Assignment 4 Solutions

Problem1: Use Method of Undetermined Coefficients to find a particular solution of the non-homogeneous equation. Find general solution of the non-homogeneous equation.

(a.)$y'' + 2y' + y = 2e^t$

First solve the homogeneous equation:

$$y'' + 2y' + y = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)(\lambda + 1) = 0$$

$$\lambda = -1 \quad \text{(repeated root)}$$

So therefore, the general solution of the homogeneous part is:

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}$$

Now we must solve for the non-homogeneous part:

$$y(t) = Ae^t,\ y'(t) = Ae^t,\ y''(t) = Ae^t$$

Plug the above into the non-homogeneous equation and solve.

$$y'' + 2y' + y = 2e^t$$

$$(Ae^t) + 2(Ae^t) + (Ae^t) = 2e^t$$

$$4Ae^t = 2e^t$$

$$4A = 2$$

$$A = \frac{1}{2}$$

So therefore the final solution is:

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2}$$

(b.)$y'' + 2y' + y = 2e^{-t}$

First solve the homogeneous equation:

$$y'' + 2y' + y = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)(\lambda+1)=0$$

$$\lambda=-1 \quad \text{(repeated root)}$$

So therefore, the general solution of the homogeneous part is:

$$y(t)=c_1e^{-t}+c_2te^{-t}$$

Now we must solve for the non-homogeneous part:

$$y(t)=At^2e^{-t},\ y'(t)=A(2te^{-t}-t^2e^{-t}),\ y''(t)=A(2e^{-t}-4te^{-t}+t^2e^{-t})$$

Plug the above into the non-homogeneous equation and solve.

$$y''+2y'+y=2e^t$$

$$A[2e^{-t}-4te^{-t}+te^{-t}+4te^{-t}-2t^2e^{-t}+t^2e^{-t}]=2e^t$$

$$A=1$$

So therefore the final solution is:

$$y(t)=c_1e^{-t}+c_2te^{-t}+t^2e^{-t}$$

(c.)$y''+2y'+5y=3\sin(2t)$
First solve the homogeneous equation:

$$y''+2y'+5y=0$$

$$\lambda^2+2\lambda+5=0$$

$$\lambda=-1\pm 2i$$

So therefore, the general solution of the homogeneous part is:

$$y(t)=c_1e^{-t}\cos 2t+c_2e^{-t}\sin 2t$$

Now we must solve for the non-homogeneous part:

$$y(t)=A\sin 2t+B\cos 2t,\ y'(t)=2A\cos 2t-2B\sin 2t,\ y''(t)=-4A\sin 2t-4B\cos 2t$$

Plug the above into the non-homogeneous equation and solve.

$$y''+2y'+5y=2e^t$$

$$(-4A\sin 2t - 4B\cos 2t) + 2(2A\cos 2t - 2B\sin 2t) + 5(A\sin 2t + B\cos 2t) = 3\sin 2t$$

$$(A - 4B)\sin 2t + (B + 4A)\cos 2t = 3\sin 2t$$

$$A - 4B = 3, \quad B + 4A = 0$$

$$A = \frac{3}{17} and B = \frac{-12}{17}$$

So therefore the final solution is:

$$y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$$

(d.) $y'' - 2y' - 3y = e^{2t}$
First solve the homogeneous equation:

$$y'' - 2y' - 3y = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda + 1)(\lambda - 3) = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = -1$$

So therefore, the general solution of the homogeneous part is:

$$y(t) = c_1 e^{3t} + c_2 e^{-t}$$

Now we must solve for the non-homogeneous part:

$$y(t) = Ae^{2t}, \quad y'(t) = 2Ae^{2t}, \quad y''(t) = 4Ae^{2t}$$

Plug the above into the non-homogeneous equation and solve.

$$y'' - 2y' - 3y = e^{2t}$$

$$4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = e^{2t}$$

$$A = \frac{-1}{3}$$

So therefore the final solution is:

