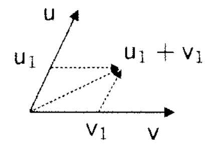
Linear Algebra Assignment 3

38. If u and v had the same length, the vector u + v represented by the diagonal of the parallelogram with sides u and v would do.



Since we have no information about lengths, we replace u and v by unit vectors u_1, v_1 with the same directions. So a suitable vector is $u_1 + v_1 = \frac{1}{\|u\|}u + \frac{1}{\|v\|}v$.

39. The diagonals are $\overrightarrow{AC} = u + v$ and $\overrightarrow{DB} = u - v$.

40. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the plane if and only if $z = -3x + 2y$. So the plane consists of all vectors of the form $\begin{bmatrix} x \\ y \\ -3x+2y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. This says that every vector in the plane is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

41. The answer is an equation of the form $4x + 2y - z = d$. Substituting $x = 1$, $y = 2$, $z = 3$, we get $d = 4 + 4 - 3 = 5$, so $4x + 2y - z = 5$ works.

42. $$u \times v = \begin{vmatrix} i & j & k \\ 3 & 0 & 2 \\ -2 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} i - \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} j + \begin{vmatrix} 3 & 0 \\ -2 & 4 \end{vmatrix} k = -8i - 7j + 12k = \begin{bmatrix} -8 \\ -7 \\ 12 \end{bmatrix}.$$

$$v \times u = \begin{vmatrix} i & j & k \\ -2 & 4 & 1 \\ 3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 0 & 2 \end{vmatrix} i - \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} j + \begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix} k = 8i - (-7)j - 12k = \begin{bmatrix} 8 \\ 7 \\ -12 \end{bmatrix} = -(u \times v).$$

43. $$u \times v = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} j + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} k = 3i - (-5)j + 7k = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}.$$ To verify that this is indeed orthogonal to u and to v, we compute $(u \times v) \cdot u = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 3 - 10 + 7 = 0$ and $(u \times v) \cdot v = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = 9 + 5 - 14 = 0$.

Finally, we have $\mathsf{v} \times \mathsf{u} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \mathsf{i} - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \mathsf{j} + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} \mathsf{k} = -3\mathsf{i} - 5\mathsf{j} + (-7)\mathsf{k} = \begin{bmatrix} -3 \\ -5 \\ -7 \end{bmatrix} = -(\mathsf{u} \times \mathsf{v}).$$

44. $\mathsf{u} \times \mathsf{v} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ -2 & 1 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$

$(\mathsf{u} \times \mathsf{v}) \times \mathsf{w} = 2\begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & 3 & -1 \\ 4 & 0 & -3 \end{vmatrix} = 2\begin{bmatrix} -9 \\ -1 \\ -12 \end{bmatrix} = \begin{bmatrix} -18 \\ -2 \\ -24 \end{bmatrix}.$

$\mathsf{v} \times \mathsf{w} = \begin{vmatrix} 0 & 1 & 3 \\ 4 & 0 & -3 \end{vmatrix} = \begin{bmatrix} -3 \\ 12 \\ -4 \end{bmatrix}.$

$\mathsf{u} \times (\mathsf{v} \times \mathsf{w}) = \begin{vmatrix} -2 & 1 & 1 \\ -3 & 12 & -4 \end{vmatrix} = \begin{bmatrix} -16 \\ -11 \\ -21 \end{bmatrix}.$

Since the cross product is not an associative operation—that is, $(\mathsf{u}\times\mathsf{v})\times\mathsf{w} \neq \mathsf{u}\times(\mathsf{v}\times\mathsf{w})$ in general—we should not be surprised by the results.

45. A normal is $\mathsf{u} \times \mathsf{v}$

$$= \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & 0 & 4 \\ -1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 2 & 0 \end{vmatrix} \mathsf{i} - \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = -8\mathsf{i} - 4\mathsf{j} + 2\mathsf{k} = \begin{bmatrix} -8 \\ -4 \\ 2 \end{bmatrix}.$$

The vector $\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ is also a normal. Since the plane in question passes through $(0, 0, 0)$, an equation is $4x + 2y - z = 0$.

