Answer Key

Math 2050 **Test 1** **February 1, 2010**

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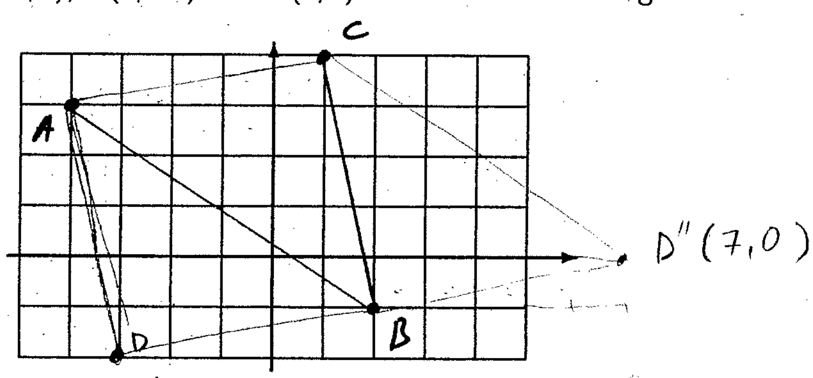
Student • D' **Student number**

1. Sketch points with coordinates $A(-4,3)$, $B(2,-1)$ and $C(1,4)$ and find the following:



[5] (a) the shortest distance between points A and B

$$\sqrt{36+16} = \sqrt{52} = 2\sqrt{13} \approx 7.2$$

$\boxed{5}$ for any of these but "Do not approximate your result" If the answer is given as 7.2. your may lose marks in the future.

Answer :

[5] (b) vectors $\vec{BA}$ and $\vec{BC}$ and their norms

$$\vec{BA} = \begin{bmatrix} -4-2 \\ 3+1 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix} \qquad \|\vec{BA}\| = \sqrt{(-6)^2+4^2} = \sqrt{52}$$

$$\vec{BC} = \begin{bmatrix} 1-2 \\ 4+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \qquad \|\vec{BC}\| = \sqrt{(-1)^2+5^2} = \sqrt{26}$$

Answer : $\|\vec{BA}\| = 2\sqrt{13}$, $\|\vec{BC}\| = \sqrt{26}$

[5] (c) the dot product of vectors $\vec{BA}$ and $\vec{BC}$

$$\vec{BA}\cdot\vec{BC} = (-6)\cdot(-1) + 4\cdot 5 = 26$$

Answer : 26

[5] (d) the smallest angle between vectors $\vec{BA}$ and $\vec{BC}$

$$\cos\theta = \frac{\vec{BA}\cdot\vec{BC}}{\|\vec{BA}\|\cdot\|\vec{BC}\|} = \frac{26}{\sqrt{52}\sqrt{26}} = \frac{\sqrt{26}}{\sqrt{52}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Answer : 45° or $\pi/4$

$$\theta = \arccos\frac{\sqrt{2}}{2} = 45^\circ$$

[5] (e) a point D such that ABCD is a parallelogram

Students may use the Figure to find D <u>or</u> use vectors e.g.

$\vec{DA} = \vec{BC}$

$\begin{bmatrix} x+4 \\ y-3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \Rightarrow \begin{matrix} x=-5 \\ y=8 \end{matrix}$

Answer: $D(-3,-2)$, $D(-5,8)$, $D(7,0)$ — Any of these is correct

[5] (f) the area of the triangle ABC

Method 1: Area $= \frac{1}{2}\|\vec{AB}\|\cdot\|\vec{BC}\|\cdot\sin\theta = \frac{1}{2}\sqrt{52}\cdot\sqrt{26}\sin 45^\circ = \sqrt{13}\cdot\sqrt{13} = 13.$

Method 2: Area $= \frac{1}{2}\|(\vec{BA}\times\vec{BC})\| = \frac{1}{2}\left\|\begin{bmatrix} -6 \\ 4 \\ 0 \end{bmatrix}\times\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}\right\| = \frac{1}{2}\left\|\begin{bmatrix} 0 \\ 0 \\ -26 \end{bmatrix}\right\| = 13$

Answer: 13

2. [5] Prove that $|a\cos\alpha + b\sin\alpha| \le \sqrt{a^2+b^2}$ for any numbers a, b, α.

