1. Since $A(I+A) = (I+A)A = I$, A is invertible with inverse $I+A$.

2. Note that $A = (AB)B^{-1}$ is the product of invertible matrices, so it too is invertible, as shown in Problem 2.2.11.

3. (a) $\left[\begin{array}{rrr|r} 2 & -1 & 2 & -4 \\ 3 & 2 & 0 & 1 \\ 1 & 3 & -6 & 5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & -6 & 5 \\ 0 & -7 & 14 & -14 \\ 0 & -7 & 18 & -14 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & -6 & 5 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 4 & 0 \end{array}\right]$.

So $z = 0$; $y - 2z = 2$, so $y = 2$; $x + 3y - 6z = 5$, so $x = -1$. The solution is $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

(b) $[A|\mathsf{b}] = \left[\begin{array}{rrr|r} 1 & 1 & 7 & 2 \\ 2 & -4 & 14 & -1 \\ 5 & 11 & -7 & 8 \\ 2 & 5 & -4 & -3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 7 & 2 \\ 0 & -6 & 0 & -3 \\ 0 & 6 & -42 & -2 \\ 0 & 3 & -18 & -7 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 7 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 6 & -42 & -2 \\ 0 & 3 & -18 & -7 \end{array}\right]$

$\to \left[\begin{array}{rrr|r} 1 & 1 & 7 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -42 & -5 \\ 0 & 0 & -18 & -\frac{17}{2} \end{array}\right]$.

The last two rows imply, respectively, that $z = \frac{5}{42}$ and that $z = \frac{17}{36}$. This cannot be. The system has no solution.

(c) $[A|\mathsf{b}] = \left[\begin{array}{rrrr|r} 2 & 2 & 2 & -8 & 2 \\ 4 & 6 & 6 & 0 & 4 \\ 6 & 6 & 10 & -4 & 2 \end{array}\right] \to \left[\begin{array}{rrrr|r} 2 & 2 & 2 & -8 & 2 \\ 0 & 2 & 4 & 16 & 0 \\ 0 & 0 & 4 & 20 & -4 \end{array}\right]$. This is upper triangular form. We complete the solutions by back substitution.

Variable $x_4 = t$ is free and $4x_3 + 20x_4 = -4$, so $x_3 = -1 - 5x_4 = -1 - 5t$. Then $x_2 = -x_3 - 8x_4 = 1 - 3t$ and $x_1 = 1 - x_2 - x_3 + 4x_4 = 1 + 12t$. The solution is $\mathsf{x} = \begin{bmatrix} 1+12t \\ 1-3t \\ -1-5t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 12 \\ -3 \\ -5 \\ 1 \end{bmatrix}$.

(d) $[A|\mathsf{b}] = \left[\begin{array}{rrr|r} 1 & -1 & 2 & 4 \end{array}\right]$. This is row echelon form. The free variables are $y = t$ and $z = s$, so $x = 4 + y - 2z = 4 + t - 2s$. In vector form the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4+t-2s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

(e) This system is homogeneous. The right hand column of 0s is not affected by the elementary row operations, so we omit it from the steps of Gaussian elimination, remembering, however, that it is there!

$$A = \begin{bmatrix} 2 & -7 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 3 & 6 & 7 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -7 & 1 & 1 \\ 3 & 6 & 7 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & -1 & 1 \\ 0 & 12 & 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables $x_3 = s$ and $x_4 = t$ are free. Back substitution gives

$$x_2 + \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 = 0, \text{ so that } x_2 = -\tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 = -\tfrac{1}{3}s + \tfrac{1}{3}t,$$

and

$$x_1 - 2x_2 + x_3 = 0, \text{ so } x_1 = 2x_2 - x_3 = 2(-\tfrac{1}{3}s + \frac{1}{3}t) - s = -\tfrac{5}{3}s + \tfrac{2}{3}t.$$

Our solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3}s + \frac{2}{3}t \\ -\frac{1}{3}s + \frac{1}{3}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -\frac{5}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$, which is the set of linear combinations of $\begin{bmatrix} -\frac{5}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$, a set which is also the set of linear combinations of $\begin{bmatrix} -5 \\ -1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}$. So we could also write the solution as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -5 \\ -1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}$.

(f) $$[A|\mathsf{b}] = \left[\begin{array}{rrrr|r} 2 & -3 & 4 & -1 & 5 \\ -1 & 1 & 0 & 1 & -1 \\ 0 & 2 & -1 & -3 & 1 \\ 3 & 0 & 1 & 4 & 7 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -1 & 0 & -1 & 1 \\ 2 & -3 & 4 & -1 & 5 \\ 0 & 2 & -1 & -3 & 1 \\ 3 & 0 & 1 & 4 & 7 \end{array}\right]$$

$$\to \left[\begin{array}{rrrr|r} 1 & -1 & 0 & -1 & 1 \\ 0 & -1 & 4 & 1 & 3 \\ 0 & 2 & -1 & -3 & 1 \\ 0 & 3 & 1 & 7 & 4 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 2 & -1 & -3 & 1 \\ 0 & 3 & 1 & 7 & 4 \end{array}\right]$$

$$\to \left[\begin{array}{rrrr|r} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 0 & 7 & -1 & 7 \\ 0 & 0 & 13 & 10 & 13 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 0 & 14 & -2 & 14 \\ 0 & 0 & 13 & 10 & 13 \end{array}\right]$$

$$\to \left[\begin{array}{rrrr|r} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 0 & 1 & -12 & 1 \\ 0 & 0 & 13 & 10 & 13 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 0 & 1 & -12 & 1 \\ 0 & 0 & 0 & 166 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -4 & -1 & -3 \\ 0 & 0 & 1 & -12 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Thus $x_4 = 0$, $x_3 - 12x_4 = 1$, so $x_3 = 1 + 12x_4 = 1$,

$$x_2 - 4x_3 - x_4 = -3, \text{ so } x_2 = -3 + 4x_3 + x_4 = -3 + 4 = 1$$

and $x_1 - x_2 - x_4 = 1$, so $x_1 = 1 + x_2 + x_4 = 1 + 1 = 2$. The solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

4. The question asks if there are scalars a and b such that $\begin{bmatrix} 2 \\ -11 \\ -3 \end{bmatrix} = a\begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + b\begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}$.

   This is $\begin{bmatrix} 0 & -1 \\ -1 & 4 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -11 \\ -3 \end{bmatrix}$. Gaussian elimination gives

$$\left[\begin{array}{cc|c} 0 & -1 & 2 \\ -1 & 4 & -11 \\ 5 & 9 & -3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & -1 & 2 \\ 0 & 29 & -58 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

   There is a unique solution: $b = -2$, $a = 3$. The given vector is indeed a linear combination of the other two.

$$\begin{bmatrix} 2 \\ -11 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}.$$