MATH 2051 MIDTERM EXAM on February 22,2008 Answers

Instructor: Margo Kondratieva.
Student:

1. (9 points)

   (a) Complete the definition:
   A linear (vector) space is a collection of vectors with the following properties:
   (1) it contains the zero vector $\vec{0}$ — such that for any vector $\vec{v}$ from the collection $\vec{0}+\vec{v} = \vec{v}+\vec{0} = \vec{v}$;
   (2) the sum of any two vectors from the collection is again in the collection;
   (3) a multiple of any vector from the collection is again in the collection.

   (b) Use you definition in part (a) to prove that:
   The collection of all vectors $\vec{v} = (x, y, z)^T$ which are orthogonal to vector

$$\vec{n} = (22, 2, 2008)^T$$

   form a linear (vector) space.
   Solution: Note that the collection include all $\vec{v} = (x, y, z)^T$ such that

$$22x + 2y + 2008z = 0.$$

   Now we will check all three properties.

   (1) $\vec{0} = (0, 0, 0)^t$ belongs to the collection because substitution of $x = y = z = 0$ into the equation above gives a true statement 0=0.
   (2) Let $\vec{v}_1 = (x_1, y_1, z_1)^T$ such that $22x_1 + 2y_1 + 2008z_1 = 0$ and $\vec{v}_2 = (x_2, y_2, z_2)^T$ such that $22x_2 + 2y_2 + 2008z_2 = 0$, that is both vectors are from the collection. By adding these two equations we obtain $22(x_1 + x_2) + 8(y_1 + y_2) + 2008(z_1 + z_2) = 0$.
   Thus the sum of the vectors

$$\vec{v}_1 + \vec{v}_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)^T$$

   also belongs to the collection.
   (3) Let $\vec{v}_1 = (x_1, y_1, z_1)^T$ such that $22x_1 + 2y_1 + 2008z_1 = 0$. Multiplying this equation by any number $k \neq 0$ we get $22kx_1 + 2ky_1 + 2008kz_1 = 0$, thus vector $k\vec{v}_1 = (kx_1, ky_1, kz_1)^T$ also belongs to the collection.
   All three properties are verified, thus this collection of vectors is a vector space.

2. (9 point)

(a) Give a definition of n **linearly independent** vectors.

Answer: None of the n vectors can be expressed as a linear combination of the rest $n-1$ vectors.

(b) Let the columns of a rectangular $m \times n$ matrix A be linearly independent vectors, and $m \neq n$. Which of the following are TRUE?

(1) $m > n$

(2) $m < n$

(3) linear independence of columns is possible only for $m = n$.

(4) system $AX = B$ has a *unique* solution for *any* $m \times 1$ matrix B.

(5) system $AX = B$ has a *parametric* solution for *some* $m \times 1$ matrices B.

(6) system $AX = B$ has a unique solution for some $m \times 1$ matrices B. Matrix B must be in the column space of A.

(c) Give an example of a matrix with four linearly independent rows. What is the rank of your matrix?

Answer: Identity 4×4 matrix is an example.
Rank is 4.

3. (8 point)

Let $\vec{f_1} = \vec{e}_1 + 2\vec{e}_2$, $\vec{f_2} = \vec{3}e_1 + 4\vec{e}_2$, and $\vec{v} = 5\vec{e}_1 + 6\vec{e}_2$,

(a) Find the matrix of coordinate transformation for the change from basis $(\vec{e}_1, \vec{e}_2)$ to basis $(\vec{f_1}, \vec{f_2})$;

Answer: $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

(b) Find the matrix of coordinate transformation for the change from basis $(\vec{f_1}, \vec{f_2})$ to basis $(\vec{e}_1, \vec{e}_2)$;

Answer: $\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix}$

(c) Find numbers a, b such that $\vec{v} = a\vec{f_1} + b\vec{f_2}$.

Answer: $a = -1, b = 2$

4. (9 points)

(a) Give a definition of **basis** in a vector space.

Answer: Basis in a space is a collection of linearly independent vectors which span the space.
It has the property that every vector in the space can be represented as a linear combination of basis in a unique way.

(b) Find a basis in the **column space** of matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 9 & 12 & 13 & 14 \end{bmatrix}$

Answer: basis in column space is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 13 \end{bmatrix}$.

(c) Find a basis in the **eigenspace** of matrix $\begin{bmatrix} 2008 & 2008 & 2008 \\ 0 & 2008 & 2008 \\ 0 & 0 & 2008 \end{bmatrix}$

Answer: Eigenvalue $\lambda = 2008$ has multiplicity 3. The only eigenvector is $X = (t, 0, 0)^T$. Thus basis consists of only one vector $(1, 0, 0)^T$.

5. (14 points)

(a) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be an orthonormal basis in a vector space S. Let (x, y, z) denote coordinates of a vector $\vec{w} \in S$ w.r.t this basis. Prove that

$$x = \vec{w} \cdot \vec{v}_1, \quad y = \vec{w} \cdot \vec{v}_2, \quad z = \vec{w} \cdot \vec{v}_3.$$

Answer: there are many ways to prove this statement. One possibility is as follows. Let

$$\vec{w} = x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3.$$

Then

$$\vec{w} \cdot \vec{v}_1 = x\vec{v}_1 \cdot \vec{v}_1 + y\vec{v}_2 \cdot \vec{v}_1 + z\vec{v}_3 \cdot \vec{v}_1.$$

Since the basis is orthonormal we have $\vec{v}_1 \cdot \vec{v}_1 = 1$, $\vec{v}_2 \cdot \vec{v}_1 = 0$, $\vec{v}_3 \cdot \vec{v}_1 = 0$. Thus $\vec{w} \cdot \vec{v}_1 = x$. Similarly we find that $\vec{w} \cdot \vec{v}_2 = y$ and $\vec{w} \cdot \vec{v}_3 = z$.

(b) Show that the following set of vectors is an **orthogonal basis** in space $S = span(\vec{f}_1, \vec{f}_2, \vec{f}_3)$

$$\vec{f}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}, \quad \vec{f}_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

Answer: Check directly that $\vec{f}_1 \cdot \vec{f}_2 = 0$, $\vec{f}_1 \cdot \vec{f}_3 = 0$, $\vec{f}_2 \cdot \vec{f}_3 = 0$.
Also the vectors are linearly independent. Thus they form an orthogonal basis in S.

(c) Obtain **orthonormal basis** from vectors $\vec{f}_1, \vec{f}_2, \vec{f}_3$ given in (b), by normalization. Call the vectors you found $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Answer:

$$\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(d) Find coordinates of $\vec{w} = (4, 9, 4, -7)^T$ w.r.t the orthonormal basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ found in (c).

Answer: Use statement from (a) to find

$$x = \vec{w} \cdot \vec{v}_1 = 3, \quad y = \vec{w} \cdot \vec{v}_2 = 6\sqrt{2}, \quad z = \vec{w} \cdot \vec{v}_3 = 9,$$