Solutions

Math 2050 **Test 2** **March 1, 2010**

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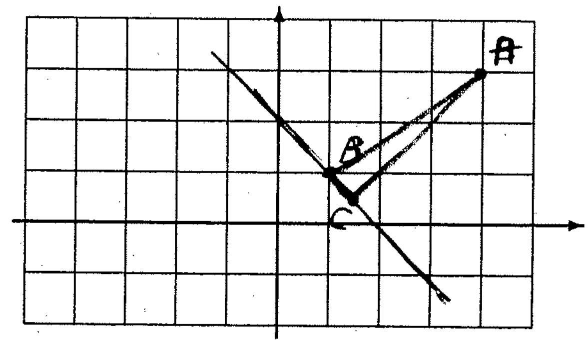
1. Consider a point with coordinates $A(4,3)$ and a line l given by equation

$x = 1 + t$
$y = 1 - t, \qquad t \in R$

[2] (a) Find the direction vector of the line l and coordinates of a point B which lies on the line.



$\vec{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad B(1,1)$

Answer :

[3] (b)Sketch points A, B and the line l.

[10] (c) Find the shortest distance between point A and the line l.

$\vec{v} = \overrightarrow{BA} = \begin{pmatrix} 4-1 \\ 3-1 \end{pmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\text{proj}_{\vec{d}} \vec{v} = \left(\frac{\vec{v}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\right)\vec{d} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \overrightarrow{BC}$; $\overrightarrow{AC} = \overrightarrow{BC} - \overrightarrow{BA} = \begin{bmatrix} -5/2 \\ -5/2 \end{bmatrix}$

distance $= \|\overrightarrow{AC}\| = \sqrt{\frac{25}{4} + \frac{25}{4}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$;

Alternaterly,
$\|AB\|^2 = 13$, $\|BC\|^2 = \frac{1}{2}$
$\|AC\| = \sqrt{13 - \frac{1}{2}} = \sqrt{\frac{25}{2}} = \frac{5\sqrt{2}}{2}$

Answer : $5\sqrt{2}/2$

2. Consider plane $x + 2y + 3z = 0$ and point $B(4,5,0)$.

[2] (a) Find a vector normal to the plane and coordinates of a point which lies in the plane.

$\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad P(0,0,0)$

Answer :

[8] (b) Find the point C of the plane closest to B and the distance from point B to point C.

Method 1 $\quad C$ is the intersection of the line $\begin{matrix} x = 4 + t \\ y = 5 + 2t \\ z = 3t \end{matrix}$ and the plane $x + 2y + 3z = 0$.

Thus $(4+t) + 2(5+2t) + 3(3t) = 0 \Rightarrow 14 + 14t = 0 \Rightarrow t = -1$

and $C(4-1, 5-2, -3) = (3,3,-3)$

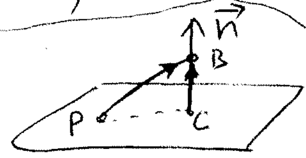
Then the distance $BC = \sqrt{(4-3)^2 + (5-3)^2 + (-3)^2} = \sqrt{14}$.

Method 2

$\overrightarrow{PB} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$; $\text{proj}_{\vec{n}} \overrightarrow{PB} = \left[\frac{\begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}\right]\cdot\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{14}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Thus the distance is the norm of this projection, $\sqrt{14}$.

Now, $\overrightarrow{CB} = \begin{pmatrix} 4 - x_c \\ 5 - y_c \\ 0 - z_c \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \begin{matrix} x_c = 3 \\ y_c = 3 \\ z_c = -3 \end{matrix} \Rightarrow C(3,3,-3)$



$CB = \text{proj}_{\vec{n}}(\overrightarrow{PB})$

3. Consider vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \vec{u} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}.$$

[5] (a) Find the projection $\vec{p}$ of vector $\vec{u}$ onto vector $\vec{v}$.

