Assignment 7 - Answers
(see #110, 111, 114 in Aswers for Assign 5)

137. The principle to follow when the given system involves letters is to proceed as if the letters were numbers. So apply Gaussian elimination to the matrix of coefficients.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ a & b & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & b-a & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & c-(b-a) \end{bmatrix}$$

(a) If $c - b + a \neq 0$, row echelon form is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and there is a unique solution: $x = y = z = 0$.

(b) It is not possible for this system to have no solution. Since it is homogeneous, there is always at least the solution, $x = y = z = 0$.

(c) If $c - b + a = 0$, row echelon form is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. In this case, z is a free variable and there are infinitely many solutions.

138. The given system is $A\mathsf{x} = \mathsf{b}$ where $A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$, $\mathsf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathsf{b} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 5 & 7 & -2 \\ 0 & 0 & 9 & 3 \end{array}\right]$.

Row echelon form is $U = \left[\begin{array}{ccc|c} 1 & 5 & 7 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$. Variable $x_2 = t$ is free. Back substitution yields $x_3 = \frac{1}{3}$ and $x_1 = -2 - 5x_2 - 7x_3 = -\frac{13}{3} - 5t$, so the solution is

$$\mathsf{x} = \begin{bmatrix} -\frac{13}{3} - 5t \\ t \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{13}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix},$$

which is of the from $\mathsf{x}_p + \mathsf{x}_h$ with $\mathsf{x}_p = \begin{bmatrix} -\frac{13}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$ and $\mathsf{x}_h = t \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$.

139. The question is whether or not the equation $c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + c_3\mathsf{v}_3 + c_4\mathsf{v}_4 = 0$ implies that the four coefficients c_1, c_2, c_3, c_4 are all 0. This vector equation is the matrix system

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Gaussian elimination proceeds

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $c_3 = t$ and $c_4 = s$ are free variables. There are lots of solutions which do not have $c_3 = 0$, for instance. The given vectors are not linearly independent. They are linearly dependent.

140. (a) The second row of EA is the sum of the second row and four times the third row of A; all other rows of EA are the same as those of A.

(b) E is called an elementary matrix.

(c) Any elementary matrix is invertible, its inverse being that elementary matrix that "undoes" E. Here, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

141. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

142. EA was formed by the operation $R3 \rightarrow R3 - 6R1$, so $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}$.

143. (a) $E = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$. Since EA is A with row two replaced by row two plus four times row one, $EF = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$. Since FA is A with row three replaced by row three minus three times row two, $FE = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -12 & -3 & 1 \end{bmatrix}$.

(b) Since E^{-1} undoes E, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since F^{-1} undoes F, $F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$. Since $(EF)^{-1} = F^{-1}E^{-1}$ and $F^{-1}A$ is A with row three replaced by row three plus three times row two, $(EF)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -12 & 3 & 1 \end{bmatrix}$. Since $(FE)^{-1} = E^{-1}F^{-1}$ and $E^{-1}A$ is A with row two replaced by row two minus four times row one, $(FE)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$.

144. $\begin{bmatrix} 2 & 5 & 1 \\ 4 & x & 1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 5 & 1 \\ 0 & x-10 & -1 \\ 0 & 1 & -1 \end{bmatrix}$, so if $x = 10$, it will be necessary to interchange rows two and three.

145. This question is designed to reinforce the important fact expressed in **6.3** and also to help you remember that matrix multiplication occurs columnwise. See (4) of 5.3. The first column of AB is A times the first column of B; this is $b_{11}\mathsf{a}_1$. The second column of AB is A times the second column of B; this is $b_{12}\mathsf{a}_1 + b_{22}\mathsf{a}_2$. In general, column k of AB is A times column k of B; this is $b_{1k}\mathsf{a}_1 + b_{2k}\mathsf{a}_2 + \cdots + b_{kk}\mathsf{a}_k$.

146. (a) A is elementary, so its inverse is the elementary matrix that "undoes" A:
$A^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$.

(b) A is elementary, so its inverse is the elementary matrix that "undoes" A:
$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

147. The given matrix is a permutation matrix P, so $P^{-1} = P^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

148. Let $M = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$. Then $LM = \begin{bmatrix} 1 & 0 \\ a+x & 1 \end{bmatrix}$, so we should let $x = -a$. The inverse of L is $\begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}$.

149. (a) The first row of EA is the same as the first row of A. The second row of EA is (Row 2 of A)−(Row 1 of A); the third row of EA is (Row 3 of A)+2 (Row 1 of A).
The first row of DA is 4 (Row 1 of A); the second row of DA is the same as the second row of A, and the third row of DA is −(Row 3 of A).
The rows of PA, in order, are the third, first and second rows of A.

(b) Since P is a permutation matrix, $P^{-1} = P^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

150. Ian was a student in my linear algebra class some years ago and he was almost always right. This matrix has no LU factorization. Lynn's first step corresponds to multiplication by $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which is not lower triangular. A lower triangular matrix can be obtained only if the row operations used in the transformation from A to U only change entries on or below the main diagonal.

151. If E is elementary, remember that EA is A transformed in exactly the way that the identity matrix was transformed to make E. This is the basis for our factorization (without calculation).