

$$4) a) q = x_1^2 + 4x_1x_2 + x_2^2 \quad Q = X^T A X \quad X = (x_1, x_2)^T$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$\begin{aligned} \det &= (1-\lambda)(1-\lambda) - 4 \\ &= \lambda^2 - 2\lambda - 3 \quad \lambda_1 = -1 \\ &= (\lambda+1)(\lambda-3) \quad \lambda_2 = 3 \end{aligned}$$

Eigenvalues

$$\lambda_1 = -1$$

$$A - \lambda_1 I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} X = 0 \Rightarrow \begin{bmatrix} x_1 & x_2 \\ 1 & 1 | 0 \\ 0 & 0 | 0 \end{bmatrix} \quad \begin{array}{l} x_1 = -t \\ x_2 = t \end{array} \quad t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{Let } t = 1 \quad \vec{e}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3$$

$$A - \lambda_2 I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 & x_2 \\ 1 & -1 | 0 \\ 0 & 0 | 0 \end{bmatrix} \quad \begin{array}{l} x_1 = s \\ x_2 = s \end{array} \quad s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \det s = 1 \quad \vec{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|\vec{e}_1| = 1 \quad |\vec{e}_2| = \sqrt{2}$$

Matrix P ($P^{-1} = P^T$)

$$\sqrt{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X = PY$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}}(-y_1 + y_2) \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}}(y_1 + y_2)$$

$$q = x_1^2 + 4x_1x_2 + x_2^2$$

$$= \left(\frac{1}{\sqrt{2}}(-y_1 + y_2)\right)^2 + 4 \left[\left(\frac{1}{\sqrt{2}}(-y_1 + y_2)\right) \cdot \left(\frac{1}{\sqrt{2}}(y_1 + y_2)\right)\right]$$

$$+ \left(\frac{1}{\sqrt{2}}(y_1 + y_2)\right)^2$$

$$q = -y_1^2 + 3y_2^2$$

$$4.) b) \quad g = x_1^2 - 2x_1x_2 + 2x_2x_3 + x_3^2$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda) \cdot \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} + (-1)(-1) \det \begin{bmatrix} -1 & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda) [\lambda^2 - \lambda - 1] - 1(1-\lambda)$$

$$= (1-\lambda)(\lambda+1)(\lambda-2)$$

$$\underline{\lambda_1=1} \quad \underline{\lambda_2=-1} \quad \underline{\lambda_3=2}$$

$$\underline{\lambda_1=1}$$

$$A - \lambda_1 I = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array} \quad \begin{array}{l} x_1 = t \\ x_2 = 0 \\ x_3 = t \end{array} \quad \begin{array}{l} x_1 = t \\ x_2 = 0 \\ x_3 = 1 \end{array} \quad t=1$$

$$\underline{\lambda_2=-1}$$

$$A - \lambda_2 I = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = -s \\ x_2 = 2s \\ x_3 = s \end{array} \quad \begin{array}{l} x_1 = -s \\ x_2 = s \\ x_3 = 1 \end{array} \quad s=-1$$

$$\underline{\lambda_3=2}$$

$$A - \lambda_3 I = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = -2 \\ x_2 = q \\ x_3 = q \end{array} \quad \begin{array}{l} x_1 = -1 \\ x_2 = 1 \\ x_3 = 1 \end{array} \quad q=1$$

Normalize eigenvectors

$$\|x_1\| = \sqrt{2}$$

$$\hat{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\|x_2\| = \sqrt{6}$$

$$\hat{e}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\|x_3\| = \sqrt{3}$$

$$\hat{e}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(6)

$$P = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{3}}y_3$$

$$x_2 = \frac{2}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3$$

$$x_3 = \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3$$

$$Q = x_1^2 - 2x_1x_2 + 2x_2x_3 + x_3^2$$

$$= \left[\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{3}}y_3 \right]^2 - 2 \left[\left(\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{3}}y_3 \right) \left(\frac{2}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \right) \right]$$

$$+ 2 \left[\left(\frac{2}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \right) \left(\frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \right) \right] + \left[\frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \right]^2$$

$$= y_1^2 - y_2^2 + 2y_3^2$$

(10)

$$5) a) 3x^2 + 4xy + y^2 = 1$$

$$b^2 - 4ac = 4^2 - 4(3)(1) = 16 - 12 > 0 \text{ so curve is } \boxed{\text{hyperbola}}$$