46. One vector perpendicular to both u and v is $\mathsf{u} \times \mathsf{v} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$

$$= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \mathsf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \mathsf{j} + \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \mathsf{k} = 6\mathsf{i} - 3\mathsf{j} + 0\mathsf{k} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix}.$$

Any multiple of this vector, for instance $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, is another.

47. One vector perpendicular to both u and v is $\mathsf{u} \times \mathsf{v} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & 3 & -1 \\ 5 & 0 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} \mathsf{i} - \begin{vmatrix} 1 & -1 \\ 5 & 1 \end{vmatrix} \mathsf{j} + \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} \mathsf{k} = 3\mathsf{i} - 6\mathsf{j} - 15\mathsf{k} = \begin{bmatrix} 3 \\ -6 \\ -15 \end{bmatrix}.$$

Any multiple of this is also perpendicular to both, for instance, $\mathsf{n} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, a vector of length $\sqrt{1+4+25} = \sqrt{30}$. Now $\frac{1}{\sqrt{30}}\mathsf{n}$ has length one and $\frac{5}{\sqrt{30}}\mathsf{n}$ has length five. One answer is $\frac{5}{\sqrt{30}}\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$. The only other possibility is the negative of this vector.

48. $\mathsf{u} \times \mathsf{v} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & 2 & 3 \\ 0 & -1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \mathsf{i} - \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \mathsf{j} + \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \mathsf{k} = 5\mathsf{i} - 1\mathsf{j} - 1\mathsf{k} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}.$$

Thus $\|\mathsf{u} \times \mathsf{v}\| = \sqrt{25+1+1} = \sqrt{27}$.
Also $\|\mathsf{u}\| = \sqrt{1+4+9} = \sqrt{14}$, $\|\mathsf{v}\| = \sqrt{0+1+1} = \sqrt{2}$ and $\mathsf{u} \cdot \mathsf{v} = 0 - 2 + 3 = 1$. The cosine of the angle θ between u and v is $\cos\theta = \dfrac{\mathsf{u} \cdot \mathsf{v}}{\|\mathsf{u}\|\,\|\mathsf{v}\|} = \dfrac{1}{\sqrt{28}}$. So $\sin^2\theta = 1 - \cos^2\theta = \dfrac{27}{28}$. Assuming $0 \le \theta \le \pi$, $\sin\theta \ge 0$, so $\sin\theta = \sqrt{\dfrac{27}{28}}$. Thus $\|\mathsf{u}\|\,\|\mathsf{v}\|\sin\theta = \sqrt{14}\sqrt{2}\sqrt{\dfrac{27}{28}} = \sqrt{27} = \|\mathsf{u} \times \mathsf{v}\|$.

49. $\overrightarrow{AB} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -2 \\ 1 \\ -7 \end{bmatrix}$. A normal vector is $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & -2 & 2 \\ -2 & 1 & -7 \end{vmatrix} = 12\mathsf{i} + 3\mathsf{j} - 3\mathsf{k} = \begin{bmatrix} 12 \\ 3 \\ -3 \end{bmatrix}$. We take $\mathsf{n} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$. The equation of the plane is $4x + y - z = d$. Since the coordinates of A satisfy the equation, we have $8 + 1 - 3 = d$, so $d = 6$ and the equation is $4x + y - z = 6$.

50. Setting $t = 0$ and then $t = 1$, we see that $A(1, 0, 2)$ and $B(-3, 3, 3)$ are also on the plane. The plane contains the arrows from P to A and from P to B, hence it is parallel to the vectors $\begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}$. So a normal vector is $\mathsf{n} = \begin{vmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ -2 & -1 & 3 \\ -6 & 2 & 4 \end{vmatrix} = \begin{bmatrix} -10 \\ -10 \\ -10 \end{bmatrix}$.

The vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is equally good. The equation of the plane is $x+y+z=d$. Substituting the coordinates of P gives $d=3$, so we get $x+y+z=3$.

