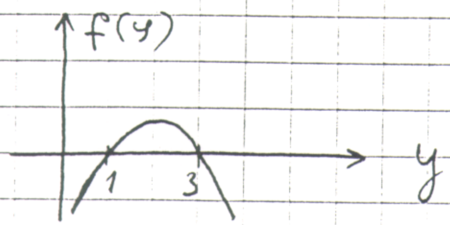
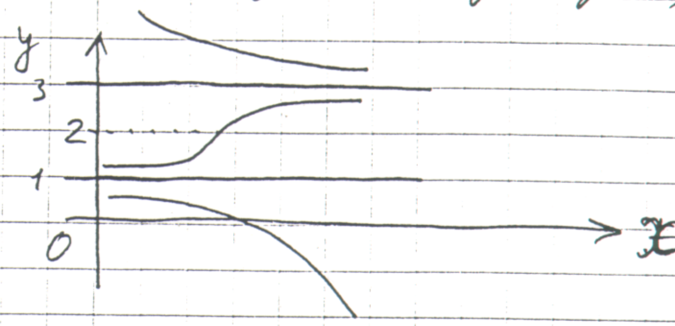
AMAT 3260 Assignment 2

Problem 1

(1) $y' = (1-y)(y-3)$

$y' = f(y)$, where $f(y) = (1-y)(y-3)$





$y(t) = 1$ unstable

$y(t) = 3$ stable

y	$y' = f$	f'	$y'' = f \cdot f'$	$y(x)$
$(-\infty; 1)$	$-$	$+$	$-$	$\cap$ $\downarrow$
$(1, 2)$	$+$	$+$	$+$	$\cup$ $\uparrow$
$(2, 3)$	$+$	$-$	$-$	$\cap$ $\uparrow$
$(3, \infty)$	$-$	$-$	$+$	$\cup$ $\downarrow$

$$\int \frac{dy}{(1-y)(y-3)} = x + C$$

$$\frac{1}{2}\int \left(\frac{1}{y-1} - \frac{1}{y-3} \right) dy = \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y-3|$$

$$\boxed{\frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y-3| = x + C}$$

Problem 1

(2.) $y' = ry - ky^2$

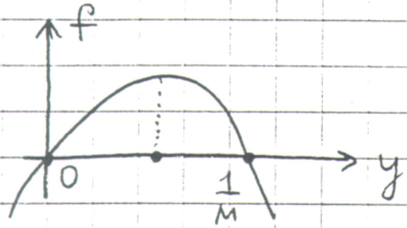
$y' = f(y)$, where $f(y) = ry\left(1 - \frac{k}{r}y\right)$

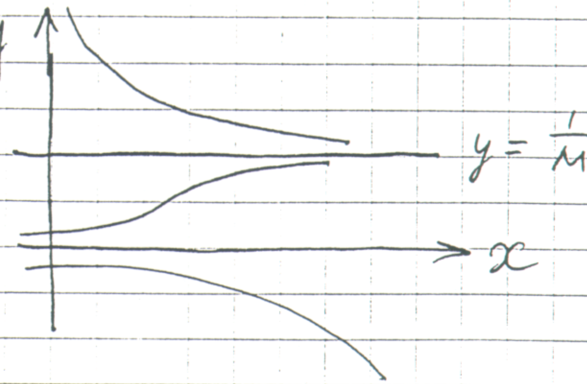
Here $k > 0$, $r > 0$ are given constants.

Denote $M = \frac{k}{r} > 0$. Then $f(y) = ry(1 - My)$

$y = 0$ (unstable)

$y(t) = \frac{1}{M}$ (stable)





y	$y' = f$	f'	$y'' = ff'$	$y(t)$
$(-\infty; 0)$	$-$	$+$	$-$	$\cap \downarrow$
$(0, \frac{1}{2M})$	$+$	$+$	$+$	$\cup \uparrow$
$(\frac{1}{2M}, \frac{1}{M})$	$+$	$-$	$-$	$\cap \uparrow$
$(\frac{1}{M}, \infty)$	$-$	$-$	$+$	$\cup \uparrow$