Angle b/w axes of symmetry $\tan 2\theta = \frac{b}{a-c}$

$$\theta = 0.5 \arctan \frac{4}{2} \approx 32^\circ (-148^\circ)$$

2nd axis

$$\theta = 90 + 0.5 \arctan 2 \approx 122^\circ (-58^\circ)$$

Asymptotes ($y=mx$)

$$3x^2 + 4xy + y^2 = 0$$

$$3 + 4m + m^2 = 0$$

$$(m+1)(m+3)=0$$

$$m_1 = -1 \quad m_2 = -3$$

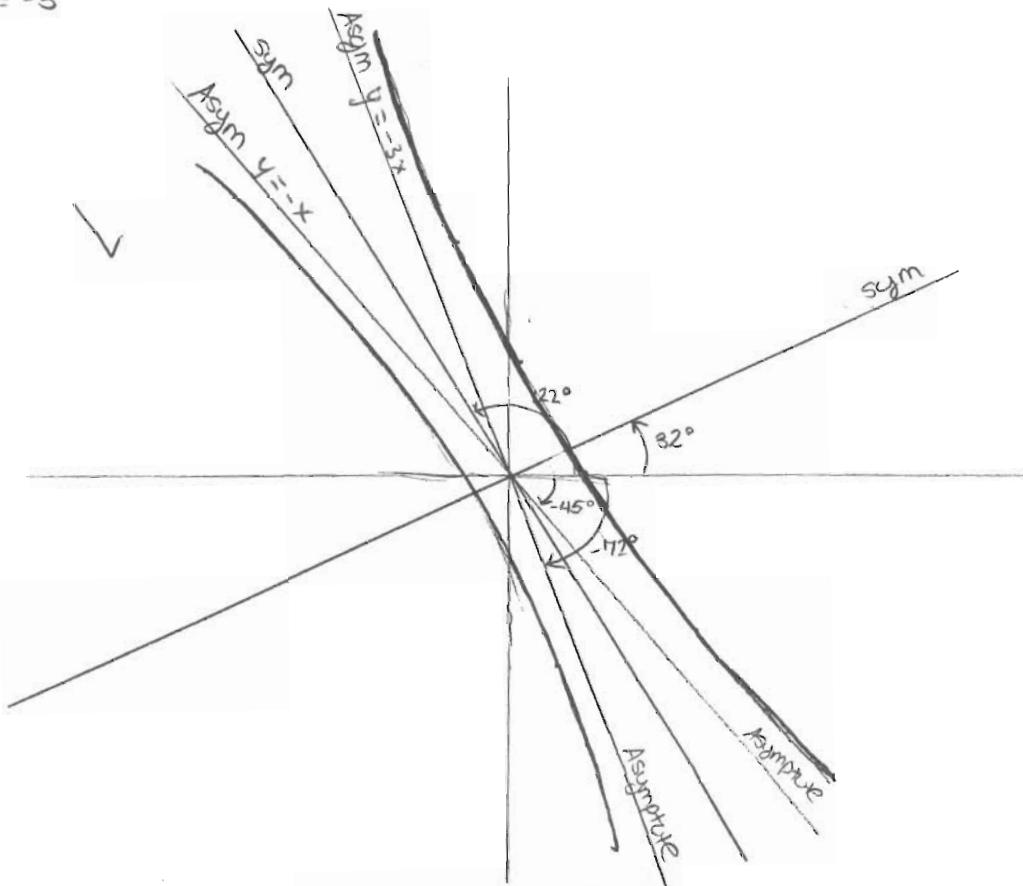
$$m_1 = \arctan(-1) \\ = -45^\circ$$

$$m_2 = \arctan(-3) \\ \approx -72^\circ$$

Pts

$y=0$	$x=0$
$x = \pm \frac{1}{\sqrt{3}}$	$y = \pm 1$

$$(0,1) \quad (0.58,0) \\ (0,-1) \quad (-0.58,0)$$



$$5) b) 3x^2 + 4xy + 2y^2 = 1$$

$$b^2 - 4ac = (4)^2 - 4(3)(2) = 16 - 24 < 0 \quad \text{Curve is an ellipse.}$$

Axes of symmetry

$$\tan 2\theta = \frac{b}{a-c}$$

$$\theta = 0.5 \arctan \frac{4}{3-2} \approx 38^\circ$$

2nd axis

$$\theta = 90^\circ + 0.5 \arctan 4 \approx 128^\circ$$

$$\alpha = \frac{1}{\sqrt{\lambda_1}} \approx 0.468 \Rightarrow \vec{F}_1$$

$$\beta = \frac{1}{\sqrt{\lambda_2}} \approx 1.51 \Rightarrow \vec{F}_2$$

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \quad \det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}$$

$$= (3-\lambda)(2-\lambda) - 4$$

$$= \lambda^2 - 5\lambda + 2$$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25-4(2)}}{2}$$

