twice, namely

$$\begin{aligned} \int e^{t/4} 2cos(2t)\, dt &= e^{t/4} sin(2t) - \int \frac{1}{4} e^{t/4} sin(2t)\, dt \\ &= e^{t/4} sin(2t) - \left(-\frac{1}{8} e^{t/4} cos(2t) - \int -\frac{1}{32} e^{t/4} cos(2t)\, dt \right), \end{aligned}$$

where both sides contain the same integral that we can solve for

$$\begin{aligned} \left(1 + \frac{1}{64}\right) \int e^{t/4} 2cos(2t)\, dt &= e^{t/4} sin(2t) + \frac{1}{8} e^{t/4} cos(2t) \\ \int e^{t/4} 2cos(2t)\, dt &= \frac{64}{65} \left(e^{t/4} sin(2t) + \frac{1}{8} e^{t/4} cos(2t) \right) + C. \end{aligned}$$

Hence the solution is

$$y(t) = 12 + \frac{64}{65} \left(sin(2t) + \frac{1}{8} cos(2t) \right) + Ce^{-t/4}.$$

Again, to satisfy the initial condition, one must set the constant C such that

$$\begin{aligned} 12+\frac{64}{65}\left(sin\left(0\right)+\frac{1}{8}cos\left(0\right)\right)+C &= 0 \\ 12+\frac{64}{65}\frac{1}{8} &= -C \end{aligned}$$

or

$$C=-12-\frac{8}{65}.$$

3. $y'+2y+3t=0$ and $y\left(0\right)=0$
   The solution of this equation is again given by

$$\begin{aligned} y\left(t\right) &= \frac{1}{e^{2t}}\left(-\int e^{2t}3tdt\right) \\ &= -e^{-2t}\left(\frac{1}{2}e^{2t}3t-\int\frac{1}{2}e^{2t}3dt\right) \\ &= -\frac{3}{2}t+e^{-2t}\left(e^{2t}\frac{3}{4}+C\right) \\ &= -\frac{3}{2}t+\frac{3}{4}+Ce^{-2t} \end{aligned}$$

In order to satisfy the initial condition, one must set C such that

$$3/4+C=0,$$

or

$$C=-3/4.$$

4. $y'+\frac{t}{y}=\frac{1}{y}$ and $y\left(1\right)=-2$
   This separable equation can be solved by the following integration

$$\begin{aligned} \int ydy &= \int\left(1-t\right)dt \\ \frac{1}{2}y^2 &= t-\frac{1}{2}t^2+C \end{aligned}$$

where the constant is such that

$$\frac{1}{2}\left(-2\right)^2=1-\frac{1}{2}+C$$

or

$$C = \frac{3}{2}.$$

Problem 4 Solve the following separable differential equations

a) $y' + y^4 sin\left(x\right) = 0$

This can be solved by the following integration

$$\begin{aligned} \int y^{-4} dy &= -\int sin\left(x\right) dx \\ \frac{-1}{3} y^{-3} &= cos\left(x\right) + C \end{aligned}$$

b)$y' = \frac{x - e^{-x}}{y + e^y}$

This can be solved by the following integration

$$\begin{aligned} \int y + e^y dy &= \int x - e^{-x} dx \\ \frac{1}{2} y^2 + e^y &= \frac{1}{2} x^2 + e^{-x} + C \end{aligned}$$

c)$xy' = \left(1 - y^2\right)^{1/2}$

This can be solved by the following integration

$$\begin{aligned} \int \frac{dy}{\sqrt{1 - y^2}} &= \int \frac{dx}{x} \\ arcsin\left(y\right) &= ln\left|x\right| + C \\ y &= sin\left(ln\left|x\right| + C\right) \end{aligned}$$

d)$y' = \frac{x^2}{1 + y^2}$

This can be solved by the following integration

$$\begin{aligned} \int \left(1 + y^2\right) dy &= \int x^2 dx \\ y + \frac{1}{3} y^3 &= \frac{1}{3} x^3 + C \end{aligned}$$

e)$y' = \frac{1 + y^2}{x^2}$

This can be solved by the following integration

$$\begin{aligned}
\int \frac{dy}{1+y^2} &= \int \frac{dx}{x^2} \\
arctan\,(y) &= -\frac{1}{x} + C \\
y &= tan\left(-\frac{1}{x} + C\right)
\end{aligned}$$

f)$y' = \frac{x^3}{y(1+x^4)}$
This can be solved by the following integration

$$\begin{aligned}
\int ydy &= \int \frac{x^3}{1+x^4}dx \\
\frac{1}{2}y^2 &= \frac{1}{4}ln\left|1+x^4\right| + C \\
y &= \pm\sqrt{\frac{1}{2}ln\left|1+x^4\right| + 2C}
\end{aligned}$$