Solution 2. Imagine a slingshot. When you put a stone in the middle and pull it back, the length of the rubber band is the same from the stone to either side of the handle and when shot, the stone moves right down the plane in question. The line joining the points is perpendicular to the plane. So we can take $\mathsf{n} = \overrightarrow{PQ} = \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}$ and have $-x+2y-4z = d$. The midpoint on the line joining P and Q is on the plane; this is the point $(\frac{1}{2}(2+1), \frac{1}{2}(-1+1), \frac{1}{2}(3-1))$, that is, $(\frac{3}{2}, 0, 1)$. Thus $d = -\frac{3}{2} - 4 = -\frac{11}{2}$. The equation is $-x+2y-4z = -\frac{11}{2}$, or $-2x+4y-8z+11 = 0$, as before.

62. We have $\operatorname{proj}_{\mathsf{v}} \mathsf{u} = c\mathsf{v}$, with $c = \dfrac{\mathsf{u}\cdot\mathsf{v}}{\mathsf{v}\cdot\mathsf{v}}$. If this vector has direction opposite v, the scalar c is negative. Since the denominator of c is positive, the numerator $\mathsf{u}\cdot\mathsf{v}$ is negative. Since $\mathsf{u}\cdot\mathsf{v} = \|\mathsf{u}\|\,\|\mathsf{v}\|\cos\theta$, we must have $\cos\theta < 0$. Thus the angle θ between u and v is *obtuse*: $\frac{\pi}{2} < \theta < \pi$.

63. The plane spanned by vectors is the set of all linear combinations of the vectors. Since $0 = 0\mathsf{u} + 0\mathsf{v}$, the zero vector is always in the plane spanned by vectors.

64. (a) The projection of u on v is $\operatorname{proj}_{\mathsf{v}} \mathsf{u} = \dfrac{\mathsf{u}\cdot\mathsf{v}}{\mathsf{v}\cdot\mathsf{v}}\mathsf{v} = \dfrac{3+2-1}{9+1+1}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \dfrac{4}{11}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

The projection of v on u is $\operatorname{proj}_{\mathsf{u}} \mathsf{v} = \dfrac{\mathsf{v}\cdot\mathsf{u}}{\mathsf{u}\cdot\mathsf{u}}\mathsf{u} = \dfrac{3+2-1}{1+4+1}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \dfrac{2}{3}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(b) The projection of u on v is $\operatorname{proj}_{\mathsf{v}} \mathsf{u} = \dfrac{\mathsf{u}\cdot\mathsf{v}}{\mathsf{v}\cdot\mathsf{v}}\mathsf{v} = -\frac{5}{29}\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$.

The projection of v on u is $\operatorname{proj}_{\mathsf{u}} \mathsf{v} = \dfrac{\mathsf{v}\cdot\mathsf{u}}{\mathsf{u}\cdot\mathsf{u}}\mathsf{u} = -\frac{5}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

65. With reference to the diagram at the start of the notes for Week 5, we let Q be any point on the line, say $Q(1,2,3)$. Let $\mathsf{w} = \overrightarrow{QP} = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}$. The line has direction $\mathsf{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the distance we want is the length of $\mathsf{w} - \mathsf{p}$, where $\mathsf{p} = \operatorname{proj}_{\mathsf{d}} \mathsf{w}$ is the projection of w on d. We have $\mathsf{p} = \operatorname{proj}_{\mathsf{d}} \mathsf{w} = \frac{\mathsf{w}\cdot\mathsf{d}}{\mathsf{d}\cdot\mathsf{d}}\mathsf{d} = -\frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

so $\mathsf{w} - \mathsf{p} = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{2}{3} \end{bmatrix} = \frac{2}{3}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ and the required distance is $\frac{2}{3}\left\|\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}\right\| = \frac{2}{3}\sqrt{6}$.

66. Find any point in the plane, say $Q(0,0,1)$. Then the distance we want is the length of the projection of $\overrightarrow{PQ}$ onto a normal n to the plane. (See the figure in the notes for

Week 5.) We have $\overrightarrow{PQ} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ and $\mathsf{n} = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$. So $\mathrm{proj}_{\mathsf{n}} \overrightarrow{PQ} = -\dfrac{6}{27}\begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} = -\dfrac{2}{9}\begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$.
This has length $\frac{2}{9}\sqrt{27} = \frac{2}{3}\sqrt{3}$.

