MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 6 **MATH 2050 sect. 3** ANSWERS

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1. For each of the following elementary matrices describe the corresponding elementary row operation and write the inverse

(a) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer. ERO: Row One and Row Three are interchanged.

Inverse operation: Row One and Row Three are interchanged.

Inverse matrix is the same: $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *Answer.* ERO: Row One is Row One **plus** two times Row Two.

Inverse operation: Row One is Row One **minus** two times Row Two.

Inverse matrix is : $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ERO: Row Three is Row Three **minus** four times Row Two.

Inverse operation: Row One is Row One **plus** two times Row Two.

Inverse matrix is : $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Row Two is Row Two times **ten**.

Inverse operation: Row Two is Row Two times **one ten's**.

Inverse matrix is : $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. Find an invertable matrix U such that $UA = R$ is in reduced row-echelon form, and express U as a product of elementary matrices

(a) $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ 1 & -3 & 3 & 2 \end{bmatrix}$

Solution: To transform matrix A into reduced REF R we apply the following sequence of EROs: $R_2 := R_2 - 3R_1$, $R_3 := R_3 - R_1$, $R_3 := R_3 - R_2$, $R_2 := R_2/5$, $R_1 := R_1 - 2R_2$. Consequently, related elemetrary matrices are: $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_5 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The reduced REF matrix is: $R = \begin{bmatrix} 1 & 0 & 3/5 & 4/5 \\ 0 & 1 & -4/5 & -2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Finally, $R = E_5E_4E_3E_2E_1A$. Thus $U = E_5E_4E_3E_2E_1 = \frac{1}{5}\begin{bmatrix} -1 & 2 & 0 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$.

It is invertable because its inverse exist and defined as the product of the inverses: $U^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}$.

(b) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$

Solution: To transform matrix A into reduced REF R we apply the following sequence of EROs: $R_3 := R_3 - 2R_1$, $R_3 := R_3 - R_2$, $R_2 := R_2 - R_3$, $R_1 := R_1 - 2R_3$. Consequently, related elemetrary matrices are: $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The reduced REF matrix is the identity matrix: $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Finally, $R = E_4E_3E_2E_1A$. Thus $U = E_4E_3E_2E_1 = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$.

It is invertable because its inverse exist and defined as the product of the inverses: $U^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$.

3. Suppose B is obtained from A by the following ERO. In each case describe how to obtain B^{-1} from A^{-1}.

   Answer. Since $B = EA$, we have $B^{-1} = A^{-1}E^{-1}$. Thus to obtain B^{-1} from A^{-1} we have to multiply A^{-1} by E^{-1} **from the right**. Now, E^{-1} will be specified for each case:

   a) interchanching rows i and j;

   E is the elementary matrix obtained from the identity by switching rows i and j. Its inverse, E^{-1}, coincides with E.

   b) multiplying row j by $a \neq 0$;

   E is the elementary matrix obtained from the identity by multiplying row j by a. Its inverse, E^{-1}, is the elementary matrix obtained from the identity by multiplying row j by $1/a$.

   c) adding a times row j to row $i \neq j$;

   E is the elementary matrix obtained from the identity by adding a times row j to row i . Its inverse, E^{-1}, is the elementary matrix obtained from the identity by adding $-a$ times row j to row i.

4. For each of the following matrices explain why it is stochastic and find the steady-state vector

   (a) $A = \begin{bmatrix} .8 & 0 & .2 \\ .1 & .6 & .1 \\ .1 & .4 & .7 \end{bmatrix}$

   Solution: The matrix is stochastic because its elements are between zero and one and its columns sum to one.

   To find the stady state we need to find a parametric solution of the homogeneous system with matrix of coefficients $I - A = \begin{bmatrix} .2 & 0 & -.2 \\ -.1 & .4 & -.1 \\ -.1 & -.4 & .3 \end{bmatrix}$. The parametric solution is $x = 2t$, $y = t$, $z = 2t$, and we need $x + y + z = 1$. This gives $t = 1/5$. Thus the stedy state is $\begin{bmatrix} 2/5 \\ 1/5 \\ 2/5 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/2 \end{bmatrix}$

   Solution: The matrix is stochastic because its elements are between zero and one and its columns sum to one.

   To find the stady state we need to find a parametric solution of the homogeneous system with matrix of coefficients $I - A = \begin{bmatrix} 1/2 & -1/4 & -1/4 \\ 0 & 1/2 & -1/4 \\ -1/2 & -1/4 & 1/2 \end{bmatrix}$. The parametric solution is $x = 3t$, $y = 2t$, $z = 4t$, and we need $x + y + z = 1$. This gives $t = 1/9$. Thus the stedy state is $\begin{bmatrix} 1/3 \\ 2/9 \\ 4/9 \end{bmatrix}$.

5. Bob has a linear algebra class on MWF each week. Bob makes it to LA class on time one Monday out of four. (Don't be a Bob!) If he is late one day he is twice as likely to come to the next LA class on time. If he is on time one LA class he is as likely to be late as not the next LA class. Find the probability of his being late and that of his being on time on Fridays.

   Solution: There two mutually exluded events: *Bob is late* and *Bob is on time*. Thus the state has two components: x-probability that Bob is late, and y-probability that he is on time. Initial state of observation $S_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is on Monday. Since Bob makes it to LA class on time one Monday out of four, $x_0 = 3/4$, $y_0 = 1/4$.

   The transition matrix A is 2×2. "If Bob is late one day he is twice as likely to come to the next LA class on time"– thus gives the first column of A. "If he is on time one LA class he is as likely to be late as not the next LA class"- this gives the second column of A, and we have $A = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$. The state on Wednesday is $S_1 = AS_0$; the state on Friday is $S_2 = AS_1 = A^2 S_0 = \begin{bmatrix} 7/16 \\ 9/16 \end{bmatrix}$. Thus the probability that Bob is late on Friday is $7/16$, and be on time $9/16$.

6. Compose a word problem which leads to the Markov chain with one of the stochastic matrices given in Problem 4 above.