179. (a) $c_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, $c_{21} = -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -(-2) = 2$, $c_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(-1) = 1$.

(b) Expanding by cofactors of the first row, $\det A = 1(-1) + 0(-1) + 1(3) = 2$.

(c) A is invertible since $\det A \neq 0$.

(d) $A^{-1} = \dfrac{1}{\det A}C^T = \dfrac{1}{2}\begin{bmatrix} -1 & 2 & -1 \\ -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}$.

180. (a) A matrix is not invertible if and only if its determinant is 0. The determinant of the given matrix is $x^2 + 3x + 2 = 0$; so the matrix is singular if and only if $x = -1$ or $x = -2$.

(b) Expanding by cofactors across the second row, we see that the matrix has determinant $-3\begin{vmatrix} 1 & x \\ 4 & 7 \end{vmatrix} = -3(7 - 4x)$, so the matrix is singular if and only if $x = \frac{7}{4}$.

181. (a) Let $A = \begin{bmatrix} -1 & 2 & x \\ 0 & 3 & y \\ 2 & -2 & z \end{bmatrix}$. The $(1,1)$ cofactor is $19 = 3z + 2y$. The $(2,1)$ cofactor is $-14 = -(2z + 2x)$ and the $(2,2)$ cofactor is $-11 = -z - 2x$. We get $x = 4$, $y = 5$, $z = 3$. Now let $C = \begin{bmatrix} 19 & 10 & r \\ -14 & -11 & s \\ -2 & 5 & t \end{bmatrix}$. Then $r = -6$, $s = -(2 - 4) = 2$ and $t = -3$.

(b) We compute

$$AC^T = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 3 & 5 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} 19 & -14 & -2 \\ 10 & -11 & 5 \\ -6 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -23 & 0 & 0 \\ 0 & -23 & 0 \\ 0 & 0 & -23 \end{bmatrix} = -23I$$

and conclude that $A^{-1} = -\frac{1}{23}\begin{bmatrix} 19 & -14 & -2 \\ 10 & -11 & 5 \\ -6 & 2 & -3 \end{bmatrix}$.

182. As a triangular matrix, $\det P(a) = 2(a^2 + a)$. Thus $\det P(a) = 0$ if and only if $a^2 + a = 0 = a(a+1)$; that is, if and only if $a = 0$ or $a = -1$. Singular means "not invertible," so $P(a)$ is singular if and only if $a = 0$ or $a = -1$.

183. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The cofactor matrix is $C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$, so $d = -1$, $-c = 7$, $-b = 2$, $a = 4$. Thus $A = \begin{bmatrix} 4 & -2 \\ -7 & -1 \end{bmatrix}$.

184. Since $C = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$, $C^T = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$, so $A^{-1} = \dfrac{1}{\det A}C^T = \dfrac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$.

Thus $A = (A^{-1})^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

185. (a) 0 (two equal rows)

(b) 236 (third row has been multiplied by 2)

(c) 0 (first column is twice the fourth)

(d) −118 (rows two and three were interchanged)

(e) 118 (this matrix is the transpose of the one given)

186. $\det(-3A) = (-3)^5 \det A = -243 \det A = 4$, so $\det A = -\dfrac{4}{243}$,

$\det B^{-1} = \dfrac{1}{\det B}$, so $\det B = \dfrac{1}{\det B^{-1}} = \dfrac{1}{2}$,

$\det AB = \det A \det B = -\dfrac{4}{243} \cdot \dfrac{1}{2} = -\dfrac{2}{243}$.

187. The determinant of a triangular matrix is the product of its diagonal entries, so $\det A = 60$. Also, $\det A^{-1} = \dfrac{1}{\det A} = \frac{1}{60}$ and $\det A^2 = (\det A)^2 = 3600$.

188. (a) Expanding by cofactors of the first column gives

$$-1\begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & -1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 3 & -1 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 \\ 1 & 1 & 3 \\ 1 & 1 & 2 \end{vmatrix} = -1(-4) - 2(0) - (2) = 2.$$

(b) $$\det A = \begin{vmatrix} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 6 & 8 \end{vmatrix}$$

$$= 4\begin{vmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 6 & 8 \end{vmatrix} = 4\begin{vmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & \frac{1}{2} \end{vmatrix} = 4(\tfrac{1}{2}) = 2.$$

189. We form the matrix $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ whose columns are the given vectors. Since $\det A = 10 \neq 0$, the vectors are linearly independent.