$$\int \frac{dy}{y(1-My)} = \int r\,dx$$

$$\int \frac{1}{y} + \frac{M}{1-My}\,dy = \ln|y| - \ln|1-My|$$

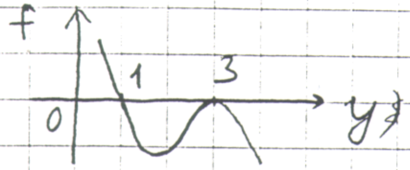
$$\boxed{\ln|y| - \ln|1-My| = rx + C}$$

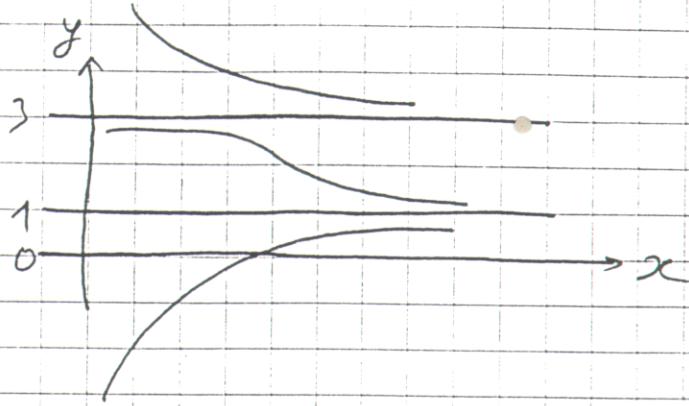
Problem 1. (3) $y' = (1-y)(3-y)^2$

$$y' = f(y), \text{ where } f(y) = (1-y)(3-y)^2$$

$y(t) = 1$ (stable)

$y(t) = 3$ (semistable)





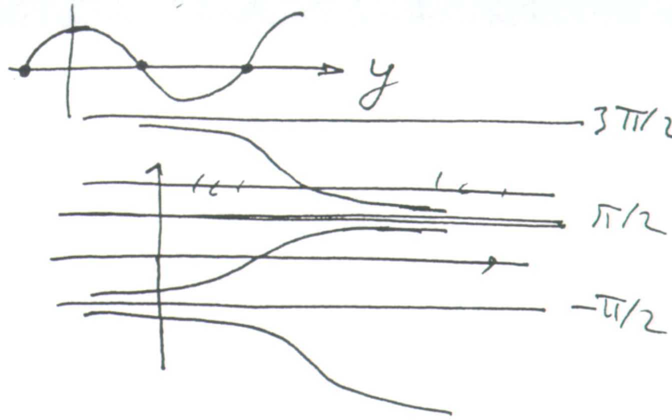
y	$y' = f$	f'	$y'' = f \cdot f'$	$y(x)$
$(-\infty, 0)$	$+$	$-$	$-$	$\cap$ $\uparrow$
$(1, 2)$	$-$	$-$	$+$	$\cup$ $\downarrow$
$(2, 3)$	$-$	$+$	$-$	$\cap$ $\downarrow$
$(3, \infty)$	$-$	$-$	$+$	$\cup$ $\downarrow$

$$\int \frac{dy}{(1-y)(3-y)^2} = \int dx$$

$$\frac{A}{1-y} + \frac{B}{3-y} + \frac{C}{(3-y)^2} = \frac{1}{(1-y)(3-y)^2}$$

$$\boxed{-\frac{1}{4}\ln|1-y| + \frac{1}{4}\ln|3-y| + \frac{1}{2}\frac{1}{y-3} = x + c}$$