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{1}{3} e^{2t}$$

Problem 2: Solve the Initial Value Problem
(a.)$y'' - 2y' + y = 4, \qquad y(0) = 1, \quad y'(0) = 1$
First solve the homogeneous equation:

$$y'' - 2y' + y = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)(\lambda - 1) = 0$$

$$\lambda_1 = \lambda_2 = 1$$

So therefore, the general solution of the homogeneous part is:

$$y(t) = c_1 e^t + c_2 t e^t$$

Now we must solve for the non-homogeneous part:

$$y(t) = A, \ y'(t) = 0, \ y''(t) = 0$$

Plug the above into the non-homogeneous equation and solve.

$$y'' - 2y' + y = 4$$

$$(0) - 2(0) + (A) = 4$$

$$A = 4$$

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So therefore the general solution is:

$$y(t) = c_1 e^t + c_2 t e^t + 4$$

Now we must plug in the initial values to determine c_1 and c_2:

$$y(t) = c_1 e^t + c_2 t e^t + 4$$

$$y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t$$

$$y(0) = c_1 + 4 = 1$$

$$\Longrightarrow c_1 = -3$$

$$y'(0) = c_1 + c_2 = 1$$

$$\implies c_2 = 4$$

So therefore the final solution to the initial value problem is:

$$y(t) = -3e^t + 4te^t + 4$$

(b.) $y'' - 2y' + y = te^t + 4, \qquad y(0) = 1, \quad y'(0) = 1$

First solve the homogeneous equation:

$$y'' - 2y' + y = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)(\lambda - 1) = 0$$

$$\lambda_1 = \lambda_2 = 1$$

So therefore, the general solution of the homogeneous part is:

$$y(t) = c_1 e^t + c_2 te^t$$

Now we must solve for the non-homogeneous part, which we can solve separately:

We can first solve $y'' - 2y' + y = 4$, which was already done in the previous question. So therefore we can go ahead and solve

$$y'' - 2y' + y = te^t$$

$$y(t) = At^3e^t,\ y'(t) = A(3t^2e^t + t^3e^t),\ y''(t) = A(6te^t + 6t^2e^t + t^3e^t)$$

Plug the above into the non-homogeneous equation and solve.

$$y'' - 2y' + y = te^t$$

$$A[6t + 6t^2 + t^3 - 6t^2 - 2t^3 + t^3)e^t = te^t$$

$$A = \frac{1}{6}$$

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So therefore the general solution is:

$$y(t) = c_1 e^t + c_2 te^t + \frac{1}{6}t^3e^t + 4$$

So therefore the final solution to the initial value problem is:

$$y(t) = -3e^t + 4te^t + \frac{1}{6}t^3e^t4$$

Problem 3: Use Method of Variation of Parameters to find a solution of the non-homogeneous equation. Find general solution of the non-homogeneous equation

(a.)$y'' + 4y = 3\csc 2t, \qquad 0 < t < \frac{\pi}{2}$

First solve the homogeneous equation:

$$y'' + 4y = 0$$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

So therefore, the general solution of the homogeneous part is:

$$y(t) = c_1 \cos 2t + c_2 \sin 2t$$

Now first solve for Wronskian of the equation and then solve for c_1 and c_2:

$$W[\cos 2t, \sin 2t] = \cos 2t(2\cos 2t) + 2\sin 2t(\sin 2t)$$

$$= 2\cos^2 2t + 2\sin^2 2t = 2$$

$$c_1 = \int \csc 2t \cos 2t dt = \frac{\ln|\sin 2t|}{2}$$

$$c_2 = \int \csc 2t \sin 2t dt = t$$

$$y(t) = \frac{3\sin 2t}{4}\ln|\sin 2t| - \frac{3t\cos 2t}{2}$$

So therefore the final solution is:

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{3\sin 2t}{4}\ln|\sin 2t| - \frac{3t\cos 2t}{2}$$

(b.) $y'' + 4y = 3\sec^2(2t), \qquad \frac{-\pi}{4} < t < \frac{\pi}{4}$

First solve the homogeneous equation:

$$y'' + 4y = 0$$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

So therefore, the general solution of the homogeneous part is:

$$y(t) = c_1 \cos 2t + c_2 \sin 2t$$

Now first solve for Wronskian of the equation and then solve for c_1 and c_2:

$$W[\cos 2t, \sin 2t] = 2\cos 2t(\cos 2t) + 2\sin 2t(\sin 2t)$$

$$= 2\cos^2 2t + 2\sin^2 2t = 2$$

$$c_2 = \int \sec^2(2t)\cos 2t dt = \frac{\ln|\tan 2t + \sec 2t|}{2}$$

$$c_1 = \int \sec^2(2t)\sin 2t dt = -\frac{1}{2\cos 2t}$$

So therefore the final solution is:

