Midterm Solutions

Question 1: Solve initial value problem.

$$y' + 3y - e^t = 0, \quad y(0) = 0.$$

Solution:

$$y' + 3y = e^t$$

$$p(t) = constant = 3, \quad g(t) = e^t$$

$$y(t) = A(t)e^{-at} = A(t)e^{-3t}$$

$$A(t) = \int g(t)e^{at}dt$$

$$= \int (e^t)(e^{3t})dt = \int e^{4t}dt = \frac{e^{4t}}{4} + c$$

$$y(t) = \left(\frac{e^{4t}}{4} + c\right) e^{-3t}$$

$$= \frac{e^t}{4} + ce^{-3t}$$

Therefore, general solution is:

$$y(t) = \frac{e^t}{4} + ce^{-3t}$$

Pluging in t=0;

$$y(0) = \frac{e^0}{4} + ce^0 = 0$$

$$\Longrightarrow \frac{1}{4} + c = 0$$

Therefore, $c = \frac{-1}{4}$ and the final soultion is:

$$y(t) = \frac{e^t}{4} - \frac{e^{-3t}}{4}$$

Question 2: Solve initial value problem.

$$y' = \frac{yx^5}{2 + x^6}, \quad y(1) = 1.$$

Solution:

$$y' = \frac{yx^5}{2+x^6}$$

$$\int \frac{1}{y} dy = \int \frac{x^5}{2+x^6} dx$$

let $u = 2 + x^6$, $du = 6x^5 dx$

$$\ln|y| = \int \frac{1}{6u} du$$

$$\ln|y| = \frac{1}{6} \int \frac{1}{u} du$$

$$\ln|y| = \frac{1}{6}(\ln|u|) + c$$

$$\ln|y| = \frac{\ln|2+x^6|}{6} + c$$

Therefore, general soultion is:

$$\ln|y| = \frac{\ln|2+x^6|}{6} + c$$

Pluging in $x = 1$ and $y = 1$:

$$\ln|1| = \frac{\ln|2+1|}{6} + c$$

$$0 = \frac{\ln|3|}{6} + c$$

Therefore, $c = \frac{ln|3|}{6}$ and the final solution is:

$$y(x) = e\left(\frac{\ln|2+x^6|}{6} - \frac{\ln|3|}{6}\right)$$

Simplified to:

$$y(x) = \left(\frac{2+x^6}{3}\right)^{\frac{1}{6}}$$

Question 3: Consider equation

$$y' = y^2 - 1.$$

(a.)Plot RHS vs y and find equilibrium solutions and their types of stability.

Solution:

$$Let f(y) = y^2 - 1$$
$$F(y) = (y-1)(y+1)$$

The equilibrium soultions are:

$$y = 1, \quad y = -1$$

$$\text{If} \quad y(t) < -1 \quad \text{then} \quad f(y) > 0 \Longrightarrow y' > 0 \Longrightarrow y(t) \uparrow$$

$$\text{If} \quad y(t) > 1 \quad \text{then} f(y) > 0 \Longrightarrow y' > 0 \Longrightarrow y(t) \uparrow$$

$$\text{If} \quad -1 < y(t) < 1 \quad \text{then} f(y) < 0 \Longrightarrow y' < 0 \Longrightarrow y(t) \downarrow$$

From the above information, we can say that $y = 1$ is unstable and $y = -1$ is stable.

(b.)Investigate growth/decay and concavity of the solution for different initial values. Plot the solution $y(t)$ on the (t, y)-plane.

(c.)Find the solution given by (implicit) formula.

