Assignment 8

#151 Here, $L = E_1E_2E_3$ with $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$.

$$L^{-1} = E_3^{-1}E_2^{-1}E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -13 & -5 & 1 \end{bmatrix}.$$

152. $U = E_1E_2$, where $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$U^{-1} = E_2^{-1}E_1^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

153. We have $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & -2 \\ 0 & 3 & 3 \end{bmatrix} = EA = U$ with $E = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Thus $A = E^{-1}U = LU$ with $L = E^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

154. (a) $A \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U$ with $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

(b) From (a), remembering the single multiplier, we get $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

(c) $A\mathsf{v} = -2\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$.

155. Here is a way to move A to row echelon form U that uses only the third elementary row operation: $\begin{bmatrix} -3 & 3 & 6 \\ 2 & 5 & 10 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R2 \to R2 - (-\frac{2}{3})R1} \begin{bmatrix} -3 & 3 & 6 \\ 0 & 7 & 14 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{R3 \to R3 - \frac{1}{7}(R2)} \begin{bmatrix} -3 & 3 & 6 \\ 0 & 7 & 14 \\ 0 & 0 & 2 \end{bmatrix} = U.$

The corresponding elementary matrices are $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We have $E_2E_1A = U$, so $A = LU$ with

$$L = (E_2E_1)^{-1} = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{7} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & \frac{1}{7} & 1 \end{bmatrix}.$$

156. (a) We bring A to row echelon form using only the third elementary row operation. This can be done in one step:
$A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & 4 \\ 0 & 8 \end{bmatrix} = U$ with $L = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$.

(b) The first step in Gaussian elimination is $A \to \begin{bmatrix} 2 & -6 & 5 \\ 0 & 0 & 1 \\ 0 & -3 & 3 \end{bmatrix}$.
Now an interchange of rows is required, so there is no LU factorization.

157. (a) First we solve $L\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$. Using forward substitution, we have $2y_1 = -2$, so $y_1 = -1$, and then $6y_1 + 5y_2 = 9$, so $y_2 = 3$. Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Now we solve $U\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Using back substitution, we obtain $y = 3$; then, since $x + \frac{1}{2}y = -1$, we get $x = -\frac{5}{2}$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ 3 \end{bmatrix}$.

(b) We have just shown that $A\begin{bmatrix} -\frac{5}{2} \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$, thus $\begin{bmatrix} -2 \\ 9 \end{bmatrix} = -\frac{5}{2}\begin{bmatrix} 1 \\ 6 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 8 \end{bmatrix}$.

158. We have $A = LU$ with $L = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 3 \\ 0 & 12 \end{bmatrix}$.

First, we solve $L\mathsf{y} = \mathsf{b}$ for $\mathsf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. This is the system $\begin{aligned} -2y_1 \quad &= -3 \\ y_1 + y_2 &= \ \ 4 \end{aligned}$.

By forward substitution, we obtain $y_1 = \frac{3}{2}$ and $y_2 = 4 - y_1 = \frac{5}{2}$, so $\mathsf{y} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \end{bmatrix}$.

Now we solve $U\mathsf{x} = \mathsf{y}$ by back substitution. We have $\begin{aligned} x_1 + \ 3x_2 &= \tfrac{3}{2} \\ 12x_2 &= \tfrac{5}{2} \end{aligned}$ so $x_2 = \frac{5}{24}$ and $x_1 = \frac{3}{2} - 3x_2 = \frac{3}{2} - \frac{5}{8} = \frac{7}{8}$. Our solution is $\mathsf{x} = \begin{bmatrix} \frac{7}{8} \\ \frac{5}{24} \end{bmatrix}$.

159. (a) $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 3 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 3 & 2 & -3 \\ 0 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 3 & 2 & -3 \\ 0 & 0 & -\frac{1}{3} & 0 \end{array}\right]$.

Letting $\mathsf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the equations corresponding to this last (upper triangular) system are

$$\begin{aligned} x - \ y \qquad &= \ \ 1 \\ 3y + 2z &= -3 \\ -\tfrac{1}{3}z &= \ \ 0. \end{aligned}$$

So $z = 0$. Then $3y = -3$ so $y = -1$. Then $x = 1 + y = 0$, so $\mathsf{x} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

(b) From the preceding elimination, we see that $U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$.