51. The direction of the line is $\begin{bmatrix}-1\\1\end{bmatrix}$, so the slope is -1 and the equation is of the form $y=-x+b$. The point $(1,2)$ lies on the line, so $2=-1+b$, $b=3$. The line has equation $x+y=3$.

52. The line has slope 2, so $\begin{bmatrix}1\\2\end{bmatrix}$ is a direction vector. The line contains $(0,-3)$, so the vector equation is $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\-3\end{bmatrix}+t\begin{bmatrix}1\\2\end{bmatrix}$.

53. (a) We need triples (x,y,z) that satisfy both equations. Setting $x=0$ gives $y+z=5$, $-y+z=1$, so $z=3$ and $y=2$. This gives the point $A(0,2,3)$. Setting $y=0$ gives $2x+z=5$, $x+z=1$, so $x=4$ and $z=-3$. This gives the point $B(4,0,-3)$. Setting $z=0$ gives $2x+y=5$, $x-y=1$, so $x=2$ and $y=1$. This gives the point $C(2,1,0)$. Many other points are possible, of course.

(b) **Solution 1.** The line has direction $\overrightarrow{AB}=\begin{bmatrix}4\\-2\\-6\end{bmatrix}$ and hence also $\begin{bmatrix}2\\-1\\-3\end{bmatrix}$. A vector equation is $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\2\\3\end{bmatrix}+t\begin{bmatrix}2\\-1\\-3\end{bmatrix}$.

Solution 2. The line has direction perpendicular to the normal vector of each plane, so the cross product of the normal vectors gives a direction vector. This cross product is $\begin{vmatrix}\mathsf{i} & \mathsf{j} & \mathsf{k}\\ 2 & 1 & 1\\ 1 & -1 & 1\end{vmatrix}=2\mathsf{i}-\mathsf{j}-3\mathsf{k}=\begin{bmatrix}2\\-1\\-3\end{bmatrix}$. As in Solution 1, we still need to find a point on the line; $A(0,2,3)$ will do. The equation is $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\2\\3\end{bmatrix}+t\begin{bmatrix}2\\-1\\-3\end{bmatrix}$, as before.

[Solutions to this question may look different from the one we have obtained, of course. A correct answer must be the equation of a line whose direction vector is a multiple of $\begin{bmatrix}2\\-1\\-3\end{bmatrix}$ and that passes through a point whose coordinates satisfy the equation of each plane.]

54. If (x,y,z) is on the line, then, for some t, $x=1+2t$, $y=-2+5t$, $z=3-t$. If such a point is on the plane, then $4=x-3y+2z=(1+2t)-3(-2+5t)+2(3-t)=13-15t$, so $15t=9$, $t=\frac{3}{5}$ and $(x,y,z)=(\frac{11}{5},1,\frac{12}{5})$.

55. $\begin{bmatrix}2\\-3\\5\end{bmatrix}$ is a normal and hence perpendicular to π.

56. (a) Since $\overrightarrow{AB} = \begin{bmatrix}1\\1\\0\end{bmatrix} = \overrightarrow{CD}$, $ABCD$ is a parallelogram. Since

$$\overrightarrow{AB} \times \overrightarrow{AC} == \begin{bmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 1 & 1 & 0 \\ 1 & -4 & 4 \end{bmatrix} = 4\mathsf{i} - 4\mathsf{j} - 5\mathsf{k} = \begin{bmatrix}4\\-4\\-5\end{bmatrix},$$

the area is $\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \sqrt{57}$.

(b) The area of the triangle is $\frac{1}{2}\sqrt{57}$, one-half the area of the parallelogram.

57. Two sides of the triangle are $\overrightarrow{AB} = \begin{bmatrix}6\\1\end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix}7\\-6\end{bmatrix}$. Think of these as lying in 3-space. The triangle has area one half the area of the parallelogram with sides $\mathsf{u} = \begin{bmatrix}6\\1\\0\end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix}7\\-6\\0\end{bmatrix}$. This is one half the length of

$$\mathsf{u} \times \mathsf{v} = \begin{bmatrix} \mathsf{i} & \mathsf{j} & \mathsf{k} \\ 6 & 1 & 0 \\ 7 & -6 & 0 \end{bmatrix} = 0\mathsf{i} - 0\mathsf{j} - 43\mathsf{k} = \begin{bmatrix}0\\0\\-43\end{bmatrix}.$$

The area is $\frac{1}{2}\sqrt{(-43)^2} = \frac{43}{2}$.