④ $y' = \cos y$

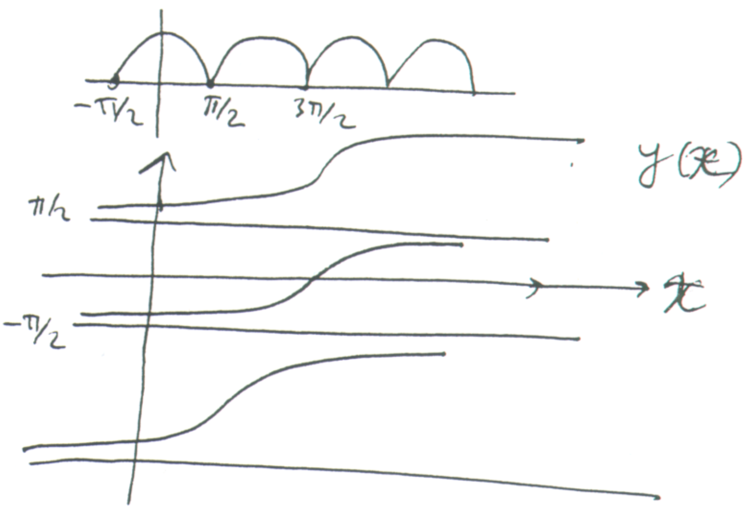


$y = \pi/2$ stable

$y = 3\pi/2$ unstable

$\ln|\sec y + \tan y| = x + C.$

⑤ $y' = \cos^2 y$

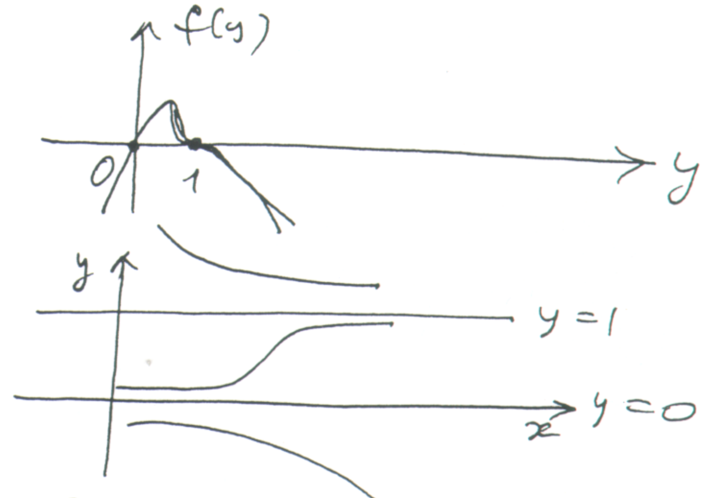


$\tan y = x + C.$

$y(t) = \frac{\pi}{2} + \pi k$ (semistable)

⑥ $y' = y - y^4$

$= y(1-y^3)$



$y = 0$ unstab.

$y = 1$ stable.

$y' y^{-4} = y^{-3} - 1$

$u = y^{-3}$

$u' = -4y^{-4} \cdot y'$

$u' = -4u + 4.$

$u = Ce^{-4x} + 1$

$$\boxed{y(x) = \frac{1}{\sqrt[3]{1 + Ce^{-4x}}}}$$

Problem 2

a) $(5x+7) + (6y+11)y' = 0$

$M_y = 0 \quad N_x = 0 \quad \Rightarrow$ Exact

$F = \frac{5}{2}x^2 + 7x + C(y)$

$F' = C' = 6y+11 \quad \Rightarrow C = 3y^2 + 11y$

$$\boxed{\frac{5}{2}x^2 + 7x + 3y^2 + 11y = C}$$

b) $(5x+7y) + (7x+11y)y' = 0$

$M_y = 7 \quad N_x = 7 \quad \Rightarrow$ Exact

$F = \frac{5}{2}x^2 + 7yx + C$

$F'_y = 7x + C' = 7x + 11y \quad \Rightarrow \quad C = \frac{11}{2}y^2$

$$\boxed{\frac{5}{2}x^2 + 7yx + \frac{11}{2}y^2 = \text{const}}$$

c) $(e^x \sin y - 2y\sin x) + (e^x \cos y + 2\cos x)y' = 0$

$M_y = e^x\cos y - 2\sin x \quad N_x = e^x \cos y - 2\sin x$. Exac

$F = e^x \cancel{\sin} y + 2y\cos x + C(y)$

$F'_y = +e^x \cancel{\cos} y + 2\cos x + C(y) = e^x\cos y + 2\cos x$

$\Rightarrow C(y) = 0 \quad \Rightarrow \quad C = \text{const}$

$$\boxed{e^x \sin y + 2y\cos x = \text{const}}$$

d) $y' = \dfrac{ax+by}{bx+cy}$; $ax+by - (bx+cy)y' = 0$

$M_y = b \quad N_x = -b \quad$ Not Exact

e) $y' = -\frac{ax+by}{bx+cy}$ $ax + by + (bx+cy)y' = 0$

$M_y = b$ $N_x = b.$

$F = \frac{ax^2}{2} + byx + C$

$F'_y = bx + C' = bx + cy$ $C = \frac{cy^2}{2}$

$$\boxed{\frac{ax^2}{2} + bxy + c\frac{y^2}{2} = \text{const}}$$

f) $\frac{x\,dx}{(x^2+y^2)^{5/2}} + \frac{y\,dy}{(x^2+y^2)^{5/2}} = 0$

$M_y = -\frac{5}{2}\frac{x \cdot 2y}{(x^2+y^2)^{7/2}}$ $N_x = -\frac{5}{2}\frac{y \cdot 2x}{(x^2+y^2)^{7/2}}$ Exact