To find the point $A(x, y, z)$ of the plane closest to P, note that $\overrightarrow{PA}$ is the projection of $\overrightarrow{PQ}$ on n, that is, $\overrightarrow{PA} = -\dfrac{2}{9}\begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{10}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix}$. Since $\overrightarrow{PA} = \begin{bmatrix} x-2 \\ y-3 \\ z \end{bmatrix}$, we get $A = (\frac{8}{9}, \frac{29}{9}, -\frac{2}{9})$.

67. (a) Let $\mathsf{p} = \mathrm{proj}_{\mathsf{v}}\,\mathsf{u}$. Then $\mathsf{u} - \mathsf{p}$ is orthogonal to v and in the plane spanned by u and v, so we can use these two vectors. We have $\mathsf{p} = \mathrm{proj}_{\mathsf{v}}\,\mathsf{u} = \dfrac{\mathsf{u}\cdot\mathsf{v}}{\mathsf{v}\cdot\mathsf{v}}\mathsf{v} = \dfrac{2}{5}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathsf{u} - \mathsf{p} = \begin{bmatrix} -\frac{9}{5} \\ 1 \\ \frac{18}{5} \end{bmatrix}$. In applications, we should probably multiply by 5, obtaining $\begin{bmatrix} -9 \\ 5 \\ 18 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ as a desired pair of orthogonal vectors.

(b) We proceed as in Example 4.18. First we find two nonparallel vectors in the plane, for example, $\mathsf{u} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. Let $\mathsf{p} = \mathrm{proj}_{\mathsf{v}}\,\mathsf{u}$. Then $\mathsf{u} - \mathsf{p}$ is orthogonal to v and in the plane spanned by u and v, which is precisely the given plane, so we can use these two vectors. We have $\mathsf{p} = \mathrm{proj}_{\mathsf{v}}\,\mathsf{u} = \frac{\mathsf{u}\cdot\mathsf{v}}{\mathsf{v}\cdot\mathsf{v}}\mathsf{v} = \frac{6}{5}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\mathsf{u} - \mathsf{p} = \begin{bmatrix} \frac{3}{5} \\ 2 \\ \frac{6}{5} \end{bmatrix}$. In applications, we should probably multiply by 5, obtaining $\begin{bmatrix} 3 \\ 10 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ as a desired pair of orthogonal vectors. Many answers are possible: any pair of orthogonal vectors whose components satisfy $2x - 3y + 4z = 0$ is a correct answer.

68. (a) We find two nonparallel vectors u and v in the plane, find the projection p of u on v, and take v and $\mathsf{u} - \mathsf{p}$ as our vectors. Let $\mathsf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\mathsf{p} = \mathrm{proj}_{\mathsf{v}}\,\mathsf{u} = \frac{\mathsf{u}\cdot\mathsf{v}}{\mathsf{v}\cdot\mathsf{v}}\,\mathsf{v} = \frac{-2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

and $\mathsf{u} - \mathsf{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ are suitable orthogonal vectors.

Many answers are possible, of course. Correct answers consist of two orthogonal vectors whose components satisfy $2x - y + z = 0$.

(b) In part (a), we learned that $\mathsf{e} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathsf{f} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are orthogonal vectors spanning π, so we just write the answer:

$$\operatorname{proj}_\pi \mathsf{w} = \frac{\mathsf{w}\cdot\mathsf{e}}{\mathsf{e}\cdot\mathsf{e}}\,\mathsf{e} + \frac{\mathsf{w}\cdot\mathsf{f}}{\mathsf{f}\cdot\mathsf{f}}\,\mathsf{f} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}.$$

69. (a) First we find two orthogonal vectors in the plane: $\mathsf{u} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ come quickly to mind. These are not orthogonal but, as illustrated in Example 4.18, we can get orthogonal vectors e, f by taking $\mathsf{e} = \mathsf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathsf{f} = \mathsf{u} - \mathsf{p}$, with p the projection of u on v. We have

$$\mathsf{p} = \operatorname{proj} -\mathsf{vu} = \frac{\mathsf{u}\cdot\mathsf{v}}{\mathsf{v}\cdot\mathsf{v}}\,\mathsf{v} = -\frac{1}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ 0 \\ -\frac{2}{5} \end{bmatrix}$$

and

$$\mathsf{f} = \mathsf{u} - \mathsf{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{1}{5} \\ 0 \\ -\frac{2}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ 2 \\ \frac{2}{5} \end{bmatrix} = \frac{2}{5}\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix},$$

