MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Am I ready for Final? **Mathematics 3260** **Winter 2004**

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In this course we were talking about the following problems and ways to solve them.

1. First order Linear ODE with variable coefficients.

$$y' + p(t)y = g(t).$$

Let $P(t) = \int p(t)dt$ be untiderivaive of $p(t)$, and $G(t) = \int g(t)e^{P(t)}dt$. Then *general solution* has the form

$$y(t) = e^{-P(t)}(C + G(t)), \quad \text{where} \quad C = const.$$

Example : $y' + 2y = \sin t$. Here $p(t) = 2$, $g(t) = \sin t$.

Thus $P(t) = \int 2\,dt = 2t$, $G(t) = \int e^{2t}\sin t\,dt = e^{2t}(2\sin t - \cos t)/5$ and finally,

$$y(t) = e^{-2t}(C + e^{2t}(2\sin t - \cos t)/5) = Ce^{-2t} + \frac{2\sin t - \cos t}{5}.$$

(see Assignment 1 for more examples and *initial value problems.*)

2. First order Separable ODE.

$$y' = f(y,t), \quad f(y,t) = g(y)h(t).$$

Then

$$\frac{dy}{dt} = g(y)h(t) \quad \text{or} \quad \int \frac{dy}{g(y)} = \int h(t)\,dt$$

Example : $y' = 2yt$ Then

$$\int \frac{dy}{y} = \int 2tdt \quad \text{thus} \quad \ln|y| = t^2 + C.$$

Often we find solution $y(t)$ in the implicite form. Sometimes we are lucky to solve for y. Here we have

$$y(t) = C_1 e^{t^2}.$$

(see Assignment 1 for more examples and *initial value problems.*)

3. Bernoulli equations. This is a special class of first order **non-linear** equations

$$y' = ay + by^n, \quad n \neq 0, 1,$$

Use subsitution $v = y^{1-n}$ to obtain **linear** equation for $v(t)$

$$v' = (1-n)av + (1-n)b$$

This one you can solve. Then do back substitution to find $y(t) = (v(t))^{1/(1-n)}$.

Example: $y' = 2y + y^5$. Here $a = 2, b = 1$, $n = 5$, so $v = y^{-4}$ and

$$v' = -8v - 4. \quad \text{Thus} \quad v(t) = Ce^{-8t} - \frac{1}{2}.$$

Finally,

$$y(t) = v^{-1/4} = (Ce^{-8t} - \frac{1}{2})^{-1/4}.$$

4. First order Autonomous ODE.

$$y' = f(y).$$

The RHS is independent on t. In this case we can tell a lot just looking and the function $f(y)$. E.g. in $f(y_*) = 0$ then $y(t) = y_*$ is stationary solution of the equation. We can also tell whether $y(t)$ in increasing/decreasing, concave or convex. See more examples in Assignment 2.

5. Exact Equations.

$$M(x,y)dx + N(x,y)dy = 0, \quad \text{where} \quad \partial_y M = \partial_x N.$$

In this case you can find function $F(x,y)$ such that

$$\partial_x F = M(x,y), \quad \partial_y F = N(x,y).$$

Then your solution in implicit form is $F(x,y) = const.$

Example: $(x + 3y)dx + (3x + y^2)dy = 0$. Here $M = x + 3y$, $N = 3x + y^2$ and $\partial_y M = \partial_x N = 3$. Then $F(x,y) = \int M(x,y)dx = x^2/2 + 3yx + C(y)$, where $C(y)$ is any function of y. From condition $\partial_y F = N(x,y)$ we get equation for $C(y)$: $C' = y^2$. Thus $C(y) = y^3/3$ and the answer is

$$\frac{x^2}{2} + 3xy + y^3 3 = const.$$

See more examples in Assignment 2.

5a. Integrating factors. Some equations become exact after multiplication by certain function. In general in is very difficult to find such a function. In some special cases you can try one of the following:

$$\mu(x) = \exp \int \frac{M_y - N_x}{N}\, dx, \quad \text{if } \frac{M_y - N_x}{N} \text{ depends only of } x,$$

or

$$\mu(y) = \exp \int \frac{-M_y + N_x}{M}\, dy, \quad \text{if } \frac{M_y - N_x}{M} \text{ depends only of } y.$$

6. Second order Homogeneous ODE with constant coeffitients

$$y'' + by' + cy = 0, \quad \text{where} \quad b, c = const.$$

First, you solve the characteristic equation

$$\lambda^2 + b\lambda + c = 0, \quad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

There are three different cases:

case 1. $\lambda_1 \neq \lambda_2$, both real. Then *general solution* is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \quad \text{where} \quad C_1, C_2 = const.$$

case 2 $\lambda_1 = \lambda_2 = -b/2$ Then *general solution* is

$$y(t) = C_1 e^{-bt/2} + C_2 t e^{-bt/2}, \quad \text{where} \quad C_1, C_2 = const.$$

case 3 $\lambda_{1,2} = A \pm iB$ complex roots. Then *general solution* is

$$y(t) = e^{At}(C_1 \cos(Bt) + C_2 \sin(Bt)), \quad \text{where} \quad C_1, C_2 = const.$$

See examples in Assignment 3.