$F = \frac{1}{3(x^2+y^2)^{3/2}} \Rightarrow \boxed{x^2+y^2 = \text{const}}$

Problem 3

$(ye^{-3xy} + x^2)\,dx - bxe^{-3xy}\,dy = 0$

$M_y = e^{-3xy} - 3xye^{-3xy}$

$N_x = -be^{-3xy} + b\,x \cdot 3y\,e^{-3xy} \Rightarrow \boxed{b=-1}$

$F = \frac{1}{(-3y)}\cdot y e^{-3xy} + \frac{x^3}{3} = -\frac{1}{3}e^{-3xy} + \frac{x^3}{3}$

$F'_y = xe^{-3xy} \Rightarrow C = \text{const}$

$$\boxed{x^3 - e^{-3xy} = \text{const}}$$

Problem 4

a) $y\,dx + (2x - ye^y)\,dy = 0 \qquad \mu(x,y) = y$

$y^2 dx + 2xy - y^2 e^y dy = 0.$

$M_y = 2y \qquad N_x = 2y$

$F = y^2 x + C(y) \quad ; \quad C' = -y^2 e^y$

$-C = + \int y^2 e^y dy = y^2 e^y - 2\int y e^y dy$

$= y^2 e^y - 2(y-1)e^y$

$= (y^2 - 2y + 2)\, e^y$

$$\boxed{y^2 x - (y^2 - 2y + 2)e^y = \text{const}}$$

b) $(x+2)\sin y + (x\cos y)\,y' = 0. \qquad \mu = xe^x$

$xe^x(x+2)\sin y + \underbrace{x^2 e^x \cos y}\, y' = 0$

$M_y = xe^x(x+2)\cos y \quad , \quad N_x = (2xe^x + x^2 e^x)\cos y$

$F = x^2 e^x \sin y + C(x)$

$F'_x = (2x + x^2)e^x \sin y = x(x+2)e^x \sin y \Rightarrow C = \text{con}$

$$\boxed{x^2 e^x \sin y = \text{const}}$$

Problem 5

(a) $1 + \left(\frac{x}{y} - \sin y\right) y' = 0$

$M_y = 0 \qquad N_x = \frac{1}{y}$

$\frac{M_y - N_x}{M} = -\frac{1}{y} \Rightarrow \mu = e^{\ln y} = y$

$y + (x - y\sin y) y' = 0$

$M_y = 1 \qquad N_x = 1$ Exact

$F = xy + C \,;\; F_y' = x + C' = x - y\sin y$

$C = -\int y \sin y \, dy = -\sin y + y\cos y$

$$\boxed{xy - \sin y + y\cos y = \text{const}}$$

(b) $y\,dx + (3x - e^y)\,dy = 0$

$M_y = 1 \qquad N_x = 3$

$\frac{M_y - N_x}{M} = -\frac{2}{y} \Rightarrow \mu = e^{2\ln y} = y^2$

$y^3 dx + (3xy^2 - y^2 e^y)\,dy = 0$

$M_y = 3y^2 \qquad N_x = 3y^2 \Longrightarrow$ Exact

$F = y^3 x + C \qquad F_y' = 3xy^2 + C' = 3xy^2 - y^2 e^y$

$C = -(y^2 - 2y + 2) e^y$

$$\boxed{y^3 x - (y^2 - 2y + 2) e^y = \text{const}}$$

c) $(3xy+y^2) + (x^2+xy)y' = 0$ $\qquad \mu = \dfrac{1}{xy(2x+y)}$ $\qquad$ (8

$$\frac{3x+y}{x(2x+y)} + \frac{x+y}{y(2x+y)}\, y' = 0$$

$$M_y = \frac{x(2x+y) - x(3x+y)}{x^2(2x+y)^2} = \frac{2x^2 + \cancel{xy} - 3x^2 - \cancel{xy}}{x^2(2x+y)^2} = -\frac{1}{(2x+y}$$

$$N_x = \frac{y(2x+y) - 2y(x+y)}{y^2(2x+y)^2} = \frac{2yx + y^2 - 2yx - 2y^2}{y^2(2x+y)^2} = -\frac{1}{(2x+y)}$$