a vector that is indeed perpendicular to e. (Did you check?) Since the purpose is simply to find orthogonal vectors that span π, we can make life much simpler by replacing f by a new $\mathsf{f} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$. Now the projection of $\mathsf{w} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ on the plane is $\operatorname{proj}_\pi \mathsf{w} = \frac{\mathsf{w}\cdot\mathsf{e}}{\mathsf{e}\cdot\mathsf{e}}\,\mathsf{e} + \frac{\mathsf{w}\cdot\mathsf{f}}{\mathsf{f}\cdot\mathsf{f}}\,\mathsf{f} = \frac{5}{5}\mathsf{e} + \frac{0}{30}\mathsf{f} = \mathsf{e}$. The projection of w on π is $\mathsf{e} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

(b) In general, we take as before $\mathsf{e} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathsf{f} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ as orthogonal vectors that span π. With $\mathsf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we have

$$\begin{aligned} \operatorname{proj}_\pi \mathsf{w} &= \frac{\mathsf{w}\cdot\mathsf{e}}{\mathsf{e}\cdot\mathsf{e}}\mathsf{e} + \frac{\mathsf{w}\cdot\mathsf{f}}{\mathsf{f}\cdot\mathsf{f}}\mathsf{f} = \frac{x+2z}{5}\mathsf{e} + \frac{-2x+5y+z}{30}\mathsf{f} \\ &= \frac{x+2z}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \frac{-2x+5y+z}{30}\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x-y+z}{3} \\ \frac{-2x+5y+z}{6} \\ \frac{2x+y+5z}{6} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2x-2y+2z \\ -2x+5y+z \\ 2x+y+5z \end{bmatrix}. \end{aligned}$$

70. Two orthogonal vectors in π are $\mathsf{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathsf{f} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, so

$$\operatorname{proj}_\pi \mathsf{w} = \frac{\mathsf{w}\cdot\mathsf{e}}{\mathsf{e}\cdot\mathsf{e}}\mathsf{e} + \frac{\mathsf{w}\cdot\mathsf{f}}{\mathsf{f}\cdot\mathsf{f}}\mathsf{f} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{13}{6} \\ \frac{17}{6} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -13 \\ 17 \\ 2 \end{bmatrix}.$$

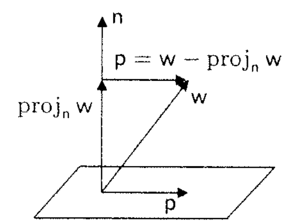
71. Write $\mathsf{w} = a\mathsf{e} + b\mathsf{f}$. Then $\mathsf{w}\cdot\mathsf{e} = a\mathsf{e}\cdot\mathsf{e}$, so $a = \frac{\mathsf{w}\cdot\mathsf{e}}{\mathsf{e}\cdot\mathsf{e}}$. Similarly $b = \frac{\mathsf{w}\cdot\mathsf{f}}{\mathsf{f}\cdot\mathsf{f}}$, so $\mathsf{w} = \frac{\mathsf{w}\cdot\mathsf{e}}{\mathsf{e}\cdot\mathsf{e}}\mathsf{e} + \frac{\mathsf{w}\cdot\mathsf{f}}{\mathsf{f}\cdot\mathsf{f}}\mathsf{f}$.

72. (a) The lines have directions $\mathsf{d}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ and $\mathsf{d}_2 = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}$. Since d_2 is a scalar multiple of d_1, the lines are parallel.

(b) The distance between parallel lines is the distance from any point on one line to the other line. We take $Q = (0,1,4)$ as a point on the first line and $P = (4,-2,2)$ on the second line and proceed as we did in Week 5. Let $\mathsf{w} = \overrightarrow{PQ} = \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$. The second line has direction $\mathsf{d} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ and the distance we want is the length of $\mathsf{w} - \mathsf{p}$, where $\mathsf{p} = \operatorname{proj}_\mathsf{d} \mathsf{w}$ is the projection of w on d: $\mathsf{p} = \operatorname{proj}_\mathsf{d} \mathsf{w} = \frac{\mathsf{w}\cdot\mathsf{d}}{\mathsf{d}\cdot\mathsf{d}}\mathsf{d} = -\frac{15}{9}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$, $\mathsf{w} - \mathsf{p} = \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix} + \frac{5}{3}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{4}{3} \end{bmatrix} = \frac{2}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$, so the required distance is $\frac{2}{3}\left\| \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \right\| = \frac{2}{3}(3) = 2$.