190. (a) $\begin{vmatrix} -2 & 1 & 2 \\ 1 & 3 & 6 \\ -4 & 5 & 9 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 6 \\ -2 & 1 & 2 \\ -4 & 5 & 9 \end{vmatrix}$ interchanging rows one and two

$= -\begin{vmatrix} 1 & 3 & 6 \\ 0 & 7 & 14 \\ 0 & 17 & 33 \end{vmatrix}$ using the third elementary row operation to put 0s in the first column under the leading 1

$= -7\begin{vmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 17 & 33 \end{vmatrix}$ factoring 7 from row two

$= -7\begin{vmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix}$ using the third elementary row operation to put a 0 in the $(2, 3)$ position

$= (-7)(-1) = 7$ since the determinant of a triangular matrix is the product of its diagonal entries.

(b) $\begin{vmatrix} -3 & 0 & 1 & 1 \\ 3 & 1 & 2 & 2 \\ -6 & -2 & -4 & 2 \\ 1 & -1 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ -6 & -2 & -4 & 2 \\ -3 & 0 & 1 & 1 \end{vmatrix}$

interchanging rows one and four

$= -\begin{vmatrix} 1 & -1 & 0 & -1 \\ 0 & 4 & 2 & 5 \\ 0 & -8 & -4 & -4 \\ 0 & -3 & 1 & -2 \end{vmatrix}$ using the third elementary row operation to put 0s in the first column under the leading 1

$= \begin{vmatrix} 1 & -1 & 0 & -1 \\ 0 & -8 & -4 & -4 \\ 0 & 4 & 2 & 5 \\ 0 & -3 & 1 & -2 \end{vmatrix}$ interchanging rows two and three

$= -8\begin{vmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 4 & 2 & 5 \\ 0 & -3 & 1 & -2 \end{vmatrix}$ factoring -8 from row two

$= -8\begin{vmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 3 \\ 0 & 0 & \frac{5}{2} & -\frac{1}{2} \end{vmatrix}$ using the third elementary row operation to put 0s under the leading 1 in column two

$$= 8\begin{vmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 3 \end{vmatrix}$$ interchanging rows three and four

$$= 8(\tfrac{5}{2})(3) = 60$$ since the determinant of a triangular matrix is the product of the diagonal entries.

191. $(\det P)(\det A) = \det PA = (\det L)(\det U) = 1(-35) = -35$ and $\det P = \pm 1$, so $\det A = \pm 35$.

192. a zero row; two equal rows; one row a scalar multiple of another. one row is a linear combination of others.

193. It is sufficient to show that $\det A = 0$. Let the columns of A be $a_1\mathsf{u} + b_1\mathsf{v}$, $a_2\mathsf{u} + b_2\mathsf{v}$, $a_3\mathsf{u} + b_3\mathsf{v}$ for scalars $a_1, b_1, a_2, b_2, a_3, b_3$. Write $\det(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3)$ for the determinant of a matrix whose columns are the vectors $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3$. Then

$$\begin{aligned} \det A &= \det(a_1\mathsf{u} + b_1\mathsf{v}, a_2\mathsf{u} + b_2\mathsf{v}, a_3\mathsf{u} + b_3\mathsf{v}) \\ &= \det(a_1\mathsf{u}, a_2\mathsf{u} + b_2\mathsf{v}, a_3\mathsf{u} + b_3\mathsf{v}) + \det(b_1\mathsf{v}, a_2\mathsf{u} + b_2\mathsf{v}, a_3\mathsf{u} + b_3\mathsf{v}) \end{aligned}$$

by linearity of the determinant in the first column. Using linearity of the determinant in columns two and three, eventually we obtain

