# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

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Assignment 7 **MATH 2050** Answers

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1. (a) Give a definition of an eigenvalue and corresponding eigenvector of a matrix.

$$AX = \lambda X, \qquad X \neq 0.$$

(b) Explain the method (step by step) how to find all the eigenvalues and corresponding eigenvectors of a matrix.

1. Solve $\det(A - \lambda I) = 0$ for $\lambda$ to find all eigen values.
2. For each value of $\lambda$ found at step 1, solve homogeneous system $(A - \lambda I)X = 0$ for $X$ to find corresponding eigenvector.
3. Check your eigen pair to satisfy the definition.

(c) Give an exapmle of a problem when knowing the eigenvalues and eigenvectors of a matrix can be useful to find the solution explicitly.

Knowing the eigenvalues $\lambda_1, \lambda_2$ and eigenvectors $X_1, X_2$ of a $(2 \times 2)$-matrix $A$ can be useful to find a power of this matrix, say $A^{100}$ in a faster way:

$$A^{100} = PD^{100}P^{-1}, \quad D^{100} = \begin{bmatrix} \lambda_1^{100} & 0 \\ 0 & \lambda_2^{100} \end{bmatrix}, \quad P = [X_1|X_2].$$

2. Find the characteristic polynomial, eigenvalues, eigenvectors and (if possible) an invertable matrix $P$ such that $P^{-1}AP$ is diagonal. If the later is not possible, explain why.

(a) $A = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$

*Answer:*

characteristic polynomial: $\lambda^2 - 9\lambda + 14$;

eigenvalues, eigenvectors: $\lambda_1 = 7$, $X_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$; $\lambda_2 = 2$, $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$;

matrix $P = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$

*Answer:*

characteristic polynomial: $\lambda^2 - 4\lambda + 4$;

eigenvalues, eigenvectors: $\lambda_1 = \lambda_2 = 2$, $X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(multiple root and only one eigenvector.)

matrix $P$ does not exist.

(c) $A = \begin{bmatrix} 2 & -16 & -2 \\ 0 & 5 & 0 \\ 2 & -8 & -3 \end{bmatrix}$ *Answer:*

characteristic polynomial: $-\lambda^3 + 4\lambda^2 + 7\lambda - 10$;

eigenvalues, eigenvectors:

$\lambda_1 = 1,\ X_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$; $\lambda_2 = 5,\ X_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$; $\lambda_3 = -2,\ X_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$;

matrix $P = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 2 & 1 & -12 \\ 0 & 1 & 11 \\ 1 & 1 & 4 \end{bmatrix}$

*Answer:*

characteristic polynomial: $-\lambda^3 + 7\lambda^2 - 15\lambda = 9$;

eigenvalues, eigenvectors: $\lambda_1 = 1,\ X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$; $\lambda_2 = \lambda_3 = 3,\ X = \begin{bmatrix} -13 \\ 11 \\ 2 \end{bmatrix}$;

(for the multiple root there is only one eigenvector.)

matrix $P$ does not exist.

3. Let a matrix $A$ have eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 3$ with corresponding eigenvectors $X_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the matrix $A$.

   *Answer:* $A = PDP^{-1} = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & -3 & -2 \\ 5 & 5 & 4 \end{bmatrix}$. Find an invertable matrix $P$ such that $P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

   *Answer:* In general form $P = \begin{bmatrix} -t & 0 & -q \\ t & -s & q \\ -t & s & 0 \end{bmatrix}$ for any nonzero values $t, s, q$. For example,

   $P = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ if parameters $s, t, q$ all are set equal to 1.

5. Consider a linear dynamical system $V_{k+1} = AV_k$ for $k \geq 0$. Find exact formula for $V_k$. Approximate $V_k$ for large values of $k$.

(a) $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$, $V_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

*Answer:*

$$V_k = \frac{7^k}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2^k \frac{3}{5} \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

$V_k \approx \frac{7^k}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for large values of $k$.

(b) $A = \begin{bmatrix} 2 & 0 & 2 \\ -16 & 5 & -8 \\ -2 & 0 & -3 \end{bmatrix}$ $V_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

6. True or False? Explain.

a) every square matrix is *diagonalizable* (i.e. *similar* to a diagonal matrix).

*Answer:* **False**. Problem 2b gives an example of square but non-diagonalizable matrix.

b) any $n \times n$-matrix has at most $n$ distinct eigenvalues.

*Answer:* **True**. The characteristic polynomial has degree $n$, thus at most $n$ distinct roots are possible.

c) if $\lambda \neq 0$ is an eigenvalue of $A$ and $A$ is invertable then $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

*Answer:* **True**. Multiply equation $AX = \lambda X$ by $A^{-1}$ from the left; then devide both sides by *lambda*. You get $A^{-1}X = \lambda^{-1}X$.