73. The converse of Theorem 5.2 says that if a vector w is orthogonal to every vector in the plane π spanned by u and v, then w is orthogonal to u and to v.

74. (a) The picture certainly makes Don's theory appear plausible and the following argument establishes that it is indeed correct. Don takes for the projection the vector $\mathsf{p} = \mathsf{w} - \operatorname{proj}_{\mathsf{n}} \mathsf{w}$. By definition of "projection on a plane," we must show that $\mathsf{w} - \mathsf{p}$ is orthogonal to every vector in π. This is immediately clear since $\mathsf{w} - \mathsf{p} = \mathsf{w} - (\mathsf{w} - \operatorname{proj}_{\mathsf{n}} \mathsf{w}) = \operatorname{proj}_{\mathsf{n}} \mathsf{w}$ is a normal to the plane.



(b) A normal to π is $\mathsf{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$. Thus $\operatorname{proj}_{\mathsf{n}} \mathsf{w} = \dfrac{\mathsf{w} \cdot \mathsf{n}}{\mathsf{n} \cdot \mathsf{n}} \mathsf{n} = \dfrac{2}{14} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \dfrac{1}{7} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and

$\mathsf{p} = \mathsf{w} - \operatorname{proj}_{\mathsf{n}} \mathsf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \dfrac{1}{7} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \dfrac{1}{7} \begin{bmatrix} 4 \\ 9 \\ 6 \end{bmatrix}$.

75. (a) The lines have directions $\mathsf{d}_1 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ and $\mathsf{d}_2 = \begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix}$. Neither vector is a scalar multiple of the other, so the lines are not parallel. Suppose they intersect at (x, y, z). Then there would exist t and s such that

$$\begin{aligned} x &= 3 + 4t = 2 + 2s \\ y &= 3t \quad\;\; = -1 + 6s \\ z &= -1 + t = -3 + 7s. \end{aligned}$$

The third equation says $-3 + 3t = -9 + 21s$. Since $3t = -1 + 6s$, we have $-3 - 1 + 6s = -9 + 21s$, so $15s = 5$, $s = \frac{1}{3}$ and (third equation) $t = -2 + 7s = \frac{1}{3}$ too. But $s = t = \frac{1}{3}$ do not satisfy the first equation. No s, t exist, so the lines do not intersect.

(b) Each line is perpendicular to a normal to the plane, so the cross product of the two direction vectors is a normal:

$$\mathsf{n} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{j} \\ 4 & 3 & 1 \\ 2 & 6 & 7 \end{vmatrix} = 15\mathsf{i} - 26\mathsf{j} + 18\mathsf{k} = \begin{bmatrix} 15 \\ -26 \\ 18 \end{bmatrix}.$$

The plane has equation $15x - 26y + 18z = d$, and since it contains the point $(3, 0, -1)$, $d = 15(3) + 18(-1) = 27$. The equation is $15x - 26y + 18z = 27$.

(c) The shortest distance between the lines is the distance from any point on ℓ_2, say $P(2, -1, -3)$, to π. Take a point on π, say $Q(3, 0, -1)$. Then $\overrightarrow{PQ} = [1, 1, 2]$. We have $\operatorname{proj}_{\mathsf{n}} \overrightarrow{PQ} = \dfrac{\overrightarrow{PQ} \cdot \mathsf{n}}{\mathsf{n} \cdot \mathsf{n}} \mathsf{n} = \dfrac{25}{1225} \mathsf{n} = \dfrac{1}{49} \mathsf{n}$, so the distance we want, the length of this vector, is $\frac{1}{49}\sqrt{1225}$.

76. (a) This is not defined.

(b) $\|\mathsf{y}\|^2 = 0^2 + (-1)^2 + 1^2 + 2^2 = 6$, so $\|\mathsf{y}\| = \sqrt{6}$.

77. We need $a - 1 = 1 - b$, $2b = -6$, $3 = -b$, and $a - b = 8$, so $b = -3$, $a = 8 + b = 5$.

78. Since $3\mathsf{u} - 2\mathsf{v} = \begin{bmatrix} -3-2c \\ 3a-2 \\ 3b-2 \\ 6-2a \\ 8 \end{bmatrix}$, we must have $-3 - 2c = -1$, $3a - 2 = 1$, $3b - 2 = -2$, and $6 - 2a = 4$. We obtain $a = 1$, $b = 0$, $c = -1$.

