Math 2051 W2008 Margo Kondratieva

Quiz 3 on Wed Feb 13: Answers

NAME and student number

[2] 1. Give definition of **rank** of a matrix.

Answer: Either of the following can serve as a definition of rank:
-Rank of a matrix is the number of leading 1s in a REF of the matrix.
-Rank of a matrix is the dimension of its column space.
-Rank of a matrix is the dimension of its row space.

[4] 2. a) Find basis in **column space** and **null space** of matrix A.
b) Give an example of vector B for which the system $AX = B$ has a solution, and find **all** the solutions corresponding to B.

$$A = \begin{bmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{bmatrix}.$$

Answer: Use elementary row operations to rewrite the matrix in REF

$$\begin{bmatrix} 1 & -1 & 5 & -2 & 2 \\ 0 & 0 & 1 & -0.75 & 0.25 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The leading 1s are in column 1 and 3, thus the basis in the column space is $\vec{u} = A_1 = (1, 2, 0, -1)^T$, and $\vec{v} = A_3 = (5, -2, -12, 7)^T$.

The solution of corresponding homogeneous $AX = 0$ system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = q \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -0.75 \\ 0 \\ -0.25 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1.75 \\ 0 \\ 0.75 \\ 1 \\ 0 \end{bmatrix}.$$

Let $X = (2, 0, 0, 0, 0)^T$ then $B = AX = 2A_1 = (2, 4, 0, -2)^T$. The parametric solution corresponding to this B is the sum of the X we have chosen plus the null space of A:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -0.75 \\ 0 \\ -0.25 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1.75 \\ 0 \\ 0.75 \\ 1 \\ 0 \end{bmatrix}.$$

[4] 3. **Prove** the following statement and **give an example** of how this statement can be used.
If $\vec{v}_1, \vec{v}_2$ is an orthonormal basis then any $\vec{w} \in span(\vec{v}_1, \vec{v}_2)$ has coordinates $x_1 = \vec{w} \cdot \vec{v}_1$, $x_2 = \vec{w} \cdot \vec{v}_2$ relative to this basis.

Answer:

Way 1: If $\vec{v}_1, \vec{v}_2$ is an orthonormal basis then

$$\vec{v_1} \cdot \vec{v_2} = 0, \quad \vec{v_1} \cdot \vec{v_1} = \vec{v_2} \cdot \vec{v_2} = 1.$$

Consider matrix A whose columns are vectors $\vec{v}_1$ and $\vec{v}_2$. Observe that since the basis is orthonormal, we have $A^T A = I$, where I is 2×2 identity matrix.

In order to find coordinates of $\vec{w}$ in basis $\vec{v}_1, \vec{v}_2$, that is to find x_1 and x_2 such that

$$\vec{w} = x_1 \vec{v}_1 + x_2 \vec{v}_2,$$

one needs to solve the system

$$AX = \vec{w}.$$

To solve the system $AX = \vec{w}$ we multiply both sides by A^T from the left and use $A^T A = I$. Then we have $X = A^T \vec{w}$. Recall that rows of matrix A^T are vectors $\vec{v}_1, \vec{v}_2$, thus $x_1 = \vec{w} \cdot \vec{v}_1$, $x_2 = \vec{w} \cdot \vec{v}_2$.

Way 2: Draw a picture of two orthogonal vectors and an arbitrary vector $\vec{w}$ in the plane. Observe that

$$\vec{w} = proj_{v_1}(\vec{w}) + proj_{v_2}(\vec{w}).$$

Recall that orthogonal projection $proj_g(\vec{f})$ of a vector $\vec{f}$ onto vector $\vec{g}$ is given by formula

$$proj_g(\vec{f}) = \frac{\vec{f} \cdot \vec{g}}{\vec{g} \cdot \vec{g}} \vec{g}.$$

Since the basis is orthonormal, $\vec{v_1} \cdot \vec{v_1} = \vec{v}_2 \cdot \vec{v}_2 = 1$ and we have $proj_{v_1}(\vec{w}) = (\vec{w} \cdot \vec{v}_1)\vec{v}_1$ and $proj_{v_2}(\vec{w}) = (\vec{w} \cdot \vec{v}_2)\vec{v}_2$. This gives

$$\vec{w} = (\vec{w} \cdot \vec{v}_1)\vec{v}_1 + (\vec{w} \cdot \vec{v}_2)\vec{v}_2,$$

which means that the coordinates of $\vec{w}$ relative to basis $\vec{v}_1, \vec{v}_2$ are $x_1 = \vec{w} \cdot \vec{v}_1$, $x_2 = \vec{w} \cdot \vec{v}_2$.