Solution:

$$y' = y^2 - 1$$
$$\int \frac{1}{y^2-1} dy = \int dx$$
$$\int \frac{1}{(y-1)(y+1)} dy = \int dx$$

Now must solve LHS by using partial fractions:

$$\frac{A}{y-1}+\frac{B}{y+1}$$
$$\Longrightarrow \frac{Ay+A+By-B}{(y-1)(y+1)}$$
$$\Longrightarrow \frac{(A+B)y+A-B}{(y+1)(y-1)}$$

Therfore, $A+B=0$ and $A-B=1$ and you get $A=\frac{1}{2}$ and $B=\frac{-1}{2}$
Now the LHS becomes:

$$\int \frac{1}{2(y-1)}dy - \int \frac{1}{2(y+1)}dy$$

So therefore, the implicit formula is:

$$\frac{\ln|y-1|}{2} - \frac{\ln|y+1|}{2} = x + c$$

Question 4: Determine whether or not the equation is exact. Solve it.

$$x + 2y - (3y - 2x)y' = 0.$$

Solution:
$M(x,y) = x + 2y$ and $N(x,y) = -(3y - 2x)$
In determining if the equation is exact you must find the partial derivatives of M and N:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$2 = 2$$

Therefore equation is exact.

Now you must find the function F(x,y):

First determining the integral of M with respect to $x(F' = M)$:

$$F(x,y) = \int x + 2y dx$$
$$= \frac{x^2}{2} + 2xy + C(y)$$

Now determining F' with respect to y, where F' must equal N:

$$F'(x,y) = 0 + 2x + C'(y) = N = -3y + 2x$$

Above the $2x$ term cancels and we are left with:

$$C'(y) = -3y$$

Therefore, the final solution is:

$$\frac{x^2}{2} + 2xy - \frac{3y^2}{2} = constant$$

Question 5: Find an integrating factor and transform the equation into an exact one. **Do not solve it**.

$$(2y - \sin xe^x)dx + xdy = 0.$$

Solution:

$M(x,y) = 2y - \sin(xe^x)$ and $N(x,y) = x$

First, show that the equation is not exact and an integrating factor is needed:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$2 = 1$$

Therfore, equation is not exact.

Now we must determine the integrating factor:

$$\mu' = \frac{M' - N'}{N}$$

$$= \frac{2-1}{x}$$

$$= \frac{1}{x}$$

$$\int \frac{1}{x} dx = \ln|x|$$

$$\mu(x) = e^{\ln|x|}$$

$$= x$$

Above M' is determined with respect to y and N' is determined with respect to x.

We now multiple the integrating factor by the equation that was given to get:

$$x[(2y - \sin(xe^x))dx + xdy = 0]$$
$$(2xy - (x)\sin(xe^x))dx + x^2dy = 0$$

We now check the equation again to see if it is exact or not(if the equation is not exact then we know that something was done wrong when determining the integrating factor):

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$2x = 2x$$

Therfore, equation is exact and we are done.

Question 6: Consider differential equation

$$9y - 6y' + y'' = 0.$$

(a.) Find the fundumental set of solutions.

Solution:

$$y'' - 6y' + 9y = 0$$

$$Let y(t) = e^{\lambda t}, \quad y' = \lambda e^{\lambda t}, \quad y'' = \lambda^2 e^{\lambda t}$$

$$\lambda^2 e^{\lambda t} - 6(\lambda e^{\lambda t}) + 9e^{\lambda t} = 0$$

$$e^{\lambda t}(\lambda^2 - 6(\lambda) + 9) = 0$$

$$(\lambda - 3)(\lambda - 3) = 0$$

$$\lambda = 3$$

Therefore the fundamental set of solutions are:

$$y_1 = e^{3t}, \qquad y_2 = te^{3t}$$

(b.)Show the solutions are linearly independent by calculating their Wronskian.

Solution:

$$W[y_1, y_2] = (y_1)(y_2') - (y_2)(y_1')$$

$$= (e^{3t})(e^{3t} + 3te^{3t}) - (te^{3t})(3e^{3t})$$

$$= e^{6t} + 3te^{6t} - 3te^{6t}$$

$$= e^{6t}$$

(c.)Find general solution of the equation.

