Exercise Set 7. Answers.

1. Find the matrix which induces projection on given vector $\vec{d} = (a, b, c)^T$ in $\mathbf{R}^3$. Check you result for the case $\vec{d} = (1, 0, 0)$.

Solution:

$$\vec{w} = proj_d(\vec{v}) = \left(\frac{\vec{v}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\right)\vec{d} = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}\vec{v} = M\vec{v}.$$

Thus the matrix is

$$M = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}.$$

This becomes for $a = 1$, $b = 0$, $c = 0$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\vec{v} = (x, y, z)^T$ then $M_1\vec{v} = (x, 0, 0)^T$, which is the projection onto $(1, 0, 0)^T$ indeed.

2. Find the matrix which induces reflection w.r.t given vector $\vec{d} = (a, b, c)^T$ in $\mathbf{R}^3$. Check you result for the case $\vec{d} = (1, 0, 0)$.

Solution: If $\vec{w}$ is reflection of $\vec{v}$ w.r.t. $\vec{d}$ then all three vectors lie in the same plane and we must have the relation: $\vec{v} + \vec{w} = 2proj_d(\vec{v})$. Then we have

$$\vec{w} = ref_d(\vec{v}) = 2proj_d\vec{v} - \vec{v} = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{bmatrix}\vec{v} = L\vec{v}.$$

In other words,

$$L = 2M - I == \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{bmatrix}$$

This becomes for $a = 1$, $b = 0$, $c = 0$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Let $\vec{v} = (x, y, z)^T$ then $L_1\vec{v} = (x, -y, -z)^T$, which is the reflection w.r.t $(1, 0, 0)^T$ indeed.

3. Find kernel and image of the following linear transformations in $\mathbf{R}^2$. Are they invertible?
- rotation by angle $\theta = \pi/3$;

Solution: Rotation matrix for $\theta = \pi/3$ has the form $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$. Determinant is equal to 1, thus the matrix is invertible. Thus the transformation is invertible (rotation in opposite direction by $\pi/3$).

Kernel is $\vec{0}$, Image is $\mathbf{R}^2$.

- reflection w.r.t $\vec{d} = (1, 2)$;

Solution:

Reflection matrix has the form $\frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$. This matrix is invertible. Thus the transformation is invertible (reflection w.r.t. the same vector).

Kernel is $\vec{0}$, Image is $\mathbf{R}^2$.

- stretching a vector by factor 2;

Solution:

Stretching matrix has the form $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and in invertible. Thus the transformation is invertible (shrinking by factor 2).

Kernel is $\vec{0}$, Image is $\mathbf{R}^2$.

- projection on vector $\vec{d} = (0, 5)$;

Solution:

Projection matrix has the form $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. It is not invertible. Thus the transformation is not invertible.

Kernel is all vector $(x, 0)^T$. Image is all vector $(0, y)^T$.

4. A student had chosen to work in a non-standard basis $\vec{f_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{f_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in $\mathbf{R}^2$. How do the matrices of rotation, projection and reflection look in this basis?

Solution: Let $\vec{v} = (x_1, x_2)^T = x_1 E_1 + x_2 E_2$, where E_1, E_2 is the standard basis. The same vector can be written in the Student's basis as $\vec{v} = (x_1, x_2)^T = x_2 \vec{f_1} + x_1 \vec{f_2}$. Thus the matrix $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Consequently, if $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ represents a vector transformation in the standard basis then $B = C^{-1}AC = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix}$ represents the same transformation in the Student's basis.

Thus we have in the Student's basis:

Rotation $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$;

Projection w.r.t $\vec{d} = aE_1 + bE_2$ is $\frac{1}{a^2+b^2}\begin{bmatrix} b^2 & ab \\ ab & a^2 \end{bmatrix}$;

Reflection w.r.t $\vec{d} = aE_1 + bE_2$ is $\frac{1}{a^2+b^2}\begin{bmatrix} -a^2+b^2 & 2ab \\ 2ab & -b^2+a^2 \end{bmatrix}$.

5. Find the area of parallelogram obtained from a unit square by a linear transformation induced by a degenerate matrix.

Answer: Area $=0$.