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{3\sin 2t}{4}\ln|\tan 2t + \sec 2t| - \frac{3}{4}$$

(c.) $y'' + 4y' + 4y = t^{-2}e^{-2t}$
First solve the homogeneous equation:

$$y'' + 4y' + 4y = 0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)(\lambda + 2)$$

$$\lambda_1 = \lambda_2 = -2$$

So therefore, the general solution of the homogeneous part is:

$$y(t) = c_1 e^{-2t} + c_2 te^{-2t}$$

Now first solve for Wronskian of the equation and then solve for c_1 and c_2:

$$W[e^{-2t}, te^{-2t}] = e^{-2t}(-2te^{-2t} + e^{-2t}) - (-2e^{-2t})(te^{-2t})$$

$$= (e^{-2t})^2$$

$$= e^{-4t}$$

$$c_1 = \int \left(\frac{t^{-2}e^{-2t}}{e^{-4t}}\right) e^{-2t} dt$$

$$= \int t^{-2} dt = \frac{-1}{t}$$

$$c_2 = \int \left(\frac{t^{-2} e^{-2t}}{e^{-4t}} \right) t e^{-2t} dt$$

$$= \int \frac{1}{t} dt = \ln |t|$$

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} - e^{-2t} \ln |t| - t e^{-2t} \frac{1}{t}$$

So therefore the final solution is:

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} - e^{-2t} \ln |t|$$

(d.)$t^2 y'' - 2y = 3t^2 - 1, \qquad t > 0$
Below we have Euler's Equation with $\alpha = 0$ and $\beta = -2$

$$t^2 y'' - 2y = 0$$

$$x = \ln |t|$$

$$y''_{xx} - y'_x - 2y = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = -1$$

$$y(x) = c_1 e^{2x} + c_2 e^{-x}$$

$$y(t) = c_1 t^2 + c_2 t^{-1}$$

Now first solve for the Wronskian of the equation and then solve for c_1 and c_2:

$$W[e^{2x}, e^{-x}] = -3e^x$$

$$\int \left(\frac{3e^{2x} - 1}{-3e^x} \right) e^{-x} dx$$

$$= -\frac{1}{3} \int 3e^{3x} - e^x dx$$

$$= -\frac{1}{3} [e^{3x} - e^x]$$

$$\int \left(\frac{3e^{2x}-1}{-3e^x}\right) e^{-x} dx$$

$$= \int -1 + \frac{e^{-2x}}{3} dx = -x - \frac{e^{-2x}}{6}$$

$$y = e^{2x}(x + \frac{e^{-2x}}{6}) + e^{-x}\left(\frac{e^x - e^{3x}}{3}\right)$$

$$= e^{2x}x + \frac{1}{6} + \frac{1}{3} - \frac{e^{2x}}{3}$$

$$y(x) = c_1 e^{2x} + c_2 e^{-x} + xe^{2x} + \frac{1}{2}$$

So therefore final solution is:

$$y(t) = c_1 t^2 + c_2 t^{-1} + \ln|t|(t^2) + \frac{1}{2}$$

Problem 4: An undamped spring-mass system with mass $m = 2$ and a spring constant $k = 8$ is suddenly set in motion at time $t = 0$ by an external force $f = 5\cos 3t$. Determine the position of the mass as a function of time and draw the graph.

$$mu'' + ku = F(t)$$

$$2u'' + 8u = 5\cos 3t$$

$$u(0) = 0, \quad u'(0) = 0$$

$$w_0 = \sqrt{\frac{k}{m}} = 2$$

$$u(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{\cos 3t}{2}$$

$$c_1 = \frac{1}{2}, \qquad c_2 = 0$$

So therefore the final solution is:

$$u(t) = \frac{1}{2}(\cos 2t - \cos 3t) = \sin\left(\frac{t}{2}\right)\sin\left(\frac{5t}{2}\right)$$