58. Since $ad - bc = 0$, we have $ad = bc$.

If $a \neq 0$ and $d \neq 0$, then $b \neq 0$, so $\dfrac{c}{a} = \dfrac{d}{b} = k$. This says $c = ka$ and $d = kb$, so $\begin{bmatrix}c\\d\end{bmatrix} = k\begin{bmatrix}a\\b\end{bmatrix}$.

Suppose $a = 0$ and $d \neq 0$. Then $b = 0$ or $c = 0$. In the case $b = 0$, then $\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}c\\d\end{bmatrix}$, while if $c = 0$ (and $b \neq 0$), then $\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}0\\d\end{bmatrix} = \frac{d}{b}\begin{bmatrix}0\\b\end{bmatrix}$.

Suppose $a \neq 0$ and $d = 0$. Again, $b = 0$ or $c = 0$. In the case $b = 0$, then $\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}c\\0\end{bmatrix} = \frac{c}{a}\begin{bmatrix}a\\0\end{bmatrix} = [c]\ a\begin{bmatrix}a\\b\end{bmatrix}$, while if $c = 0$ (and $b \neq 0$), then $\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}a\\b\end{bmatrix}$.

Finally, if $a = d = 0$, then $b = 0$ or $c = 0$. If $b = 0$, then $\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}c\\d\end{bmatrix}$, while if $c = 0$ (and $b \neq 0$), then $\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}0\\d\end{bmatrix} = \frac{d}{b}\begin{bmatrix}0\\b\end{bmatrix} = \frac{d}{b}\begin{bmatrix}a\\b\end{bmatrix}$.

In every case, one of the two given vectors is a multiple of the other.

59. If the lines intersect, say at the point (x, y, z), then (x, y, z) is on both lines. So there is a t such that

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\-2\end{bmatrix} + t\begin{bmatrix}-3\\1\\1\end{bmatrix}$$

and an s such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 11 \\ -3 \\ -5 \end{bmatrix}.$$

The question then is, do there exist parameters t and s so that

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 11 \\ -3 \\ -5 \end{bmatrix} \; ?$$

We try to solve

$$\begin{aligned} 1-3t &= -4+11s \\ t &= 1-3s \\ -2+t &= 1-5s; \end{aligned} \qquad \text{that is,} \qquad \begin{aligned} 11s+3t &= 5 \\ 3s+t &= 1 \\ 5s+t &= 3, \end{aligned}$$

and find that $t=-2$, $s=1$ is a solution. The lines intersect where

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 11 \\ -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ -4 \end{bmatrix};$$

that is, at the point $(7,-2,-4)$.

60. Since the lines are perpendicular, the direction vectors must have dot product 0: $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ b \end{bmatrix} = 0$ gives $-1-2+b=0$, so $b=3$. Since the lines intersect, there are values of t and s such that

$$\begin{aligned} 2-t &= 3+s \\ -1+2t &= 1-s \\ 3+t &= a+bs = a+3s. \end{aligned}$$

Adding the first two equations gives $1+t=4$, so $t=3$ and $3+s=2-t=-1$, so $s=-4$. Substituting $t=3$, $s=-4$ in the third equation gives $6=a-12$, so $a=18$.

61. **Solution 1.** The point $A(x,y,z)$ is on the plane if and only if $\left\|\overrightarrow{PA}\right\| = \left\|\overrightarrow{QA}\right\|$; that is, if and only if

$$\sqrt{(x-2)^2+(y+1)^2+(z-3)^2} = \sqrt{(x-1)^2+(y-1)^2+(z+1)^2};$$

that is, if and only if

$$\begin{aligned} &x^2-4x+4+y^2+2y+1+z^2-6z+9 \\ &\qquad = x^2-2x+1+y^2-2y+1+z^2+2z+1. \end{aligned}$$

We obtain $-2x+4y-8z+11=0$.