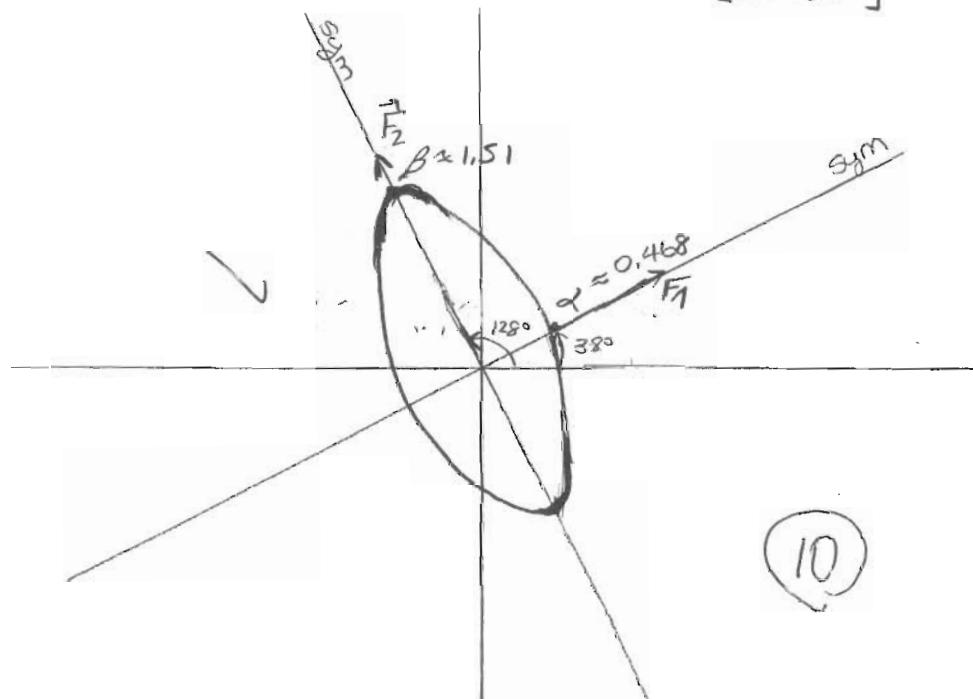
$$= \frac{5 \pm \sqrt{17}}{2}$$

$$\lambda_1 \approx 4.56$$

$$\lambda_2 \approx 0.438$$

$$\lambda_1 \Rightarrow \vec{e}_1 = \begin{bmatrix} 0.788 \\ 0.615 \end{bmatrix} \Rightarrow \vec{F}_1$$

$$\lambda_2 \Rightarrow \vec{e}_2 = \begin{bmatrix} -0.615 \\ 0.788 \end{bmatrix} \Rightarrow \vec{F}_2$$



7.) $x^T A x = 1$ A is 2×2 matrix $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ symmetric.

The unit ball in \mathbb{R}^2 contains all vectors x ($\|x\|_A = 1$) such that $x^T A x = 1$.
Taking the normalized eigenvectors of matrix A , one can compose an orthogonal matrix P ($P^{-1} = P^T$) of coordinate transformation ($x = Py$).

By the Principal Axes Theorem, A being symmetric means

$P^T A P = D$ (or $A = PDP^T$). It can then be proven that

$x^T A x = y^T D y = 1$ where D is a diagonal matrix with eigenvalues (from matrix A) on its diagonal (A is symmetric so eigenvalues are positive).
(Symmetric implies real, not necessarily positive)

$y^T D y = 1$ can then be written as $\lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$.

(3) $\lambda_1 = \frac{1}{\alpha^2}$ and $\lambda_2 = \frac{1}{\beta^2} \Rightarrow \frac{y_1^2}{\alpha^2} + \frac{y_2^2}{\beta^2} = 1$ which is the equation of an ellipse.

8.) $x^T A x = 1$ A is 3×3 positive definite $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

The unit ball in \mathbb{R}^3 contains all vectors x ($\|x\|_A = 1$) such that $x^T A x = 1$.

An orthogonal matrix of coordinate transformation (P) can be constructed from the normalized eigenvectors from A .

The Principal Axes Theorem states that since A is symmetric, $P^T A P = D$ and it can be proven that $x^T A x = y^T D y$ where D is a 3×3 matrix with eigenvalues on its main diagonal. Because A is positive definite, the eigenvalues are positive.

$y^T D y = 1$ can be written as $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1$

$\Rightarrow \frac{y_1^2}{\alpha^2} + \frac{y_2^2}{\beta^2} + \frac{y_3^2}{\gamma^2} = 1$ which is the equation of an ellipsoid.

* The eigenvectors point in the direction of the principal axes.