7. Second order Non-Homogeneous ODE with constant coeffitients.

$$y'' + by' + cy = g(t), \quad \text{where} \quad b, c = const.$$

Let $Y(t)$ be a particular solution of the non-homogeneous equation and $y_1(t), y_2(t)$ be two *linearly independent* solutions of the corresponding homogeneous equation

$$y'' + by' + cy = 0,$$

Then general solution of the non-homogeneous equation is

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + Y(t), \quad \text{where} \quad C_1, C_2 = const$$

Two functions $y_1(t), y_2(t)$ are *linearly independent* for $t \in (A, B)$ if their Wronskian $W[y_1, y_2] \equiv y_1 y_2' - y_2 y_1'$ is **not zero** for all $t \in (A, B)$.

If two linearly independent functions both satisfy a second order homogeneous equation then they form a *fundamental set of solutions* for the equation.

There are two methods to find a particular solution $Y(t)$ of a non-homogeneous equation.

7a. Method of undetermined coefficients.

Try to find $Y(t)$ in the form similar to the RHS $g(t)$, but more general. Examples: If $g(t) = e^{2t}$ then $Y(t) = Ae^{2t}$. If $g(t) = t^2$ then $Y(t) = A + Bt + Ct^2$. If $g(t) = \sin(3t)$ then $Y(t) = A\sin(3t) + B\cos(3t)$. Assume that A, B, C are some constants, substitute $Y(t)$ into the equation and try to determine the unknown coefficients A, B, C to satisfy the equation.

If you can't find them (remember they are constants), take $Y(t)$ in a little modified form (multiply whole thing by t or t^2) and try again.

See examples in Assignment 4, book Section 3.6.

7b. Variation of parameters. This is more general method then the previous one. Suppose that you know the fundumental set of solutions $y_1(t)$ and $y_2(t)$ for the corresponding homogeneous equation. Let $W(t) = W[y_1(t), y_2(t)] \equiv y_1 y_2' - y_2 y_1'$ be their Wronskian. Then particular solution $Y(t)$ of a non-homogeneous equation can be found by the formula

$$Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(t)}\, dt + y_2(t) \int \frac{y_1(t)g(t)}{W(t)}\, dt.$$

Here $g(t)$ is the RHS of the non-homogeneous equation.

See examples in Assignment 4.

8. Euler equations.

This is a special type of linear second order ODE with *non-constant* coeffitients

$$t^2 y'' + aty' + by = 0, \quad y = y(t), \quad a, b = const, \quad t > 0$$

The change of indepentent variable $x = \ln t$ transform the equation into

$$u'' + (a-1)u' + b = 0, \quad u = u(x).$$

This is linear second order ODE with *constant* coeffitients, which we can solve as shown above.

Then solution $y(t) = u(\ln t)$.

See examples in Assignment 3 Problem 8.

9. Laplace transform.

9a. Just another method.

This is another method to solve linear ODE with constant coefficients. The main idea is that if you apply Laplace transform to an ODE you will obtain an algebraic equation, which you can easily solve. Then, by doing inverse Laplace transform you obtain the solution.

The formula and notation are:

$$\hat{f}(s) = F(s) = L[f] = \int_0^\infty f(t)e^{-st}\,dt.$$

Here *Laplace transform* maps $f(t)$ to $\hat{f}(s)$, and *Inverse Laplace transform maps* $\hat{f}(s)$ to $f(t)$.

Note that the integral is improper so evaluate it you need to integrate from 0 to A and then find the limit as $A \to \infty$.