$$\begin{aligned} \det A = {} & \det(a_1\mathsf{u}, a_2\mathsf{u}, a_3\mathsf{u}) + \det(a_1\mathsf{u}, a_2\mathsf{u}, b_3\mathsf{v}) + \det(a_1\mathsf{u}, b_2\mathsf{v}, a_3\mathsf{u}) \\ & + \det(a_1\mathsf{u}, b_2\mathsf{v}, b_3\mathsf{v}) + \det(b_1\mathsf{v}, a_2\mathsf{u}, a_3\mathsf{u}) + \det(b_1\mathsf{v}, a_2\mathsf{u}, b_3\mathsf{v}) \\ & + \det(b_1\mathsf{v}, b_2\mathsf{v}, a_3\mathsf{u}) + \det(b_1\mathsf{v}, b_2\mathsf{v}, b_3\mathsf{v}) \end{aligned}$$

and each of these eight determinants is 0 because in each case, either one column is 0 or one column is a multiple of another. In the second determinant—$\det(a_1\mathsf{u}, a_2\mathsf{u}, b_3\mathsf{v})$—for example, if $a_1 \neq 0$, the second column is $\dfrac{a_2}{a_1}$ times the first.

194. (a) $\begin{vmatrix} 6 & -3 \\ 1 & -4 \end{vmatrix} = -24 + 3 = -18$

(b) We expand by cofactors along the first row.

$$\begin{vmatrix} 5 & 3 & 8 \\ -4 & 1 & 4 \\ -2 & 3 & 6 \end{vmatrix} = 5\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} - 3\begin{vmatrix} -4 & 4 \\ -2 & 6 \end{vmatrix} + 8\begin{vmatrix} -4 & 1 \\ -2 & 3 \end{vmatrix}$$
$$= 5(-6) - 3(-16) + 8(-10) = -62.$$

(c) $\begin{vmatrix} -1 & 2 & 3 & 4 \\ 3 & -9 & 2 & 1 \\ 0 & -5 & 7 & 6 \\ 2 & -4 & -6 & -8 \end{vmatrix} = 0$ because the last row is a scalar multiple of the first.

195. $\begin{vmatrix} a & -g & 2d \\ b & -h & 2e \\ c & -i & 2f \end{vmatrix} = -2 \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$ factoring -1 from the second column and 2 from the third column

$= -2 \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$ since the determinant of a matrix is the determinant of its transpose

$= +2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(-3) = 6$, interchanging rows two and three.

196. Suppose $A = LDU$. Each of L and U is triangular with 1s on the diagonal, so $\det L = \det U = 1$, the product of the diagonal entries in each case. Thus $\det A = \det L \det D \det U = \det D$. Using just the third elementary row operation to reduce $A = \begin{bmatrix} 2 & -1 & 4 & 1 \\ 1 & 1 & -10 & -2 \\ 4 & 0 & -7 & 6 \\ 6 & -3 & 0 & 1 \end{bmatrix}$ to an upper triangular matrix, we have

$$\begin{bmatrix} 2 & -1 & 4 & 1 \\ 1 & 1 & -10 & -2 \\ 4 & 0 & -7 & 6 \\ 6 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 4 & 1 \\ 0 & \frac{3}{2} & -12 & -\frac{5}{2} \\ 0 & 2 & -15 & 4 \\ 0 & 0 & -12 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 4 & 1 \\ 0 & \frac{3}{2} & -12 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{22}{3} \\ 0 & 0 & -12 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & -1 & 4 & 1 \\ 0 & \frac{3}{2} & -12 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{22}{3} \\ 0 & 0 & 0 & 86 \end{bmatrix} = U' = DU$$

with $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 86 \end{bmatrix}$. Thus $\det A = \det D = 2(\frac{3}{2})(1)(86) = 258$.

197. We have $\frac{1}{2}(I - A)A = I$, so $(I - A)A = 2I$. Taking the determinant of each side gives $\det(I - A) \det A = 2^n$. If the product of two numbers is a power of 2, each number itself is a power of 2, so the result follows.

198. $A\mathsf{v} = \begin{bmatrix} 8 \\ 16 \\ 0 \end{bmatrix} = 4\mathsf{v}$. Thus v is an eigenvector of A corresponding to the eigenvalue 4.

199. $A\mathsf{v}_1 = 5\mathsf{v}_1$, so v_1 is an eigenvector corresponding to $\lambda = 5$.

$A\mathsf{v}_2 = 0\mathsf{v}_2$, so v_2 is an eigenvector corresponding to $\lambda = 0$.

$A\mathsf{v}_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is not $\lambda \mathsf{v}_3$ for any λ, so v_3 is not an eigenvector.