$$F = \int \frac{3}{2x+y}\,dx + \int \frac{y}{x(2x+y)}\,dx$$

$$= \frac{3}{2}\ln|2x+y| + \int \frac{1}{x} + \frac{-2}{2x+y}\,dx$$

$$= \underbrace{\frac{3}{2}\ln|2x+y|} + \ln|x| - \underbrace{\ln|2x+y|} + C(y).$$

$$F' = \frac{3}{2}\,\frac{1}{2x+y} - \frac{1}{2x+y} = \frac{1}{2}\,\frac{1}{2x+y} + C'$$

$$\frac{x}{y(2x+y)} + \frac{1}{2x+y} =$$

$$= \frac{1}{2}\left(\frac{1}{y} + \frac{-1}{2x+y}\right) + \frac{1}{2x+y} = \frac{1}{2y} + \frac{1}{2}\left(\frac{1}{2x+y}\right)$$

$$= C' = \frac{1}{2y} \Rightarrow C = \frac{1}{2}\ln y$$

$\cancel{\frac{3}{2}}$ $\quad \boxed{\frac{1}{2}\ln|2x+y| + \ln|x| + \frac{1}{2}\ln|y| = \text{const}}$

Problem 5

(d) $x\,dy + (2y - e^x)\,dx = 0.$

$\cancel{M_y = 0 \quad N_x = 1 - e^x y}$

$\cancel{\frac{M_y - N_x}{M} = 1 - \frac{e^x}{x}}$

$M_y = 2 \qquad N_x = 1$

$$\frac{M_y - N_x}{N} = \frac{1}{x}$$

$$\mu = e^{\ln|x|} = x.$$

$$x^2\,dy + (2yx - xe^x)\,dx = 0.$$

$F = x^2 y + C(x) \qquad F'_x = 2xy - xe^x = 2xy + C'$

$C' = -xe^x$

$C = -(x-1)e^x$

$$\boxed{x^2 y - (x-1)e^x = \text{const}}$$

Problem 6

a) $(t+3)y' + \ln t\, y = 2t$

$y' + \frac{\ln t}{t+3} y = \frac{2t}{t+3}$; $t > 0$, $t \neq -3$

$y(1) = 2.$ $\longrightarrow$ $\boxed{(0, \infty).}$

b) $y' + (\cot t) y = \cos t$ $\qquad t \neq 0, \pi, \ldots$

$y(\pi/2) = 0$ $\longrightarrow$ $\boxed{(0, \pi)}$

c) $\frac{y'}{t} + ty = 0$

$y' = -t^2 y \quad y(-1) = 1 \longrightarrow \boxed{+e^{-(1+t)^3/3}}$ $\boxed{(-\infty, +\infty)}$

Problem 7

$$\frac{v^2}{2} = \frac{gR^2}{(R+x)} + C$$

at initial time $x = 0.1R$ and $v = v_0$

$$\frac{v_0^2}{2} = \frac{gR^2}{(R+0.1R)^2} + C \quad \Rightarrow \quad C = \frac{v_0^2}{2} - \frac{g}{1.1}R$$

$$v(x) = \pm\sqrt{\frac{2gR^2}{R+x} + v_0^2 - \frac{2g}{1.1}R}$$

$$v(x) = 0 \quad \Rightarrow \quad x = x_{max}$$

$$0 = \frac{2gR^2}{R+x_{MAX}} + v_0^2 - \frac{2g}{1.1}R$$

Escape: $x_{MAX} \to \infty$

$$v_0 = \sqrt{\frac{2gR}{1.1}} \approx 6.6 \text{ miles/sec}$$

$$\boxed{\begin{array}{l}\text{Escape velocity for a rocket started at}\\ x_0 = 0.1R \text{ is approx } 6.6 \text{ miles/sec.}\\ \qquad\qquad \text{or } 10.6 \text{ km/sec.}\end{array}}$$

If rocket starts at $x_0 = \varepsilon R$ we obtain

$$v_0 = \sqrt{\frac{2gR}{(1+\varepsilon)}}$$

[Just replase 0.1 to ε in previous calculations]

To start from the surface $v_0 = \sqrt{2gR}$

Solve $\sqrt{\frac{2gR}{(1+\varepsilon)}} = 0.8\sqrt{2gR} \Rightarrow \boxed{\varepsilon = 0.56}$

To reduce escape velocity to 80% of $\sqrt{2gR}$ we need to start at $\boxed{x_0 = 0.56R}$