Example: If $f(t) = 1$ then

$$\hat{f}(s) = \lim_{A\to\infty} -\frac{e^{-st}}{s}\Big|_{t=0}^{t=A} = \frac{1}{s}, \quad \text{for} \quad s > 0.$$

The important fact is that the derivative of a function $y(t)$ becomes just multiplication by independent variable for the transformed function $\hat{y}(s)$:

$$L[y'(t)] = \int_0^\infty y'(t)e^{-st}\,dt = s\hat{y}(s) - y(0).$$

The second derivative transforms as

$$L[y''(t)] = \int_0^\infty y''(t)e^{-st}\,dt = s^2\hat{y}(s) - sy(0) - y'(0).$$

Remember, that Laplace transform is a linear map which means that

$$L[af + bg] = aL[f] + bL[g], \quad a, b = \text{const}.$$

Thus Laplace transform of equation $y'' + 2y' = 1$ gives

$$s^2\hat{y}(s) - sy(0) - y'(0) + 2(s\hat{y}(s) - y(0)) = \frac{1}{s}.$$

Suppose the initial conditions are given: $y(0) = 1$, $y'(0) = 2$.

Then we obtain

$$s^2\hat{y}(s) - s - 2 + 2s\hat{y}(s) - 2 = \frac{1}{s} \quad \text{so} \quad \hat{y}(s) = \frac{s+4}{s^2+2s} + \frac{1}{s(s^2+2s)}$$

To make inverse Laplace transform you often need to use method of partial fractions first. In our example we will need

$$\frac{s+4}{s^2+2s} = \frac{s+4}{s(s+2)} = \frac{2}{s} - \frac{1}{s+2}$$

and

$$\frac{1}{s(s^2+2s)} = \frac{1}{s^2(s+2)} = \frac{1}{2s^2} - \frac{1}{4s} + \frac{1}{4(s+2)}$$

So

$$\hat{y}(s) = \frac{2}{s} - \frac{1}{s+2} + \frac{1}{2s^2} - \frac{1}{4s} + \frac{1}{4(s+2)} = \frac{7}{4s} - \frac{3}{4(s+2)} + \frac{1}{2s^2}$$

Using the Laplace transform table (it will be given at Final exam) we get the solution

$$y(t) = \frac{7}{4} - \frac{3}{4}e^{-2t} + \frac{t}{2}$$

One can check that it indeed satisfies the equation and initial data.

9b. Equations with Dirac Delta function and Heaviside Step function.

Laplace transform of delta function is exponent:

$$L[\delta(t-c)] = e^{-cs}.$$

Note, that for $c = 0$ you have $L[\delta(t)] = 1$.

Delta function $\delta(t-c)$ is the derivative of the step function $u_c(t)$, which is 0 for $t < c$ and 1 for $t \geq c$.

Laplace transform of the step function is

$$L[u_c(t)] = \frac{e^{-cs}}{s}, \quad s > 0.$$

Also

$$L[u_c(t) f(t-c)] = e^{-cs}\hat{f}(s), \quad s > 0.$$

Use it for the inverse transform: if $\hat{f}(s)$ gets multiplied by an exponent, then $f(t)$ has shift of the argument and times the step function. For example $\hat{y}(s) = e^{-3s}\frac{s}{s^2+4}$ then $y(t) = \cos(2(t-3))u_3(t)$.

Do not confuse with the case when $f(t)$ gets multiplied by e^{ct}. Then the Laplace transform obtaines a shift of the argument, but there is NO step function:

$$L[e^{ct} f(t)] = \hat{f}(s-c).$$

9c. Convolution.

Remember that Laplace transform of the product of two functions is NOT a product of their Laplace transforms.

$$L[fg] \neq L[f]L[g]$$

Instead, Laplace transform of the convolution $(f * g)(t)$ is equal to the product of the Laplace transforms.

$$L[f * g] = L[f]L[g], \quad \text{where} \quad (f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Thus if you need to find the inverse Laplace transform of $\hat{f}(s)\hat{g}(s)$, you first find $f(t)$ and $g(t)$ and evaluate their convolution $(f * g)(t)$.

Example. Let $\hat{y}(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2}\frac{1}{s^2+1} = \hat{f}(s)\hat{g}(s)$. Then $f(t) = t$, $g(t) = \sin t$ and $y(t) = (f * g)(t) = \int_0^t \tau \sin(t-\tau)\,d\tau$. This integral can be evaluated by parts.

More examples of solving ODEs using Laplace transform are given in assignment 5.

10. Systems of linear equations with contant coefficients.

We will consider only systems with two equations and two unknown functions. The systems have form $\vec{X}' = A\vec{X} + \vec{G}(t)$. Here $\vec{X} = (x_1, x_2)$ is an unknown vector function of t, and A is a given 2×2 matrix with constant entries. Vector $\vec{G}(t)$ is given as well.

10 a. Homogeneous systems.