79. Since $\|\mathsf{u}\| = \sqrt{3}$, the answer is $\dfrac{\mathsf{u}}{\|\mathsf{u}\|} = \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

80. Since $\|c\mathsf{u}\| = |c|\,\|\mathsf{u}\| = |c|\sqrt{6}$, we want $|c| = \dfrac{8}{\sqrt{6}}$, so $c = \pm\frac{4}{3}\sqrt{6}$.

81. $\mathsf{u}\cdot\mathsf{v} = -2$, $\|\mathsf{u}\| = \sqrt{6}$, $\|\mathsf{v}\| = \sqrt{2}$, so $\cos\theta = \dfrac{\mathsf{u}\cdot\mathsf{v}}{\|\mathsf{u}\|\,\|\mathsf{v}\|} = \dfrac{-2}{\sqrt{12}} \approx -0.577$, and $\theta \approx \arccos(0.577) \approx 2.2$ rads $\approx 125°$.

82. $\|2\mathsf{u} - \mathsf{v}\|^2 = (2\mathsf{u} - \mathsf{v})\cdot(2\mathsf{u} - \mathsf{v}) = 4\mathsf{u}\cdot\mathsf{u} - 4\mathsf{u}\cdot\mathsf{v} + \mathsf{v}\cdot\mathsf{v} = 4\|\mathsf{u}\|^2 - 4\mathsf{u}\cdot\mathsf{v} + \|\mathsf{v}\|^2$. We are given that $0 = \mathsf{u}\cdot(\mathsf{u}+\mathsf{v}) = \mathsf{u}\cdot\mathsf{u} + \mathsf{u}\cdot\mathsf{v} = \|\mathsf{u}\|^2 + \mathsf{u}\cdot\mathsf{v}$, so $\mathsf{u}\cdot\mathsf{v} = -6$. Thus $\|2\mathsf{u} - \mathsf{v}\|^2 = 4(6) - 4(-6) + 1 = 49$ and $\|2\mathsf{u} - \mathsf{v}\| = 7$.

83. $(\mathsf{u}+\mathsf{v})\cdot(\mathsf{x}-\mathsf{y}) = \mathsf{u}\cdot\mathsf{x} - \mathsf{u}\cdot\mathsf{y} + \mathsf{v}\cdot\mathsf{x} - \mathsf{v}\cdot\mathsf{y} = \cos 60° - 0 - \frac{1}{3} - \frac{\sqrt{2}}{2} = \frac{\sqrt{3}}{2} - \frac{1}{3} - \frac{\sqrt{2}}{2} = \frac{1}{2}(\sqrt{3} - \sqrt{2}) - \frac{1}{3}$.

84. The vector v is a linear combination of $\mathsf{v}_1, \dots, \mathsf{v}_5$ since $\mathsf{v} = 0\mathsf{v}_1 + 0\mathsf{v}_2 + (-1)\mathsf{v}_3 + 0\mathsf{v}_4 + 0\mathsf{v}_5$.

85. We have $\mathsf{u}\cdot\mathsf{v} = -3$, $\|\mathsf{u}\| = \sqrt{5}$, $\|\mathsf{v}\| = \sqrt{10}$. The Cauchy–Schwarz inequality is the statement $3 \le \sqrt{5}\sqrt{10}$. Note that $\sqrt{5}\sqrt{10} \approx 7.1$. Since $\mathsf{u}+\mathsf{v} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$, $\|\mathsf{u}+\mathsf{v}\| = \sqrt{9} = 3$. The triangle inequality says $3 \le \sqrt{5} + \sqrt{10}$. Note that $\sqrt{5} + \sqrt{10} \approx 5.4$.

86. Since $\mathsf{v}_4 = -2\mathsf{v}_1$, the vectors are linearly dependent: $2\mathsf{v}_1 + 0\mathsf{v}_2 + 0\mathsf{v}_3 + \mathsf{v}_4 = \mathsf{0}$.

87. If you know, please tell me.

88. **First line:** Suppose $c_1\mathsf{v}_1 + c_2\mathsf{v}_2 + \cdots + c_k\mathsf{v}_k = \mathsf{0}$ for scalars $c_1, c_2, \dots, c_k$.
Last line: Therefore $c_1 = 0$, $c_2 = 0$, $\dots$, $c_k = 0$.

89. Matrices are equal if and only if corresponding columns and hence corresponding