Last Week. Linear transformations.

Text reference: this material corresponds to parts of sections 2.5, 4.4, 7.2.

Section 5.1 Linear transformations and matrices.

Given $n \times n$-matrix A consider transformation $T_A : \mathbf{R}^n \to \mathbf{R}^n$ induced by this matrix by the following formula: $\vec{w} = T_A(\vec{v}) = A\vec{v}$, $\vec{v} \in \mathbf{R}^n$. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then for any $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ the transformation $T_A(\vec{v}) = \begin{bmatrix} x \\ 0 \end{bmatrix}$.

A transformation induced by any matrix A is **linear**. It means that two properties are obeyed:

$$T_A(\vec{v} + \vec{u}) = T_A(\vec{v}) + T_A(\vec{u}), \qquad T_A(k\vec{v}) = kT_A(\vec{v}),$$

for any vectors $\vec{u}, \vec{v} \in \mathbf{R}^n$ and any number k.

Theorem 1. Any linear transformation T is induced by a unique matrix A. This matrix has columns $T(E_1), T(E_2), ...T(E_n)$, where $E_1, E_2, ..., E_n$ is the standard basis in $\mathbf{R}^n$.

Example: Consider counterclockwise **rotation** around the origin in $\mathbf{R}^2$ by angle θ. Then $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$. Thus the matrix of this transformation is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Other examples of linear transformations in $\mathbf{R}^2$ are **projection** on given vector $\vec{d} = \begin{bmatrix} a \\ b \end{bmatrix}$ and **reflection** w.r.t. given vector $\vec{d} = \begin{bmatrix} a \\ b \end{bmatrix}$. Corresponding matrices can be seen from the following relations:

$$\vec{w} = proj_d(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \right) \vec{d} = \frac{1}{a^2 + b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \vec{v}.$$

$$\vec{w} = ref_d(\vec{v}) = 2proj_d\vec{v} - \vec{v} = \frac{1}{a^2 + b^2} \begin{bmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{bmatrix} \vec{v}.$$

Consider a parallelogram with sides formed by vectors $\vec{u}, \vec{v}$. What is the area of a parallelogram obtained by a linear transformation T_A compare to the area of the initial one? The answer is given by the following theorem.

Theorem 2.

$$area(A\vec{u}, A\vec{v}) = |\det A| \cdot area(\vec{u}, \vec{v}).$$

Proof. We again restrict our consideration to $\mathbf{R}^2$. Let M be matrix with columns $\vec{u}, \vec{v}$. Then we know from M2050 that $area(\vec{u}, \vec{v}) = |\det M|$. The parallelogram obtained after the linear transformation has sides $A\vec{u}, A\vec{v}$. Thus its area is equal to $|\det(AM)| = |\det A| \cdot |\det M|$. $\square$

Section 5.2 Kernel and image of a linear transformation.

Definition 1. *Kernel* of a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is a collection of all vectors $\vec{v} \in \mathbf{R}^n$ such that $T\vec{v} = \vec{0}$.

Note that if matrix A induces T then kernel of T is the null space of A (all X s.t. $AX = 0$).

Definition 2. *Image* of a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is a collection of all vectors $\vec{v} \in \mathbf{R}^n$ that can be obtained as the result of this transformation.

Note that if matrix A induces T then image of T is the column space of A that is all B such that the system $AX = B$ has a solution X.

Example: Let T be **projection** on given vector $\vec{d} = \begin{bmatrix} a \\ b \end{bmatrix}$. Then kernel of T is a collection of all vectors $\vec{v}$ orthogonal to $\vec{d}$, that is $\vec{v} = t \begin{bmatrix} -b \\ a \end{bmatrix}$, where t is any number. The image of T is a collection of all vectors $\vec{v}$ proportional to $\vec{d}$, that is $\vec{v} = t \begin{bmatrix} a \\ b \end{bmatrix}$, where t is any number.

Remark. A transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is a **bijection** (both one-to-one and onto) if and only if kernel of T is zero vector and image of T in $\mathbf{R}^n$. Only bijective transformations are invertible.

Section 5.3 Change of basis and matrix of linear transformation.

Consider a basis $\vec{e}_1, \vec{e}_2$ and another basis $\vec{f}_1, \vec{f}_2$ in $\mathbf{R}^2$. An arbitrary vector $\vec{v}$ has coordinates w.r.t. each of this basis $\vec{v} = x_1\vec{e}_1 + x_2\vec{e}_2 = y_1\vec{f}_1 + y_2\vec{f}_2$. Consequently we will write $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for $\vec{v}$ w.r.t the first basis and $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ for the same vector w.r.t the second basis.

Let C be a matrix of coordinate transformation from $\vec{e}_1, \vec{e}_2$ to $\vec{f}_1, \vec{f}_2$, that is $X = CY$.

Now, let T be a linear transformation in $\mathbf{R}^2$. This linear transformation transforms vector $\vec{v}$ in $\vec{w} = T(\vec{v})$. This $\vec{w}$ has coordinates $\tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$ w.r.t the first basis and $\tilde{Y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}$ w.r.t the second:

$$\vec{w} = \tilde{x}_1\vec{e}_1 + \tilde{x}_2\vec{e}_2 = \tilde{y}_1\vec{f}_1 + \tilde{y}_2\vec{f}_2.$$

The matrix A such that $\tilde{X} = AX$ is the matrix which induces the linear transformation T w.r.t. the first basis. The matrix B such that $\tilde{Y} = BY$ is the matrix which induces the linear transformation T w.r.t. the second basis.

Question: What is the relation between matrix A and matrix B which represent the same linear transformation T w.r.t. two different bases?

The answer is $B = C^{-1}AC$, which can be found by resolving matrix equations given above. That relation clarifies the meaning of the notion of similarity $A \sim B$ introduced previously: similar matrices represent the same linear transformation in different bases.

Exercise Set 7.

1. Find the matrix which induces projection on given vector $\vec{d} = (a, b, c)^T$ in $\mathbf{R}^3$. Check you result for the case $\vec{d} = (1, 0, 0)$.

2. Find the matrix which induces reflection w.r.t given vector $\vec{d} = (a, b, c)^T$ in $\mathbf{R}^3$. Check you result for the case $\vec{d} = (1, 0, 0)$.

3. Find kernel and image of the following linear transformations in $\mathbf{R}^2$. Are they invertible?

- rotation by angle $\theta = \pi/3$;
- reflection w.r.t $\vec{d} = (1, 2)$;
- stretching a vector by factor 2;
- projection on vector $\vec{d} = (0, 5)$;

4. A student had chosen to work in a non-standard basis $\vec{f}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{f}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in $\mathbf{R}^2$. How do the matrices of rotation, projection and reflection look in this basis?

5. Find the area of parallelogram obtained from a unit square by a linear transformation induced by a degenerate matrix.