If $\vec{G}(t) = \vec{0}$ the system is called homogeneous. For such a system solutions can be always found.

Note that $\vec{X}(t) = \vec{0}$ is always a particular solution on a homogeneous system. No find general solution you do the following.

First, you solve the eigen-problem $A\vec{\xi} = r\vec{\xi}$ to find the eigenvalues and eigenvectors of the matrix A.

The eigenvalues are solutions of $\det(A - rI) = 0$ where I is a diagonal unit matrix. Then for each eigenvalue r_1 and r_2 solve $(A - r_iI)\vec{\xi_i} = 0$, $i = 1, 2$ to get eigenvector ξ_i.

There are 3 major cases:

1) eigenvalues are distinct and real.

2) eigenvalues coincide

3) eigenvalues are complex.

Having two eigenvalues and corresponding eigenvector, you can form vectors $\vec{X}_1 = e^{r_1 t}\vec{\xi}_1$ and $\vec{X}_2 = e^{r_2 t}\vec{\xi}_2$. If their Wronskian (determinant of the matrix with collomns $\vec{X}_1$ and $\vec{X}_2$) is not zero, then they are linearly independent and general solution of the system $\vec{X}' = A\vec{X}$ has the form

$$\vec{X}(t) = c_1\vec{X}_1 + c_2\vec{X}_2,$$

with arbitrary constants c_1, c_2. This scheme will always work in the first case (eigenvalues are distinct real).

In the second case ($r_1 = r_2 = r$) typically you will get only one eigenvector $\vec{\xi}$ and solution $\vec{X}_1(t) = e^{rt}\vec{\xi}$. The other (linearly independent) solution search in the form $\vec{X}_2(t) = (t\vec{\xi} + \vec{\eta})e^{rt}$, substitute into equation $\vec{X}' = A\vec{X}$ to determine $\vec{\eta}$.

Then the general solution is $\vec{X}(t) = c_1\vec{X}_1(t) + c_2\vec{X}_2(t)$.

In the third case ($r_{1,2} = \lambda \pm i\mu$) vectors $\vec{X}_j = e^{r_j}\vec{\xi}_j$, $j = 1, 2$ will be complex vectors (complex conjugate to each other). To get the real solution you must rewrite it in the form

$$\vec{X}_1(t) = \vec{V}_1 + i\vec{V}_2$$

where $\vec{V}_{1,2}$ are two real vectors. Then the general real solution is

$$\vec{X}(t) = c_1\vec{V}_1(t) + c_2\vec{V}_2(t).$$

10b. Non-Homogeneous systems.

General system on a non-homogeneous system $\vec{X}' = A\vec{X} + \vec{G}(t)$ (with nonzero $\vec{G}(t)$) is the sum of its particular solution and a general solution of the corresponding homogeneous system $\vec{X}' = A\vec{X}$.

To find a particular solution of the non-homogneous system we use method of undetermined coefficients. That is, we guess the solution in a form structurally similar to vector $\vec{G}(t)$ with unknown vector coefficients, substitute it into equation $\vec{X}' = A\vec{X} + \vec{G}(t)$ with nonzero $\vec{G}(t)$ and try to determine coefficients to satisfy the equation.

See examples in your Assignment 6 Problem 5.

10c. Initial value priblem.

In such problems the value of unknown vector $\vec{X}(t)$ must be equal to a given vector $\vec{X}_0$ at initial time ($t = 0$). You first find general solution of the problem, evaluate it at $t = 0$ and set it equal to $\vec{X}_0$ to determine constants c_1, c_2. It is always possible to do because vectors $\vec{X}_1$ and $\vec{X}_2$ in the general solution are linearly independent (Wronskian is not 0, so matrix is invertable and solution c_1, c_2 is unique).

See examples in your Assignment 6 Problem 4.

10d. Phase portrait.

You can plot the vector $\vec{X}(t)$ in the plane by plotting point for each value of t. These points form a continious curve. Collections of such curves is called the phase portrait of the system. As we mentioned above, for a homogeneous system $\vec{X}(t) = \vec{0}$ is always a solution. Point at the origin symbolizes this trajectory. Depending on the eigenvalues of A a vicinity of the origin has different structure. We distinguish cases of *saddle point, node, spiral point and center*. The sign of eigenvalues determine direction along the phase curves and whether the origin is stable or unstable. Look in your book for detailes of the pictures.

Please, make sure that you understand and **can do** all the problems from the midterm test, as well as from your homework assignments, and the last year final exam.

Send me an e-mail if you need something extra or if you have a question : mkondra@math.mun.ca

Good luck!