Keeping track of the multipliers, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -\frac{1}{3} & 1 \end{bmatrix}$.

(c) The matrices L and U are invertible since each is square triangular with all diagonal entries nonzero. Thus the product $A = LU$ of L and U is invertible.

(d) The first column of A^{-1} is the vector x that satisfies $A\mathsf{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

We found this in (a); $\mathsf{x} = \begin{bmatrix}0\\-1\\0\end{bmatrix}$.

160. i. The system is $A\mathsf{x} = \mathsf{b}$ with $A = \begin{bmatrix}2 & 2\\4 & 9\end{bmatrix}$, $\mathsf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ and $\mathsf{b} = \begin{bmatrix}8\\21\end{bmatrix}$.

ii. Gaussian elimination proceeds $A \to \begin{bmatrix}2 & 2\\0 & 5\end{bmatrix} = U$ so $A = LU$ with $L = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$.

iii. We must solve $L(U\mathsf{x}) = \mathsf{b}$, so let $U\mathsf{x} = \mathsf{y} = \begin{bmatrix}y_1\\y_2\end{bmatrix}$ and first solve $L\mathsf{y} = \mathsf{b}$. The corresponding equations are

$$\begin{aligned} y_1 \quad &= 8\\ 2y_1 + y_2 &= 21 \end{aligned}$$

so forward substitution gives $y_1 = 8$ and $y_2 = 21 - 2y_1 = 5$. Now we solve $U\mathsf{x} = \mathsf{y}$. The corresponding equations are

$$\begin{aligned} 2x_1 + 2x_2 &= 8\\ 5x_2 &= 5 \end{aligned}$$

and back substitution gives $x_2 = 1$ and $2x_1 = 8 - 2x_2 = 6$, giving $x_1 = 3$. Our solution is $\mathsf{x} = \begin{bmatrix}3\\1\end{bmatrix}$.

161. i. $A \to \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & -3\end{bmatrix} = U$ and, keeping track of multipliers, $L = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\0 & 2 & 1\end{bmatrix}$.

ii. $A\mathsf{x} = \mathsf{b}$ is $L(U\mathsf{x}) = \mathsf{b}$. Let $U\mathsf{x} = \mathsf{y}$; then $L\mathsf{y} = \mathsf{b}$. Solving for $\mathsf{y} = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$, we find $y_1 = -1$, $y_1 + y_2 = 0$, so $y_2 = -y_1 = 1$ and $2y_2 + y_3 = 1$ so $y_3 = 1 - 2y_2 = -1$. Thus $\mathsf{y} = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$.

Now we solve $U\mathsf{x} = \mathsf{y}$. We have $x_3 = \frac{1}{3}$, $x_2 + 2x_3 = 1$, so $x_2 = \frac{1}{3}$ and $x_1 + x_2 + x_3 = 1$, so $x_1 = -1 - x_2 - x_3 = -\frac{5}{3}$. Thus $\mathsf{x} = \begin{bmatrix}-\frac{5}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$.

170. i. The system is $A\mathsf{x} = \mathsf{b}$ with $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathsf{b} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$.

ii. We have $[A|I] = \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 5 & -3 & -2 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right]$

$$\to \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & \frac{4}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{3}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & -\frac{1}{5} & -\frac{2}{5} & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & 0 & \frac{4}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{3}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & -1 & -2 & 5 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & -4 \\ 0 & 1 & 0 & -1 & -1 & 3 \\ 0 & 0 & 1 & -1 & -2 & 5 \end{array}\right],$$

so $A^{-1} = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix}$ and $\mathsf{x} = A^{-1}\mathsf{b} = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ -9 \\ -13 \end{bmatrix}$.

iii. The given vectors are the columns of A, so $A\mathsf{x}$ is a linear combination of these vectors—recall **6.3**. We have just shown that $A \begin{bmatrix} 11 \\ -9 \\ -13 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$, so

$$\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = 11 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 9 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - 13 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

171. $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & -1 \\ 1 & 5 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & 9 \\ 3 & -3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$.