Hint: use Cauchy-Schwartz inequality for appropriate vectors.

"See problem 36 from the Assignment set"

$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\,\|\vec{v}\|$

Take $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ then $\|\vec{u}\| = \sqrt{a^2+b^2}$

$\vec{v} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ $\quad \|\vec{v}\| = 1.$

3. [10] Find equation of the line which is the intersection of the plane $2x + z = 6$ and another plane $x - 3y = 6$

Method 1

Find two points which belong to both planes: $\begin{cases} 2x+z=6 \\ x-3y=6 \end{cases}$

point 1: $A = \boxed{\begin{matrix} x=0 \\ y=-2 \\ z=6 \end{matrix}}$ $\quad$ point 2: $B = \boxed{\begin{matrix} x=6 \\ y=0 \\ z=-6 \end{matrix}}$

$\vec{AB} = \begin{bmatrix} 6 \\ 2 \\ -12 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$ ← direction of the line

Answer: $\boxed{\begin{matrix} x = 3t \\ y = -2+t \\ z = 6-6t \end{matrix}}$

Thus the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix}$

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Method 2 $\quad \vec{d} = \vec{n_1}\times\vec{n_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\times\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$ ← direction of the line

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see problem 53 for reference".

Find point $A\cdot(0,-2,6)$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix}.$

N.B. The answer may vary depending on which point is choosen. Please check by substitution e.g.

$\begin{cases} 2x+z = 6t+(6-6t) = 6 \; ✓ \\ x-3y = 3t-3(-2+t) = 6 \; ✓ \end{cases}$

4. Consider a plane given by equation $5x - 3y + z = 20$

[5] (a) Find equation of the plane parallel to the plane given above and passing through the point $(1, -1, 1)$.

$$5x - 3y + z = d,$$

$$d = 5 + 3 + 1 = 9$$

Any multiple of this eqn is ok too e.g. $10x - 6y + 2z = 18$.

Answer : $5x - 3y + z = 9$

[5] (b) Find equation of the line which is orthogonal to the plane given above and passing through the point $(1, -1, 1)$.

Answer :
$$\begin{aligned} x &= 1 + 5t \\ y &= -1 - 3t \\ z &= 1 + t \end{aligned}$$

[5] (c) Find the point of intersection of the plane given above with the line

$x = 1 + t$
$y = 1 - t$
$z = t, \qquad t \in R$

$$5(1+t) - 3(1-t) + t = 20$$

$$9t = 18 \Rightarrow t = 2 \Rightarrow x = 3 \quad y = -1 \quad z = 2$$

Answer : $(3, -1, 2)$

5. [10] Find numbers a and b such that the two lines

$$\begin{aligned} x &= 1 + t, \\ y &= 1 - at, \\ z &= b + t, \quad t \in R \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= 1 + s, \\ y &= 1 - s, \\ z &= 2 + 3s, \quad s \in R \end{aligned}$$

intersect at the right angle. What are coordinates of the point of intersection?

- Two lines are orthogonal : $\begin{bmatrix} 1 \\ -a \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = 0$

$$\Rightarrow 1 + a + 3 = 0$$

$$\Rightarrow \boxed{a = -4.}$$

- Two lines intersect :

$$\begin{cases} 1 + t = 1 + s & \Rightarrow t = s \\ 1 + 4t = 1 - s & \Rightarrow 1 + 4t = 1 - t \Rightarrow t = 0 = s \\ b + t = 2 + 3s & \Rightarrow \boxed{b = 2} \end{cases}$$

- Point of intersection : $t = s = 0 \Rightarrow \boxed{P(1,1,2)}$

See problem 60 for reference