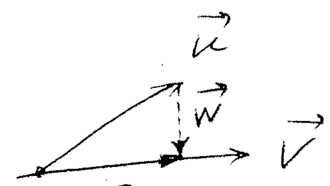
$\vec{u}\cdot\vec{v} = -1-4+12 = 7$ $\qquad$ $\vec{v}\cdot\vec{v} = 1+4+9 = 14$

$$\vec{p} = \frac{7}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 3/2 \end{pmatrix}$$

Answer : ____________

[5] (b) Find a vector $\vec{w}$ which is orthogonal to vector $\vec{v}$ and is a linear combination of vector $\vec{u}$ and vector $\vec{p}$ found in (a).

$$\vec{w} = \vec{p} - \vec{u} = \begin{pmatrix} 1/2+1 \\ 1+2 \\ 3/2-4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3 \\ -5/2 \end{pmatrix}$$



Answer : ____________ or any multiple of it.

Note that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is also a possibility.

4. Consider the plane $x - y + z = 1$ and the line

$x = 1 + t$
$y = 1 + 2t$
$z = t, \qquad t \in R.$

[3] (a) Explain why the line and the plane do not intersect.

Solve for point of intersection

$(1+t) - (1+2t) + t = 1$

$0 = 1$ no solutions $\Rightarrow$ no intersection

[7] (b) Find the shortest distance between the plane and the line

Pick any point on line $A(1,1,0)$

find the distance to the plane $x-y+z=1$

by either method given in problem 2.

For example, Method 2 gives:

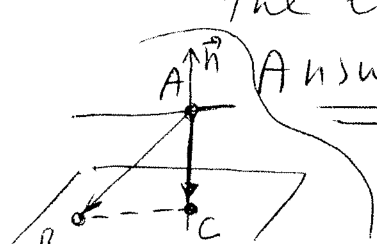
Let $B(1,1,1)$ be point from the plane

(check it out: $1-1+1 = 1$ o.k.).

Then $\vec{AB} = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; Normal to plane $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

projection $\vec{AC} = \mathrm{proj}_{\vec{n}} \vec{AB} = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

the length of the projection is $\frac{\sqrt{3}}{3}$



Answer: the distance is $\sqrt{3}/3$.

5. [5]

Express $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a linear combination of the columns of $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} 1 = a - b \\ 2 = 2b \end{matrix} \Rightarrow \begin{matrix} a = 2 \\ b = 1 \end{matrix}$

Answer : $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ +2 \end{bmatrix}$

6. [5] Is the following set of vectors linearly dependent or independent? Explain.

$$\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix},$$

Let $a\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} + b\begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix} + c\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 10a + 10b + 10c = \\ 20a + 20b + 0 = 0 \\ 30a = 0 \end{cases}$

Solution: $a = 0$, $b = 0$, $c = 0$

Since the only solution is $a=b=c=0$, the set of vectors is linearly independent

Answer :

7. [8] True or False?

False (a) For any pair of 3X3-matrices A and B, $AB \neq BA$. Diagonal matrices commute.

True (b) For all 3X3-matrices A and B, $(A+B)^T = B^T + A^T = A^T + B^T$.

False (c) Let A and B be two $N \times N$-matrices, $N \geq 2$, such that AB=0. Then either $A = 0$ or $B = 0$. Here 0 represents a zero $N \times N$-matrix. $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

True (d) For any four 3X3-matrices, $(ABC)D = A(BCD)$. Denote $M = BC$; $(AM)D = A(MD)$, associativity.

8. [7] Consider matrices

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find AB, A^T, $A+B$, and $2AB$.

$AB = \begin{pmatrix} 1 & -1 \\ 8 & 7 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$

$A+B = \begin{pmatrix} 3 & 0 \\ 4 & 4 \end{pmatrix}$

$2AB = \begin{pmatrix} 2 & -2 \\ 16 & 14 \end{pmatrix}$