172. $\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = E_1A \to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -1 \end{bmatrix} = E_2E_1A \to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = E_3E_2E_1A$

$\to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = E_4E_3E_2E_1A$, where $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $E_4 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$. Thus $A = (E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$.

173. The columns of this matrix are linearly independent, so the matrix is invertible.

174. You compute XY or YX. If $XY = I$ or $YX = I$, then X and Y are inverses.

175. No it is not. Consider, for instance, $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$.

176. (a) i. $M = \begin{bmatrix}9 & -7\\-4 & 2\end{bmatrix}$, $C = \begin{bmatrix}9 & 7\\4 & 2\end{bmatrix}$,

$$AC^T = \begin{bmatrix}2 & -4\\-7 & 9\end{bmatrix}\begin{bmatrix}9 & 4\\7 & 2\end{bmatrix} = \begin{bmatrix}-10 & 0\\0 & -10\end{bmatrix} = C^TA$$

ii. $\det A = -10$

iii. $A^{-1} = -\frac{1}{10}\begin{bmatrix}9 & 4\\7 & 2\end{bmatrix}$

(b) i. $M = \begin{bmatrix}-26 & -12 & 4\\-13 & -6 & 2\\13 & 6 & -2\end{bmatrix}$, $C = \begin{bmatrix}-26 & 12 & 4\\13 & -6 & -2\\13 & -6 & -2\end{bmatrix}$,

$$AC^T = \begin{bmatrix}2 & 3 & 4\\0 & -1 & 3\\4 & 7 & 5\end{bmatrix}\begin{bmatrix}-26 & 13 & 13\\12 & -6 & -6\\4 & -2 & -2\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = C^TA.$$

ii. Since $AC^T = (\det A)I$, we must have $\det A = 0$ in this case.

iii. A is not invertible since $AC^T = \mathbf{0}$ but $C^T \neq \mathbf{0}$. See Example 10.10.

177. Expanding by cofactors of the third row gives

$$\det A = 2\begin{vmatrix}-1 & 2\\1 & 1\end{vmatrix} + \begin{vmatrix}1 & 2\\3 & 1\end{vmatrix} + 3\begin{vmatrix}1 & -1\\3 & 1\end{vmatrix} = 2(-3) + (-5) + 3(4) = 1.$$

178. (a) The matrix of minors is $\begin{bmatrix}-2 & 41 & 7\\3 & 19 & 2\\3 & 26 & 2\end{bmatrix}$.

(b) The matrix of cofactors is $C = \begin{bmatrix}-2 & -41 & 7\\-3 & 1 & -2\\3 & -26 & 2\end{bmatrix}$.

(c) $[417] \cdot [3 - 262] = 12 - 26 + 14 = 0$. This is the dot product of the second row of A and the third column of C^T. Since AC^T is a scalar multiple of the identity, entries not on the diagonal, for example, the $(2,3)$ entry in this case, are 0.

(d) $\begin{bmatrix}2\\4\\-3\end{bmatrix} \cdot \begin{bmatrix}-41\\1\\-26\end{bmatrix} = -82+4+78 = 0$. This is the dot product of the second row of C^T and the first column of A. Since C^TA is a scalar multiple of the identity, entries not on the diagonal, for example, the $(2,1)$ entry in this case, are 0.

(e) $\begin{bmatrix}0\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-41\\1\\-26\end{bmatrix} = 1 - 26 = -25$. Since $C^TA = (\det A)I$, each diagonal entry is $\det A$, in particular, the $(2,2)$ entry, which is the dot product of the second row of C and the second column of A, that is, the second column of C^T and the second column of A. This part shows that $\det A = -25$.

179. (a) $c_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$, $c_{21} = -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -(-2) = 2$, $c_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(-1) = 1$.

(b) Expanding by cofactors of the first row, $\det A = 1(-1) + 0(-1) + 1(3) = 2$.

(c) A is invertible since $\det A \neq 0$.

(d) $A^{-1} = \dfrac{1}{\det A} C^T = \dfrac{1}{2} \begin{bmatrix} -1 & 2 & -1 \\ -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}$.