Answers for Quiz 4.

NAME and student number

1. Give a definition of *norm* of a vectors in an inner product space.

Answer: Let the inner product of two vectors be denoted by $< \vec{v}, \vec{u} >$. The norm of vector $\vec{v}$ is a nonnegative number, denoted by $||\vec{v}||$ such that $||\vec{v}||^2 =< \vec{v}, \vec{v} >$.

2. Find a symmetric matrix A s.t. $< \vec{v}, \vec{u} >= \vec{v}^T A \vec{u}$ and explain whether or not this defines an inner product:

$< \vec{v}, \vec{u} >= v_1 u_1 + 2 v_1 u_2 + 2 v_2 u_1 + 5 v_2 u_2$;

Answer:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

In order to explain whether or not this matrix defines an inner product we have to check whether or not it is positive definite. In can be done in two ways.

Way 1: $< \vec{v}, \vec{v} >= \vec{v}^T A \vec{v} = v_1^2 + 4 v_1 v_2 + 5 v_2^2 = (v_1 + 2 v_2)^2 + v_2^2 > 0$; for $\vec{v} \neq (0,0)^T$. Thus matrix A is positive definite and the formula defines inner product.

Way 2: Eigenvalues of A are $(6 \pm \sqrt{32})/2$. The both are positive. Thus matrix A is positive definite and the formula defines inner product.

3. Prove Pythagorean theorem: *Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be an orthogonal set of vectors in an inner product space. Then*

$$||\vec{e}_1 + \vec{e}_2 + \vec{e}_3||^2 = ||\vec{e}_1||^2 + ||\vec{e}_2||^2 + ||\vec{e}_3||^2.$$

Explain in words every step.

Answer:

1. By definition of norm we have

$$||\vec{e}_1 + \vec{e}_2 + \vec{e}_3||^2 =< \vec{e}_1 + \vec{e}_2 + \vec{e}_3, \vec{e}_1 + \vec{e}_2 + \vec{e}_3 >$$

2. By distributive and commutative properties of inner product we obtain

$$< \vec{e}_1 + \vec{e}_2 + \vec{e}_3, \vec{e}_1 + \vec{e}_2 + \vec{e}_3 >=$$

$$< \vec{e}_1, \vec{e}_1 > + < \vec{e}_2, \vec{e}_2 > + < \vec{e}_3, \vec{e}_3 > +2 < \vec{e}_1, \vec{e}_3 > +2 < \vec{e}_2, \vec{e}_3 > +2 < \vec{e}_1, \vec{e}_2 > .$$

3. Because the basis is orthogonal we have

$$< \vec{e}_1, \vec{e}_3 >=< \vec{e}_2, \vec{e}_3 >=< \vec{e}_1, \vec{e}_2 >= 0.$$

4. Also,

$$< \vec{e}_1, \vec{e}_1 >= ||\vec{e}_1||^2, \quad < \vec{e}_2, \vec{e}_2 >= ||\vec{e}_2||^2, \quad < \vec{e}_3, \vec{e}_3 >= ||\vec{e}_3||^2.$$

thus the statements follows.