Solution:

$$y(t) = c_1 e^{3t} + c_2 te^{3t}$$

Question 7: Find a particular solution of the non-homogeneous equation.

$$y'' - 5y' + 4y = 1 + e^{2t}.$$

Solution:
Step 1- Solve for the homogeneous equation:

$$y'' - 5y' + 4y = 0$$

$$\lambda^2 - 5(\lambda) + 4 = 0$$

$$(\lambda - 4)(\lambda - 1) = 0$$

$$\lambda_1 = 4, \quad \lambda_2 = 1$$

Therefore the solution to the homogeneous equation is:

$$y(t) = c_1 e^{4t} + c_2 e^t$$

Step 2- Solve the non-homogeneous equation separately:
First solve,

$$y'' - 5y' + 4y = 1$$

$$y(t) = A,\ y'(t) = 0,\ y''(t) = 0$$

Now plug these values into the equation that was given:

$$(0) - 5(0) + 4A = 1$$

$$\Longrightarrow 4A = 1$$

$$A = \frac{1}{4}$$

Second solve,

$$y'' - 5y' + 4y = e^{2t}$$

$$y(t) = Ae^{2t}, y'(t) = 2Ae^{2t}, y''(t) = 4Ae^{2t}$$

Now plug these values into the equation that was given:

$$(4Ae^{2t}) - 5(2Ae^{2t}) + 4(Ae^{2t}) = e^{2t}$$
$$4Ae^{2t} - 10Ae^{2t} + 4Ae^{2t} = e^{2t}$$
$$-2Ae^{2t} = e^{2t}$$
$$-2A = 1$$
$$A = \frac{-1}{2}$$

Therefore, the particular solution to the equation is:

$$y(t) = \frac{1}{4} - \frac{e^{2t}}{2}$$

Question 8: Bonus Solve.

(a)

$$y' = y + y^{100}.$$

Solution:

This is Bernouill's equation. To solve the equation we must first divide by $y^{1}00$ to get:

$$y^{-100}y' = y^{-99} + 1$$

$$v = y^{-99} \quad v' = -99y^{-100}y'$$

$$-99y^{-100}y' = -99y^{-99} - 99$$

$$v' = -99v - 99$$

$$v' + 99v = -99$$

Now the equation is first order and linear so we can solve it quite easily:

$$p(t) = constant = 99, \quad g(t) = -99$$

$$v(t) = A(t)e^{-at}$$

$$A(t) = \int g(t)e^{at}dt$$

$$A(t) = \int -99e^{99t}dt$$

$$A(t) = -e^{99t} + c$$

So therefore,

$$v(t) = e^{-99t}\left(-e^{99t} + c\right)$$

$$= -1 + ce^{-99t}$$

And $y(t) = (v(t))^{\frac{-1}{99}}$. So therefore the final solution is:

$$y(t) = \left(ce^{-99t} - 1\right)^{\frac{-1}{99}}$$

(b)

$$t^2y'' + y' + y = t.$$

Solution:
This is **not** Euler Equation.
Try substitutint the following into the equation:

$$y(t) = At + B, \quad y'(t) = A, \quad y''(t) = 0$$

$$t^2(0) + (A) + (At + B) = t$$

We get $A = 1$ and $B = -1$.
So therefore the final solution is:

$$y(t) = t - 1$$

If it was an Euler Equation you would do the following:

$$t^2y'' + ty' + y = t$$

Substitute $x = \ln t$, therfore making $t = e^x$

Plugging the above information into Euler's equation to get:

$$y""_x x + y(x) = e^x$$

$$\lambda^2 + 1 = 0$$

Therefore, $\lambda = \pm i$ so the general solution to the homogeneous equation is:

$$y(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t)$$

Now we must solve the non-homogeneous equation:

$$y''(x) + y(x) = e^x$$

$$y(x) = Ae^x,\ y'(x) = Ae^x,\ y''(x) = Ae^x$$

Plugging the above values into the equation:

$$(Ae^x) + (Ae^x) = e^x$$
$$2Ae^x = e^x$$
$$2A = 1$$

And, $A = \frac{1}{2}$

Therefore, the final solution is:

$$y(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t) + \frac{1}{2}t$$